PHYSICAL REVIEW D VOLUME 32, NUMBER 8 15 OCTOBER 1985

Gauge invariance for spin-3 fields

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(Received 18 October 1983)

The problem of constructing a manifestly covariant field theory for spin-3 fields with interactions is reconsidered using results from the light-cone treatment of higher-spin fields. In the case of self-interactions, structure relations for the gauge algebra are studied. It is shown that the possible field-independent structure relations that could be derived from the gauge transformations given to first order in the coupling constant do not obey the Jacobi identity. Minimal and nonminimal couplings to Yang-Mills fields are also studied. In this case, the gauge covariance of the field equation fails.

I. INTRODUCTION

The theory of free fields of higher spin, both massive[1-3] and massless,[4,5] is well established. The introduction of interactions has, however, met with considerable difficulties. Minimal coupling of massless spin-$\frac{5}{2}$ fields to electromagnetism or gravity fails.[6] Even for spin-$\frac{3}{2}$ fields, the coupling to an electromagnetic field leads to problems with noncausal propagation.[7]

Minimal and nonminimal coupling of spin-3 fields to gravity has also been investigated.[8] In this case, the spin-3 gauge invariance of the action fails.

As regards self-interactions, this problem does not seem to have been extensively considered in the literature. One proposed approach is the generalized Gupta program[4,9] for massless fields of integer spin.

It is clear that a manifestly covariant interacting theory, if it exists, must be a gauge theory of some kind. This is necessary since the field $\phi_{\mu_1\cdots\mu_s}$, totally symmetric in its indices, which can be seen to contain spin s, has $\binom{s+3}{s}$ components in four dimensions, but it should describe only two degrees of freedom corresponding to the helicities $\lambda=\pm s$. The alternative to a gauge theory is to give up manifest Lorentz covariance, and try to formulate the theory directly in terms of physical field components. In fact, in a light-cone frame, it is possible to construct interactions at least to first order in the coupling constant for arbitrary massless fields of integer spin.[10] In light of this result it is natural to ask whether a covariant formulation can be found which implements, in a Lorentz-covariant way, the properties of the light-cone interaction terms.

Higher-spin resonances do exist in nature, but they are then massive and considered as bound states. As regards massless fields, it is known from S-matrix arguments that they cannot mediate long-range forces.[11] This, however, rests on the assumption of a covariant formalism. Thus, the motivation for a study of higher-spin massless fields is mainly their theoretical interest. It can also be mentioned that higher-spin fields have been considered in connection with the question of extending supersymmetry beyond $N=8$.

In this paper we will only study the case of spin 3.

II. PROPERTIES OF MASSLESS SPIN-3 FIELDS

The massless spin-3 field will be described by a totally symmetric tensor field $\phi_{\mu_1\mu_2\mu_3}$. The wave equation[2,4,5]

$$W_{\mu_1\mu_2\mu_3}=\Box\phi_{\mu_1\mu_2\mu_3}-\sum_{\mu\ 1}\partial_{\mu_1}\partial_\sigma\phi^\sigma{}_{\mu_2\mu_3}+\sum_{\mu\ 1}\partial_{\mu_1}\partial_{\mu_2}\phi_\sigma{}^\sigma{}_{\mu_3}=0 \tag{1}$$

is invariant under the gauge transformation

$$\delta\phi_{\mu_1\mu_2\mu_3}=\sum_{\mu\ 1}\partial_{\mu_1}\xi_{\mu_2\mu_3} \tag{2}$$

with

$$\xi_\sigma{}^\sigma=0\ , \tag{3}$$

since under this transformation

$$\delta W_{\mu_1\mu_2\mu_3}=3\partial_{\mu_1}\partial_{\mu_2}\partial_{\mu_3}\xi_\sigma{}^\sigma=0\ . \tag{4}$$

It has been shown[4] that with a traceless gauge parameter the gauge group is large enough to remove all but the two physical helicities.

