# Quantum measurements of finite duration

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The interaction of a quantum system with a measuring apparatus is usually considered to be quasi-instantaneous. In this paper, we discuss several aspects of measurements of *finite* duration. It is shown that the observation of a quantity which is not a constant of motion may yield readings different from its eigenvalues. Moreover, the free motion of the measuring apparatus may limit its accuracy.

## I. INTRODUCTION

Measurements performed on quantum systems can be described at various levels of detail. Their simplest description is the projection postulate: The observation of a dynamical variable $A$ can yield only one of its eigenvalues $a_k$ and, immediately thereafter, the quantum system is in the corresponding eigenstate $u_k$. The probability for this particular outcome, if the quantum system was in a state $\psi$ prior to the measurement, is $|(u_k,\psi)|^2$. The above evolution, which is called the "collapse" of the wave function, is manifestly nonlinear and therefore cannot be described by the Schrödinger equation.[1,2]

A more sophisticated approach is to give dynamical degrees of freedom to the measuring apparatus. The characteristic feature of a reliable apparatus is that it can interact with a physical system in such a way that a property of the system is replicated in a property of the apparatus. This is true in classical physics too. A *measurement* can therefore be defined as the *correlation* of a dynamical variable of the apparatus with one of the physical system.

In quantum theory, a measurement, as defined above, must be distinguished from an *observation,* which is the *selection of a particular value* of a dynamical variable (of the apparatus and/or of the observed system).[3] While a measurement is a dynamical process which can be described by the Schrödinger equation, an observation (a collapse) is not amenable to a dynamical description. The present work is solely concerned with properties of measurements. Any physical or philosophical issue related to the observation process, such as the existence of an objective reality, is beyond the scope of this article.

In order to describe dynamically a measurement, one writes a Hamiltonian such as

$$H=H_s+H_a+H_i \ , \tag{1}$$

where $H_s$ refers to the system being measured, $H_a$ to the measuring apparatus, and $H_i$ to their interaction. The analysis is simplest if the interaction is very brief and is contrived in such a way that the apparatus evolves from an initial state $\phi_0$ to a set of final states $\phi_k$, which are *macroscopically distinguishable* and are correlated to the eigenstates of the dynamical variable being measured:[1]

$$\psi\phi_0 \to \sum c_k u_k \phi_k \ , \tag{2}$$

where $c_k=(u_k,\psi)$. The evolution (2) is unitary and, since it correlates the final states of the apparatus with those of the physical system, it fulfills our definition of a measurement. Clearly, it is *not* an observation because the apparatus is left in a superposition of states. It does not yet point to a definite result.

If the apparatus is macroscopic and cannot be perfectly insulated from its environment, the phase coherence in the right-hand side of (2) is rapidly destroyed and one gets a mixture rather than a pure state.[4,5] Even then, the apparatus does not point to a particular result. All possibilities are represented, with their probabilities $|c_k|^2$. In this paper, we shall ignore the irreversible evolution which follows Eq. (2), because it is not essential in the problem which we consider.

In order to discuss an actual observation, one can assume that the apparatus with states $\phi_k$ is observed by *another* apparatus, which is *not* described dynamically.[6] In other words, the projection postulate is applied to the combined system and first apparatus. The final state thus is one of the products $u_k\phi_k$, and this result occurs with probability $|c_k|^2$.

Alternatively, one can assume that the second apparatus is also described dynamically, and has states $\chi_k$, so that

$$\sum c_k u_k \phi_k \chi_0 \to \sum c_k u_k \phi_k \chi_k \ . \tag{3}$$

It is then observed by a third apparatus, etc. However, there can be no Schrödinger dynamical description for the final stage of this chain of apparatuses, at which a single result is obtained. This final stage is described by a probability rule, namely, the projection postulate.