It is straightforward to reduce this free theory to a light-cone gauge. Choose the nine gauge conditions

$$\phi^{+\mu\nu}-\tfrac{1}{4}\eta^{\mu\nu}\phi^+{}_\sigma{}^\sigma=0\ . \tag{5}$$

This is equivalent to

$$\begin{aligned}&\phi^{+++}=\phi^{+--}=\phi^{++i}=\phi^{+-i}=\phi^{+12}=0\ ,\\&\phi^{++-}=-\phi^{+11}=-\phi^{+22}\ .\end{aligned} \tag{6}$$

The equation of motion yields

$$\phi^{++-}=\phi^{+11}=\phi^{+22}=0 \tag{7}$$

and

$$\phi^{-ij}=\frac{\partial_k}{\partial^+}\phi^{kij}\ ,$$

$$\phi^{--i}=\frac{\partial_j\partial_k}{\partial^{+2}}\phi^{kji}\ ,$$

$$\phi^{---}=\frac{\partial_i\partial_j\partial_k}{\partial^{+3}}\phi^{kji}\ , \tag{8}$$

$$\phi^{11i}=-\phi^{22i}\ .$$

Owing to the tracelessness of ϕ^{ijk}, the transverse components describe precisely two degrees of freedom. The complex light-cone field is conveniently defined as

$$\phi=\sqrt{1/2}(\phi^{111}+i\phi^{112})\ . \tag{9}$$

Now, in the light-cone treatment of the spin-3 theory we have shown that a three-point interaction can be constructed if the field belongs to the adjoint representation of a Lie group. Otherwise, the interaction term vanishes. In fact, we have found the action[10] to be

$$S=\int d^4x\left[\tfrac{1}{4}\bar{\phi}^a\Box\phi^a-\alpha f^{abc}\partial^{+3}\bar{\phi}^a\times\left[\frac{\bar{\partial}^3}{\partial^{+3}}\phi^b\phi^c-3\frac{\bar{\partial}^2}{\partial^{+2}}\phi^b\frac{\bar{\partial}}{\partial^+}\phi^c+3\frac{\bar{\partial}}{\partial^+}\phi^b\frac{\bar{\partial}^2}{\partial^{+2}}\phi^c-\phi^b\frac{\bar{\partial}^3}{\partial^{+3}}\phi^c\right]\right]+\text{c.c.}+O(\alpha^2)\ . \tag{10}$$

From this we also learn that the coupling constant has mass dimension -2 and that the three-point coupling contains three transverse derivatives.

III. THE QUESTION OF EXTENDING THE GAUGE ALGEBRA TO FIRST ORDER IN THE COUPLING CONSTANT

In Ref. 5 it was shown that the free field theory of higher-spin fields can be nicely formulated in terms of generalized Christoffel symbols. These symbols generalize in a natural way the field strength $F_{\mu\nu}$ for spin 1, and the affine connection $\Gamma^{\lambda}_{\mu\nu}$ and curvature $R^{\lambda}_{\mu\nu\tau}$ for spin 2 in the linearized theory. It was hoped that the generalized Christoffel symbols would be useful when studying the question of interactions.

Since we are here only interested in self-interactions, the spin-1 field will be the Yang-Mills potentials A^a_μ. The spin-2 field is the graviton $h_{\mu\nu}$, defined through the usual split $g_{\mu\nu}=\eta_{\mu\nu}+\kappa h_{\mu\nu}$.

A simple approach to spin-3 theory would be to first bring out the similarities between Yang-Mills theory and gravity, and then try to generalize these to spin 3 using the information provided by the light-cone treatment. One important property in this context is then, of course, gauge covariance. For Yang-Mills theory, the nature of gauge covariance is clear.[12,13] However the situation is somewhat more complicated for gravity. It is well known that gravity can be treated as the gauge theory of the Poincaré group,[13,14] although a slight modification of Einstein's theory arises in that the spin of matter couples (kinematically) to the torsion tensor. In this treatment both the vierbein field e^i_μ and the spin connections ω^{ij}_μ appear as gauge fields corresponding to local translations and local Lorentz rotations, respectively. But as a consequence of the field equations, ω^{ij}_μ is not independent of e^i_μ.

It is quite probable that an interacting theory of spin-3 fields would show a still richer structure. In the free spin-3 theory there are three gauge potentials, since apart from the field $\phi^a_{\mu_1\mu_2\mu_3}$ itself, we also have $\Gamma^{(1)a}_{\alpha:\mu_1\mu_2\mu_3}$ and $\Gamma^{(2)a}_{\alpha\beta:\mu_1\mu_2\mu_3}$ (where the notation is explained in Appendix A). The gauge-covariant (invariant in the free theory) object is $\Gamma^{(3)a}_{\alpha\beta\gamma:\mu_1\mu_2\mu_3}$. However, all independent degrees of freedom reside in the field ϕ. It is tempting to suppose that these features would be present also in an interacting theory. Now the concept of gauge potentials is linked to the procedure of making a global invariance local. One is thus led to pose the question: which global invariance is gauged by the potentials ϕ, $\Gamma^{(1)}$, and $\Gamma^{(2)}$? This seems to be the central question in spin-3 theory. On the other hand, it might be the case that only $\phi^a_{\mu_1\mu_2\mu_3}$ plays the rôle of the gauge field corresponding to the parameter $\xi^a_{\mu\nu}(x)$, and that $\Gamma^{(1)}$ and $\Gamma^{(2)}$ have no significance as gauge potentials.

Here, the problem will be approached in a simple, pragmatic way by studying to what extent the Abelian gauge algebra of the free theory can be deformed[9] into a non-Abelian algebra which closes to first order in the coupling constant α. To get a hint of what we should look for, let us exploit the systematics inherent in the use of the generalized Christoffel symbols. For Yang-Mills theory and gravity, the gauge transformations are to first order in the coupling constant

$$\delta A^a_\mu=\partial_\mu\xi^a-gf^{abc}\xi^bA^c_\mu\equiv D_\mu\xi^a\ , \tag{11}$$

$$\begin{aligned}\delta h_{\mu\nu}&=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu+\kappa\xi^\sigma(\partial_\sigma h_{\mu\nu}-\partial_\mu h_{\sigma\nu}-\partial_\nu h_{\sigma\mu})\\&=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu+\kappa\xi^\sigma\Gamma^{(1)}_{\sigma:\mu\nu}\\&\equiv D_\mu\xi_\nu+D_\nu\xi_\mu\ .\end{aligned} \tag{12}$$

A natural candidate for an extension of the gauge transformation to $O(\alpha)$ for spin 3 would be

$$\begin{aligned}\delta\phi^a_{\mu_1\mu_2\mu_3}&=\sum_{\mu\ 1}(\partial_{\mu_1}\xi^a_{\mu_2\mu_3}-\alpha f^{abc}\xi^{b\sigma\rho}\Gamma^{(2)c}_{\sigma\rho:\mu_1\mu_2\mu_3})\\&\equiv\sum_{\mu\ 1}D_{\mu_1}\xi^a_{\mu_2\mu_3}\ .\end{aligned} \tag{13}$$

The rationale for this choice of the form of the transformation is the following.

(i) On dimensional grounds there must be two derivatives on the field in the nonlinear part of D_μ, since α has mass dimension -2. These derivatives could in principle act in a lot of different ways, but $\Gamma^{(2)}$ is a well-defined object which in a natural way generalizes $\Gamma^{(1)}$ which appears in the covariant derivative for spin 2.

(ii) The commutator of two covariant derivatives should be a gauge-covariant object. With the above choice of D_μ one finds that the commutator is a linear combination of $\Gamma^{(3)}$.

However, in order to attempt a construction of an interacting spin-3 theory, a more general ansatz for $\delta\phi$ should be used. Schematically this would contain terms of the forms

$$\alpha f^{abc}\xi^b\partial\partial\phi^c ,$$
$$\alpha f^{abc}\partial\xi^b\partial\phi^c ,$$
$$\alpha f^{abc}\partial\partial\xi^b\phi^c ,$$

where space-time indices are suppressed. By commuting such gauge transformations (if they exist), the structure relations can be read off. By demanding the commutator of gauge transformations to be a gauge transformation, we would get to first order in the coupling constant

$$[\delta_\xi,\delta_\eta]\phi^a_{\mu_1\mu_2\mu_3}=\sum_{\mu\ 1}\partial_{\mu_1}\omega^a_{\mu_2\mu_3} , \tag{14}$$

where $\omega^a_{\mu_1\mu_2}$ defines the new parameter. When doing this calculation one must also know how the parameters transform.