The virtue of Eqs. (2) or (3) is to show the self-consistency of the projection postulate.[1] In the above scenario, it does not matter whether the latter is used directly for the states of the system, or after the evolution (2), or after (3), etc. It is therefore important to examine *under what conditions the evolution generated by a Hamiltonian such as (1) indeed yields correlations such as those of (2).* For example, it is well known[7-9] that if the choice of the Hamiltonian is restricted by additive conservation laws, it may be impossible to generate the evolution (2). These restrictions, however, have little practical importance because Eq. (2) may be arbitrarily well approximated in the limit of large quantum numbers (i.e., for macroscopic apparatuses).

In order to obtain the evolution (2) it is customary to assume that $H_i$ is of the form

$$H_i = ABf(t) , \tag{4}$$

where $A$ is the dynamical variable being measured, $B$ involves only variables of the apparatus, and $f(t)$ has a very large value and a very narrow support near $t=t_0$, the "instant" of the measurement.[10] In fact, it is assumed that $H_i$ is so large during the brief interaction that $H_s$ and $H_a$ can be safely ignored during that period. This drastic simplification may not always be justified. Coupling constants occurring in nature are finite, and sometimes very small.[11] It may therefore be necessary to couple the measured system and the apparatus during a *finite,* possibly long time. The purpose of this paper is to investigate what happens during such a measurement of finite duration. Not unexpectedly, it is found that the perfect correlation of Eq. (2) is impaired, just as when the Hamiltonian was restricted by additive conservation laws.[7-9]

In Sec. II, we assume for simplicity that $H_s = H_a = 0$, and we follow in detail the evolution leading to (2). If we assume that the apparatus can be instantaneously observed (by a second, hypothetical apparatus) at any moment during this evolution, we obtain a kind of "continuous collapse" of the wave function.[12]

In Secs. III and IV, we consider what happens if $H_s$ and $H_a$, respectively, cannot be neglected during the duration of the interaction. Finally, Sec. V discusses the interplay of $H_s$, $H_a$, and $H_i$.

Throughout this paper, we use natural units: $\hbar = 1$.

## II. EXAMPLE OF CONTINUOUS MEASUREMENT

Suppose that we want to measure $\sigma_z$ of a spin-$\frac{1}{2}$ system. This can be done by coupling it to a "pointer" which is simply a free particle (position $q$, momentum $p$). We thus have $H_a = p^2/2m$. In this section, and the next one, we shall assume that $m$ is so large that $H_a$ can be neglected.

A suitable interaction Hamiltonian is

$$H = H_i = \sigma_z V(t) p , \tag{5}$$

where $V(t)$ is a given function of time.[10] We thus have, in the Heisenberg representation, $\dot{q} = i[H,q] = \sigma_z V$, so that the pointer moves with a velocity $\pm V(t)$, according to whether $\sigma_z = \pm 1$.

The initial state of the combined system is

$$\psi = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \phi(q) . \tag{6}$$

After a time $t$, it becomes (we are again using the Schrödinger representation)

$$e^{-i\int H\,dt}\psi = \begin{bmatrix} \alpha e^{-iLp} \\ \beta e^{iLp} \end{bmatrix} \phi(q) \tag{7}$$

$$= \begin{bmatrix} \alpha\phi(q-L) \\ \beta\phi(q+L) \end{bmatrix} , \tag{8}$$

where $L(t) = \int V(t)dt$. It is convenient to rewrite this as a density matrix

$$\rho(q,q') = \begin{bmatrix} |\alpha|^2 \phi(q-L)\phi^*(q'-L) & \alpha\beta^*\phi(q-L)\phi^*(q'+L) \\ \beta\alpha^*(q+L)\phi^*(q'-L) & |\beta|^2\phi(q+L)\phi^*(q'+L) \end{bmatrix} . \tag{9}$$