From the commutator (14) one could then extract the structure relations

$$[\xi,\eta]^a_{\mu\nu}=\omega^a_{\mu\nu} . \tag{15}$$

The notation is best explained by recording the structure relations for the non-Abelian gauge algebra for spin 1 and 2, respectively,

$$[\xi,\eta]^a=gf^{abc}\xi^b\eta^c , \tag{16}$$

$$[\xi,\eta]_\mu=\kappa(\xi^\lambda\partial_\lambda\eta_\mu-\eta^\lambda\partial_\lambda\xi_\mu) . \tag{17}$$

The point is that the structure relations which can be derived from the first-order gauge transformations must be independent of the field. Therefore, all possible forms of such structure relations can be written down and it can be checked whether they fulfill the Jacobi identities

$$\sum_{\text{cycl}}[[\xi,\eta],\psi]\equiv[[\xi,\eta],\psi]+[[\eta,\psi]\xi]+[[\psi,\xi],\eta]=0 . \tag{18}$$

In the cases of spin 1 and 2, inserting (16) and (17) in (18) it is seen that the identity holds. The question is whether the same is true for the spin-3 structure relations. To study this question, we will rely on the fact that to lowest order in the coupling constant, the structure relations are field independent as noted above. Therefore we can make a general study of such structure relations without actually having to construct the gauge transformations.

IV. STRUCTURE RELATIONS FOR SPIN 3 (TO FIRST ORDER IN α)

We will not here attempt to construct the first-order gauge transformation, but rather proceed directly to a study of the structure relations that could be derived assuming that the transformations exist. To be definite, the following assumptions are made about the properties of the structure relations.

(i) They are independent of the spin-3 field.

(ii) They contain two derivatives.

(iii) They involve f^{abc}. We will check whether structure relations fulfilling these requirements obey the Jacobi identities (18). Here it should be pointed out that the structure relations for spin 1, which are field independent, and the structure relations for spin 2, which are field independent in lowest order (i.e., when they are extracted from first-order gauge transformations), obey the Jacobi identities by themselves without recourse to explicit field transformation laws. This is a reflection of the underlying group-theory basis for these theories. Note that when we say that the structure relations should fulfill the Jacobi identities, we mean that the commutator brackets should fulfill the identities.

We will prove the following proposition: There are no structure relations which fulfill requirements (i)—(iii) that obey the Jacobi identities by themselves.

To prove this, we first write down all possible forms of the structure relations compatible with (i)—(iii). These can then be summarized in two general formulas:

$$[\xi,\eta]^a_{\mu\nu}=\alpha f^{abc}\Omega^{\sigma_1\cdots\sigma_6}_{\mu\nu}[\eta^b_{\sigma_1\sigma_2}\partial_{\sigma_3}\partial_{\sigma_4}\xi^c_{\sigma_5\sigma_6}-(\eta\leftrightarrow\xi)] , \tag{19}$$

$$\begin{aligned}[\xi,\eta]^a_{\mu\nu}&=\alpha f^{abc}\Sigma^{\sigma_1\cdots\sigma_6}_{\mu\nu}[\partial_{\sigma_1}\eta^b_{\sigma_2\sigma_3}\partial_{\sigma_4}\xi^c_{\sigma_5\sigma_6}-(\eta\leftrightarrow\xi)]\\&=2\alpha f^{abc}\Sigma^{\sigma_1\cdots\sigma_6}_{\mu\nu}\partial_{\sigma_1}\eta^b_{\sigma_2\sigma_3}\partial_{\sigma_4}\xi^c_{\sigma_5\sigma_6} .\end{aligned} \tag{20}$$

Here, $\Omega^{\sigma_1\cdots\sigma_6}_{\mu\nu}$ and $\Sigma^{\sigma_1\cdots\sigma_6}_{\mu\nu}$ denote linear combinations of products of four Kronecker delta symbols, or

$$\Omega^{\sigma_1\cdots\sigma_6}_{\mu\nu}=\sum_i a_i(\delta\delta\delta\delta)^{\sigma_1\cdots\sigma_6}_{i\mu\nu} , \tag{21}$$

$$\Sigma^{\sigma_1\cdots\sigma_6}_{\mu\nu}=\sum_i b_i(\delta\delta\delta\delta)^{\sigma_1\cdots\sigma_6}_{i\mu\nu} . \tag{22}$$

Thus the symbols Ω and Σ simply yield all different contractions of indices.