From here, the discussion can proceed with various levels of sophistication. We can, if we have enough skill, keep track of all the correlations between the spin dynamical variables and those of the apparatus. In that case, the pure state remains pure—it is not replaced by a statistical mixture.[13-15] At the other extreme, we can focus our attention on the spin and completely ignore the pointer. For example, we may ask what is the expectation value of a spin variable $A$ (any Hermitian 2 by 2 matrix). It is given by the standard rule

$$\langle A(t) \rangle = \int \int_{-\infty}^{\infty} \psi(t)^\dagger A \delta(q-q') \psi(t) dq\, dq' , \tag{10}$$

where $\delta(q-q')$ is the unit operator in the $q$ representation. Formally, this corresponds to tracing out $q$ from the density matrix:

$$\rho(q,q') \rightarrow \int \int_{-\infty}^{\infty} \rho(q,q')\delta(q-q')dq\, dq' . \tag{11}$$

Consider in particular the case where the pointer is known to be initially in a finite domain of $q$; i.e., the initial wave function $\phi(q)$ has compact support. Then, when $L(t)$ is large enough, so that the spin states "up" and "down" have become correlated with well-separated positions of the pointer, the off-diagonal elements in the right-hand side of (11) vanish and we have

$$\rho = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix} . \tag{12}$$

The pure state has thus been transformed into a mixture. Equation (12) is sometimes incorrectly called the collapse. However, as explained above, it cannot describe an individual observation. Rather, it represents the statistical properties of an ensemble of spin systems when we completely disregard the properties of the associated apparatuses.

What actually happens in observations is intermediate between these two extreme cases: We do observe the final positions of the pointer, but not its other dynamical variables. In particular, we do not correlate the latter with variables of the spin system. The statistical outcome of a large number of measurements then is neither a pure spin state, nor a mixture of spin states, but what may be called a "compound."[16] We ask *pairs* of questions such as: "What is the probability of finding the pointer in some given range $q_1 < q < q_2$ and, *in that case,* what are the properties of the spin?" We then have, instead of (11),

$$\rho(q,q') \to R = \int_{q_1}^{q_2} \rho(q,q)dq \ . \tag{13}$$

Here $R$ is a $2\times2$ matrix which is analogous to a density matrix. However, it is not normalized to unit trace. Rather, its trace is the probability of finding the pointer in the designated domain. The expectation value of any spin variable $A$ when the pointer is found between $q_1$ and $q_2$ is thus

$$\langle A(t)\rangle = \mathrm{Tr}(AR)/\mathrm{Tr}R \ . \tag{14}$$

Let us again consider two extreme cases. If $t=0$, so that $L=0$, we get

$$R = \int_{q_1}^{q_2} |\phi(q)|^2 dq \begin{bmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{bmatrix} . \tag{15}$$

This is indeed the expected result: In the right-hand side of (15), the integral is the probability of finding the pointer initially between $q_1$ and $q_2$, and the matrix represents the initial pure spin state $\binom{\alpha}{\beta}$.

At the other extreme, make $t$ large enough—i.e., $L(t)$ large enough—so that $\phi(q\pm L)=0$ for $q \gtrless 0$. Then, if, for example, we take $0<q<\infty$ (that is, we look to see if the pointer is to the right of the origin), we have

$$R = |\alpha|^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} . \tag{16}$$

The probability of finding the pointer with $q>0$ is $|\alpha|^2$ and then the spin state is $\sigma_z=+1$. Likewise, had we tested $-\infty<q<0$, we would have had a probability $|\beta|^2$ of finding the pointer there and the spin state would have been $\sigma_z=-1$.

The latter results thus represent the outcomes of *observations.* The $R$ matrix supplies both the probability of finding the pointer in a given range *and* the corresponding spin state. The non-Schrödinger part of the evolution is Eq. (13), which represents the collapse of the original density matrix (9) into a new density matrix. We emphasize that Eq. (13) is *not* a dynamical process—contrary to Eq. (2) which could be generated by a Hamiltonian. Equation (13) only formalizes the (arbitrary) selection, by an external agent, of a given range of pointer positions.