Furthermore, Σ has the additional symmetry

$$\Sigma^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6}_{\mu\nu}=\Sigma^{\sigma_4\sigma_5\sigma_6\sigma_1\sigma_2\sigma_3}_{\mu\nu} \tag{23}$$

as is seen in (20). [In fact, the proof can be carried through even without this restriction. The important thing is that Σ is *not* antisymmetric in $(\sigma_1\sigma_2\sigma_3)(\sigma_4\sigma_5\sigma_6)$ since in that case the structure relation would vanish.]

Of course, the most general structure relation is a linear combination of (19) and (20), but as we will see, (19) is excluded by itself, hence (19) and (20) can be analyzed separately.

Now let us first focus on (19). Upon computing $\sum_{\text{cycl}}[[\xi,\eta],\psi]$, there will turn up terms with the generic form $\sum_{\text{cycl}}\psi\eta\partial\partial\partial\xi$. These terms have the form

$$\begin{aligned}&\alpha^2(f^{abc}f^{cde}\Omega^{\rho_1\cdots\rho_6}_{\mu\nu}\Omega^{\sigma_1\cdots\sigma_6}_{\rho_5\rho_6}\\&\quad-f^{adc}f^{cbe}\Omega^{\sigma_1\sigma_2\rho_3\cdots\rho_6}_{\mu\nu}\Omega^{\rho_1\rho_2\sigma_3\cdots\sigma_6}_{\rho_5\rho_6})\\&\qquad\times\sum_{\text{cycl}}(\psi^b_{\rho_1\rho_2}\eta^d_{\sigma_1\sigma_2}\partial_{\rho_3}\partial_{\rho_4}\partial_{\sigma_3}\partial_{\sigma_4}\xi^e_{\sigma_5\sigma_6}) .\end{aligned} \tag{24}$$

If

$$\Omega^{\rho_1\cdots\rho_6}_{\mu\nu}\Omega^{\sigma_1\cdots\sigma_6}_{\rho_5\rho_6}\neq\Omega^{\sigma_1\sigma_2\rho_3\cdots\rho_6}_{\mu\nu}\Omega^{\rho_1\rho_2\sigma_3\cdots\sigma_6}_{\rho_5\rho_6}$$

then the terms are clearly nonzero. If, on the other hand,

$$\Omega_{\mu\nu}^{\rho_1\cdots\rho_6}\Omega_{\rho_5\rho_6}^{\sigma_1\cdots\sigma_6}=c\,\Omega_{\mu\nu}^{\sigma_1\sigma_2\rho_3\cdots\rho_6}\Omega_{\rho_5\rho_6}^{\rho_1\rho_2\sigma_3\cdots\sigma_6}$$

with some constant c, $\Omega\Omega$ factor out and in that case

$$f^{abc}f^{cde}-cf^{adc}f^{cbe}\neq 0\ .$$

Thus it is clear that it is the combined presence of the constants f^{abc} and Ω that prevents cancellations between the two groups of terms. In the absence of derivatives, for spin 1, we get zero due to the ordinary Jacobi identity for f^{abc}. If there were no f^{abc}, we could arrange to get cancellations between the two groups of terms by demanding equality between the products of two Ω above.

Finally, it has to be proved that

$$\Omega_{\mu\nu}^{\rho_1\cdots\rho_6}\Omega_{\rho_5\rho_6}^{\sigma_1\cdots\sigma_6}\neq 0\ .$$

This is somewhat awkward to prove and for this reason the proof is put aside in Appendix B.

Next, turn to (20). In that case all terms have to be computed since they are all of the same form $\sum_{\text{sym}}\partial\psi\,\partial\eta\,\partial\partial\xi$. The result is

$$4\alpha^2(f^{abc}f^{cde}\Sigma_{\mu\nu}^{\rho_1\cdots\rho_6}\Sigma_{\rho_5\rho_6}^{\sigma_1\cdots\sigma_6}+f^{aec}f^{cbd}\Sigma_{\mu\nu}^{\sigma_4\sigma_5\sigma_6\rho_4\rho_5\rho_6}\Sigma_{\rho_5\rho_6}^{\rho_1\rho_2\rho_3\sigma_1\sigma_2\sigma_3})\times\sum_{\text{cycl}}(\partial_{\rho_1}\psi^b_{\rho_2\rho_3}\partial_{\rho_4}\partial_{\sigma_1}\eta^d_{\sigma_2\sigma_3}\partial_{\sigma_4}\xi^e_{\sigma_5\sigma_6})\ . \quad (25)$$

The proof that this expression is nonzero proceeds in exactly the same way.