In this paper, we investigate the *time dependence* of Eq. (13), for fixed $q_1$ and $q_2$. To do this, we have to consider Eq. (8) at intermediate times, as the pointer moves from its initial position to the final one. The physical meaning of this mathematical procedure is this: a second apparatus may suddenly measure the first one, before the first one has fully accomplished its task, that is, before the pointer wave functions corresponding to up and down have been displaced enough to be nonoverlapping. What is then the state of the "partially measured" spin?

A numerical example is the best way to illustrate the result. Assume for simplicity that $V$ is a constant and that the initial state of the pointer is given by

$$\phi(q) = (2a)^{-1/2} \quad \text{for} \quad -a<q<a \ , \tag{17}$$

and $\phi(q)=0$ elsewhere. The measurement will then be completed when $L=Vt\geq a$. We are interested in the situation when $L<a$. For example, we ask: "What is the probability of finding $q>0$, and *in that case,* what are the properties of the spin?"

Suppose that the spin was initially in a state with $\sigma_x=1$, i.e., $\alpha=\beta=2^{-1/2}$. We obtain, from Eqs. (9) and (13),

$$R = \frac{1}{4}\begin{bmatrix} 1+b & 1-b \\ 1-b & 1-b \end{bmatrix} , \tag{18}$$

where $b=L/a<1$. Any such $R$ can be written as[1]

$$R = w_1P_1 + w_2P_2 \ , \tag{19}$$

where $w_1$ and $w_2$ are non-negative numbers, and $P_1$ and $P_2$ are projection operators onto *orthogonal* states. In the present case, we have

$$w_j = \tfrac{1}{4}[1\pm(1-2b+2b^2)^{1/2}] \ , \tag{20}$$

and the corresponding orthogonal eigenstates are

$$\begin{bmatrix} b\pm(1-2b+2b^2)^{1/2} \\ 1-b \end{bmatrix} , \tag{21}$$

up to a normalization factor. These states represent spins polarized in *opposite* directions in the $xz$ plane. A straightforward calculation gives

$$\frac{\langle\sigma_z\rangle}{\langle\sigma_x\rangle} = \frac{b}{1-b} = \frac{Vt}{a-Vt} \ , \tag{22}$$

which could also be obtained directly from (18).

The result is shown in Fig. 1, where the two spin states are represented by vectors of lengths proportional to $w_1$ and $w_2$, pointing in opposite directions. (Notice that $w_1+w_2=\frac{1}{2}$, in the special case under consideration, where $\alpha=\beta$. In general, this will not be a constant.) As expected, we have $\langle\sigma_x\rangle=1$ at $t=0$ (the beginning of the measurement) and $\langle\sigma_z\rangle=1$ at the end of the measurement.

It is clear that the time dependence given in Eq. (22) or, implicitly, in Eq. (18), does not represent a kind of evolution that one could hope to follow, moment by moment, on a single spin-$\frac{1}{2}$ particle in the laboratory. In order to observe the time dependence, one would prepare a number of identical ensembles of spin-$\frac{1}{2}$ particles, and let each ensemble evolve for a different amount of time before ob-

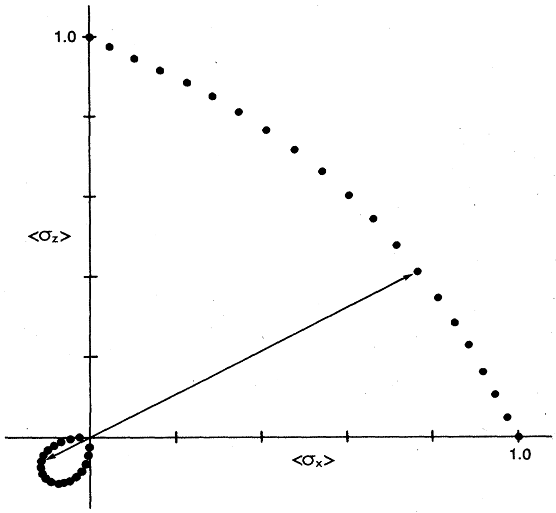


FIG. 1. The initial, final, and intermediate states of the spin, as its density matrix continuously evolves from $\langle \sigma_z \rangle = 1$ to $\langle \sigma_x \rangle = 1$. The dots represent 20 intermediate equidistant steps. At intermediate times, we have a mixture of orthogonal spin states, as shown by the opposite arrows. The length of each arrow is equal to the statistical weight of the corresponding spin state.

serving the pointer's position. For each ensemble, the state of those particles for which the pointer's position was found to be positive is correctly described by Eqs. (18)—(22). The dependence of this state on the waiting time can be thought of as a kind of evolution. Indeed, it is no more or less genuine an evolution than that described by a solution of the Schrödinger equation. In both cases, one needs to prepare a number of identical ensembles in order to observe the time dependence. The difference, of course, is that in our example one would not be able to assign an evolution to a subensemble of particles until the observation "right-vs-left" had been made on the pointer.

## III. MEASUREMENT OF TIME-DEPENDENT VARIABLES

In this section we consider what happens if we attempt to measure, in a way which is not instantaneous, a dynamical variable that is not a constant of motion. This problem is not specific to quantum theory. It may arise in everyday life, e.g., when a photographer takes a snapshot of a moving object. However, quantum theory introduces some novel features, because a measurement is not only a passive observation, but also the *preparation* of a new state. In particular, the result of the measurement is not, in general, the time average of the observed quantity during the measurement.

In order to disentangle this problem from the one discussed in the preceding section, we consider here only measurements which have been brought to completion. As an example to illustrate the situation, let $H_s = \omega\sigma_x$, so that

$$H = \omega\sigma_x + V(t)p\sigma_z \; . \tag{23}$$

We are here attempting to measure $\sigma_z$ of the spin-$\frac{1}{2}$ particle, while the latter also "wants" to precess with angular velocity $\omega$ around the $x$ axis. Assume for simplicity that $V(t)\neq 0$ only for $0<t<T$, and is constant during that interval. Also assume that at $t=0$ we have

$$\psi_0 = 2^{-1/2}\begin{bmatrix}1\\1\end{bmatrix}\phi(q) \; , \tag{24}$$

where $\phi(q)$ is peaked around $q=0$. (An initial state which is not an eigenstate of $\sigma_x$ would lead to slightly more complicated calculations, but not to any qualitatively new features.) The question which we address is the same as in the preceding section, namely: "What is the probability of finding $q>0$ at time $t=T$, and if so, what is the state of the spin?" (For $t>T$, that state will then precess around the $x$ axis with angular velocity $\omega$. This is a trivial evolution which does not concern us.)

The evolution for $0<t<T$ is given by the unitary operator

$$U(p) = e^{-i\int H\,dt} = e^{-i(\omega T\sigma_x + VTp\sigma_z)} \; . \tag{25}$$

As $p$ is constant, this is simply a precession with constant angular velocity $\mathbf{\Omega} = (\omega, 0, Vp)$. Notice, however, that $p$ is not sharp, because $\phi(q)$ is a narrow wave packet, so that different components of the state vector precess with different angular velocities. In fact, the narrower the wave packet, the larger the values of $p$ which are involved. In particular, if $\delta q \ll V/\omega$, most of the wave function has $V|p| \gg \omega$ so that $\mathbf{\Omega}$ is nearly aligned with the $z$ axis. In that case, as shown in detail in the calculation below, the free precession generated by $H_s = \omega\sigma_x$ is nearly stopped (even if $T$ is long and in particular $\omega T > 2\pi$). The calculation below also shows that if we write the initial wave function as the sum of two components having definite $\sigma_z$, these components retain nearly constant amplitudes. This phenomenon is analogous to the quantum Zeno paradox.[17,18]