The failure to find field-independent structure relations for spin-3 gauge transformations does not mean that transformation laws of the form (13) are necessarily wrong. For spin 1 and 2 it is possible to extract the structure of the gauge algebra from the gauge transformations given to first order in the coupling constant. For spin 3 this seems not to be the case. This is an indication that there is no field-independent gauge algebra and that the spin-3 theory lacks a group theory basis of the type that exists for lower spin.

V. COUPLING TO YANG-MILLS FIELDS

The free field theory of spin 3 has the same internal symmetry as the light-cone theory, namely,

$$\delta_g\phi^a_{\mu_1\mu_2\mu_3}=gf^{abc}\xi^b\phi^c_{\mu_1\mu_2\mu_3}\ , \quad (26)$$

where g and ξ^a are dimensionless. It is interesting to check whether this symmetry can be made local resulting in minimal coupling to Yang-Mills fields. Following the standard recipe we introduce the gauge-covariant derivatives

$$D_\mu^{ab}=\delta^{ab}\partial_\mu-gf^{abc}A^c_\mu \quad (27)$$

into the equation of motion (or Lagrangian) for the spin-3 field. The minimally coupled equation of motion then becomes

$$W^a_{\mu_1\mu_2\mu_3}=D_\sigma D^\sigma\phi^a_{\mu_1\mu_2\mu_3}-\sum_{\mu\ 1}D_{\mu_1}D^\sigma\phi^a_{\sigma\mu_2\mu_3}+\frac{1}{2}\sum_{\mu\ 1}\{D_{\mu_1},D_{\mu_2}\}\phi^{a\sigma}_{\sigma\mu_3}=0\ . \quad (28)$$

The complete theory must be covariant under the following set of transformations

$$\begin{aligned}&\delta_g A^a_\mu=D_\mu\xi^a,\quad \delta_s A^a_\mu=0\ ,\\&\delta_g\phi^a_{\mu_1\mu_2\mu_3}=-gf^{abc}\xi^b\phi^c_{\mu_1\mu_2\mu_3}\ ,\\&\delta_s\phi^a_{\mu_1\mu_2\mu_3}=\sum_{\mu\ 1}D_{\mu_1}\xi^a_{\mu_2\mu_3}\ .\end{aligned} \quad (29)$$

These transformations form an algebra

$$\begin{aligned}&[\delta_g,\delta_s]=\delta_s\ ,\\&[\delta_g,\delta_g]=\delta_g\ ,\\&[\delta_s,\delta_s]=0\ .\end{aligned} \quad (30)$$

However, the equation (28) is not covariant under the transformation δ_s. This is due to the fact that the covariant derivatives do not commute. Typically, $W^a_{\mu_1\mu_2\mu_3}$ varies into terms proportional to the Yang-Mills field strengths. There is, however, one nonminimal term, covariant under δ_g, which can be added to the field equation:

$$J^a_{\mu_1\mu_2\mu_3}=\sum_{\mu\ 1}gf^{abc}F^{b\sigma}_{\mu_1}\phi^c_{\sigma\mu_2\mu_3}\ . \quad (31)$$

Unfortunately, this is not enough to cancel all the unwanted terms. The result is

$$\delta(W^a_{\mu_1\mu_2\mu_3}+J^a_{\mu_1\mu_2\mu_3})=-gf^{abc}\sum_{\mu\ 1}[\,W^b_{\mu_1}\xi^c_{\mu_2\mu_3}+F^b_{\mu_1\sigma}(D^\sigma\xi_{\mu_2\mu_3})^c+(\partial_{\mu_1}F^b_{\sigma\mu_2}+\partial_{\mu_2}F^b_{\sigma\mu_1})\xi^{c\sigma}_{\mu_3}]\ , \quad (32)$$

where $W^a_{\mu_1}$ is the free equation of motion for the Yang-Mills field. The presence of the field strengths in the variation of the field equation thus seems to be a common feature to attempts to couple higher-spin fields.[5–7]