Explicitly, we have, from (25),

$$U(p) = \cos(\theta^2 + L^2p^2)^{1/2} - i(\theta\sigma_x + Lp\sigma_z)\frac{\sin(\theta^2+L^2p^2)^{1/2}}{(\theta^2+L^2p^2)^{1/2}} \; , \tag{26}$$

where $\theta = \omega T$ and $L = VT$. This $U$ is a *nonlocal operator* (in $q$ space), acting on $\psi_0$. To evaluate the result, we write

$$\phi(q) = (2\pi)^{-1}\int \exp[ip(q-q')]\phi(q')dq'dp \; ,$$

and define a kernel

$$K(q-q') = \frac{1}{2\pi}\int U(p)e^{ip(q-q')}dp \tag{27}$$

$$= A(q-q') + \sigma_x B(q-q') + \sigma_z C(q-q') \; . \tag{28}$$

The initial state (24) thus evolves into

$$\psi(q)=2^{-1/2}\int\begin{bmatrix}A(q-q')+B(q-q')+C(q-q')\\A(q-q')+B(q-q')-C(q-q')\end{bmatrix}\times\phi(q')dq' . \quad (29)$$

Explicitly (see Appendix)

$$A(x)+B(x)\pm C(x)=\delta(x\mp L)+K_{\pm}(x) , \quad (30)$$

where

$$K_{\pm}(x)=-\frac{b}{2}\left[\frac{L\pm x}{L\mp x}\right]^{1/2}J_1(b(L^2-x^2)^{1/2}) - \frac{ib}{2}J_0(b(L^2-x^2)^{1/2}) \text{ if } |x|\leq L \quad (31)$$

$$=0 \text{ if } |x|>L . \quad (32)$$

Here, $b=\theta/L=\omega/V$. Figure 2 is a plot of the real and imaginary parts of $A+B+C$ for $b=L=1$.

When this result is substituted in (29), we thus obtain

$$\psi(q)=2^{-1/2}\begin{bmatrix}\phi(q-L)\\\phi(q+L)\end{bmatrix} + \int_{q-L}^{q+L}\begin{bmatrix}K_+(q-q')\\K_-(q-q')\end{bmatrix}\phi(q')dq' . \quad (33)$$

The first term in (33) is identical to (8), with $\alpha=\beta=2^{-1/2}$. It thus corresponds to a well-done measurement, uncluttered by motion of the observed object. This motion seems to have been frozen.[17,18]

The effect of the nonlocal part of the kernel (31) can be interpreted as follows: The pointer ought to move to the right or left, depending on whether $\sigma_z=+1$ or $-1$. But $\sigma_z$ is not constant. A wave function with $\sigma_z=+1$ will acquire $\sigma_z=-1$ components, because of $H_s=\omega\sigma_x$. Therefore, the pointer zigzags and its final possible positions are not concentrated at $q=\pm L$, but spread continuously between $-L$ and $+L$.

For a given final position of the pointer, what is the state of the spin? Ideally, we would want $\sigma_z=+1$ if $q>0$, and $\sigma_z=-1$ if $q<0$. Actually, the spin state is given by Eq. (13), where the reduced density matrix corresponding to (33) is integrated from $q$ to $q+dq$ (if the pointer was found in that range). This is, in general, a mixture of states with opposite spins. Only when $q\simeq\pm L$ does the $\delta$ function contribute, and it is, of course, the main contribution. In that case, the mixture is a nearly pure state, with spins aligned in the $\pm z$ direction. For intermediate positions of the pointer, the mixture involves other orientations. In particular, if we are using this imperfect apparatus to select two beams with opposite polarizations, we may be tempted to use a large domain of $q$ in order to increase the intensity of these beams. The result is that the beams will be contaminated with particles of the wrong polarization.