V. CONCLUSION

The limitations of the present work must be stressed. We have started from the assumption that the spin-3 field is described by a totally symmetric field $\phi_{\mu_1\mu_2\mu_3}$ and that the would-be interacting theory shares some definite properties with the light-cone formulation of the theory. It might be that this is too narrow a framework. It could be that it is necessary to use a field with more auxiliary components than $\phi^a_{\mu_1\mu_2\mu_3}$, or that only more than one higher-spin field could cooperatively form a covariant interacting theory. However, within the framework of this paper, the existence of a covariant self-interacting theory for spin-3 fields is still an open question. The most pressing question to answer is whether a gauge-invariant action can be found along the lines outlined in Sec. III.

ACKNOWLEDGMENT

I thank Ingemar Bengtsson, Lars Brink, and Olof Lindgren for discussions. After completion of this work, a paper [F. A. Berends, G. J. H. Burgers, and H. van Dam, Z. Phys. C **24**, 247 (1984)] has appeared in which the three-point vertex for spin 3 is constructed, and furthermore a report by the same authors [IFP234-UNC, 1984 (unpublished)] in which problems with the gauge algebra are pointed out.

APPENDIX A

For the sake of completeness, the definitions of the generalized Christoffel symbols are given here. The mth symbol is defined in terms of the $(m-1)$th symbol according to

$$\Gamma^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}\equiv\partial_{\rho_1}\Gamma^{(m-1)}_{\rho_2\cdots\rho_m;\mu_1\cdots\mu_s}-\frac{1}{m}\sum_{\mu\ 1}\partial_{\mu_1}\Gamma^{(m-1)}_{\rho_2\cdots\rho_m;\rho_1\mu_2\cdots\mu_s}\ .$$

The first symbol is given by identifying

$$\phi_{\mu_1\cdots\mu_s}=\Gamma^{(0)}_{\mu_1\cdots\mu_s}\ .$$

Their properties under gauge transformations

$$\delta\phi_{\mu_1\cdots\mu_s}=\sum_{\mu\ 1}\partial_{\mu_1}\xi_{\mu_2\cdots\mu_s}$$

are given by

$$\delta\Gamma^{(m)}_{\rho_1\cdots\rho_m;\mu_1\cdots\mu_s}=(-1)^m(1+m)\sum_{\mu\ m+1}\partial_{\mu_1}\cdots\partial_{\mu_{m+1}}\xi_{\rho_1\cdots\rho_m\mu_{m+2}\cdots\mu_s}$$

from which it is seen that $\Gamma^{(s)}_{\rho_1\cdots\rho_s;\mu_1\cdots\mu_s}$ and $\Gamma^{(m)\sigma}_{\sigma\ \ \rho_3\cdots\rho_m;\mu_1\cdots\mu_s}$ are gauge invariant. For further properties consult Ref. 5.

The summation symbol $\sum_{\mu\ k}$ denotes a sum of the terms following it consisting of $\binom{s}{k}$ terms and such that it is symmetric in the indices μ_i.

The order of the Christoffel symbols (m) is suppressed in some formulas in the paper, as it is evident from the index structure which symbol is meant.

APPENDIX B

We give a proof of the statement

$$S\equiv\Omega^{\rho_1\cdots\rho_6}_{\mu\nu}\Omega^{\sigma_1\cdots\sigma_6}_{\rho_5\rho_6}\neq 0\ .$$

Introduce the following notation for Ω,

$$\Omega^{\rho_1\cdots\rho_6}_{\mu\nu}=\delta^{\rho_j}_{\rho_i}\delta^{\rho_l}_{\rho_k}\delta^{\rho_m}_{(\mu}\delta^{\rho_n}_{\nu)}\sum_a\Omega^{ijklmn}_a\ ,$$

where Ω^{ijklmn}_a are constants depending on the values of i,j,k,l,m,n. Note that $i\neq j\neq k\neq l\neq m\neq n$ and that they take values 1,2,3,4,5,6. Write also

$$\Omega^{\sigma_1\cdots\sigma_6}_{\rho_5\rho_6}=\delta^{\sigma_{j'}}_{\sigma_{i'}}\delta^{\sigma_{l'}}_{\sigma_{k'}}\delta^{\sigma_{m'}}_{(\rho_5}\delta^{\sigma_{n'}}_{\rho_6)}\sum_b\Omega^{i'j'k'l'm'n'}_b\ .$$

In the sum S, there will be distinct classes of terms according to whether $(m,n)=(5,6)$, $(l,m)=(5,6)$, $(l,n)=(5,6)$, $(j,m)=(5,6)$, $(j,n)=(5,6)$, or $(j,l)=(5,6)$. Owing to the symmetry in $(\rho_5\rho_6)$ it does not matter whether $m=5$, $n=6$, or $m=6$, $n=5$, etc., it just yields a factor 2. Thus, we get the following expression for S,

$$\begin{aligned}S=\;&2\left[\delta^{\rho_j}_{\rho_i}\delta^{\rho_l}_{\rho_k}\sum_a\Omega^{ijkl56}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\mu\nu}+2\left[\delta^{\rho_j}_{\rho_i}\delta^{\rho_n}_{(\mu}\sum_a\Omega^{ijk56n}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_k\nu)}\\&+2\left[\delta^{\rho_j}_{\rho_i}\delta^{\rho_m}_{(\mu}\sum_a\Omega^{ijk5m6}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_k\nu)}+2\left[\delta^{\rho_l}_{\rho_k}\delta^{\rho_n}_{(\mu}\sum_a\Omega^{i5kl6n}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_i\nu)}\\&+2\left[\delta^{\rho_l}_{\rho_k}\delta^{\rho_m}_{(\mu}\sum_a\Omega^{i5klm6}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_i\nu)}+2\left[\delta^{\rho_m}_{(\mu}\delta^{\rho_n}_{\nu)}\sum_a\Omega^{i5k6mn}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_i\rho_k}\ .\end{aligned}$$

Furthermore, nothing is changed by interchanging $(i,j)\leftrightarrow(k,l)$ and $m\leftrightarrow n$, hence

$$S=2\left[\delta^{\rho_j}_{\rho_i}\delta^{\rho_l}_{\rho_k}\sum_a\Omega^{ijkl56}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\mu\nu}+8\left[\delta^{\rho_j}_{\rho_i}\delta^{\rho_n}_{(\mu}\sum_a\Omega^{ijk56n}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_k\nu)}+2\left[\delta^{\rho_m}_{(\mu}\delta^{\rho_n}_{\nu)}\sum_a\Omega^{i5k6mn}_a\right]\Omega^{\sigma_1\cdots\sigma_6}_{\rho_i\rho_k}\ .$$

Now since $\Omega\neq 0$ and all terms in S have a different index structure, no cancellations can occur, and therefore $S\neq 0$.

To be more precise, the first term is proportional to $\Omega^{\sigma_1\cdots\sigma_6}_{\mu\nu}$ which is nonzero and the factor in front is a sum of different products of Kronecker deltas. This follows from the obvious symmetry properties of Ω^{ijkl56}. It is symmetric in (ij), (kl), and $(ij)(kl)$. Therefore we have the terms

$$\delta^{\rho_2}_{\rho_1}\delta^{\rho_4}_{\rho_3}\Omega^{123456}+\delta^{\rho_3}_{\rho_1}\delta^{\rho_4}_{\rho_2}\Omega^{132456}+\delta^{\rho_4}_{\rho_1}\delta^{\rho_3}_{\rho_2}\Omega^{142356}\ .$$

Going back to (24) we see that these terms multiply something that is symmetric in $\rho_1\rho_2$ and $\rho_3\rho_4$, thus we could arrange to get zero by choosing $\Omega^{132456}=-\Omega^{142356}$, $\Omega^{123456}=0$. But since the structure relations themselves are symmetric in $\rho_1\rho_2$ and $\rho_3\rho_4$, this solution only means these contributions to the structure relations vanish. The first set of terms therefore yields a nonzero result.

In an analogous way one can go through the other two sets of terms and show that no cancellations occur.

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