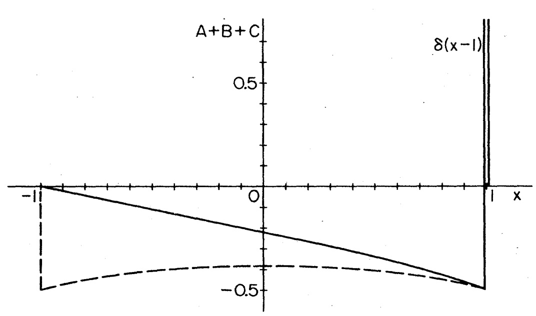


FIG. 2. The real part (solid curve) and imaginary part (broken curve) of the kernel (30) for $b=\omega/V=1$ and $L=VT=1$.

The amount of contamination crucially depends on the width of $\phi(q)$. Consider, for example, $\phi(q)$ given by (17), which, for $a\ll L$, is a kind of square root of the $\delta$ function. We then have

$$\int K_{\pm}(q-q')\phi(q')dq'\simeq(2a)^{1/2}K_{\pm}(q) , \quad (34)$$

which can be made arbitrarily small as $a\rightarrow 0$. The measurement is, therefore, nearly perfect, as predicted in our qualitative discussion of (Eq. (25): A very narrow $\phi(q)$ implies very large values of $p$ so that, in the Hamiltonian (21), the free part $\omega\sigma_x$ is overwhelmed by the interaction $Vp\sigma_z$. It does not matter whether the measurement is instantaneous or extends over a finite time. In any case, the components of the initial state along the eigenvectors of the measured variable are constant.[17,18]

These results are radically different from what could be expected on purely classical grounds, where an apparatus sensitive to $\sigma_z$ would simply give the time average of $\sigma_z$ from $t=0$ to $t=T$. In the quantum model, such a time average cannot be measured with the simple apparatus described above, with constant $V(t)$. A "back-action evasion" method[11] might be possible, but is beyond the scope of this paper.

## IV. FREE MOTION OF APPARATUS

In this section, we assume $H_s=0$ and consider the mutual influence of $H_a$ and $H_i$. The pointer now has a finite mass $m$ so that

$$H=(p^2/2m)+Vp\sigma_z . \quad (35)$$

Because of the free motion of the pointer (the spreading of its wave packet) the measurement cannot be as perfect as for infinite $m$. The correlation in Eq. (2) deteriorates as time passes and is completely lost if we wait too long to observe it.

For example, let the initial wave function be a Gaussian

$$\psi(q,0)=(2\pi a^2)^{-1/4}\exp(-q^2/4a^2) . \quad (36)$$

A free pointer (i.e., with $H_a$ only) would evolve into[19]

$$\psi(q,t)=(2\pi)^{-1/4}\left[a+\frac{it}{2ma}\right]^{-1/2}\times\exp\left[-q^2\Big/4\left[a^2+\frac{it}{2m}\right]\right] . \quad (37)$$

Since $H_a$ and $H_i$ commute, the final wave function is thus (for $\alpha=\beta=2^{-1/2}$, as before)

$$\psi=2^{-1/2}\begin{bmatrix}\psi(q-L,t)\\ \psi(q+L,t)\end{bmatrix} . \tag{38}$$

Here, $L=Vt$ if $t\leq T$ and $L=VT$ if $t\geq T$.

The discussion can now proceed as in Sec. II. If we again ask what is the probability of finding the pointer with $q>0$, and if so, what are the properties of the spin, we find a reduced density matrix similar to (18):

$$R=[2\pi(a^2+t^2/4m^2a^2)]^{1/2}\begin{bmatrix}X_+ & Y\\ Y^* & X_-\end{bmatrix} , \tag{39}$$

where

$$X_\pm=\tfrac{1}{2}\int_0^\infty \exp[-(q\mp L)^2/(2a^2+t^2/2m^2a^2)]dq, \tag{40}$$

and

$$Y=\tfrac{1}{2}\int_0^\infty \exp\left[-\frac{(q-L)^2}{4(a^2+it/2m)}-\frac{(q+L)^2}{4(a^2-it/2m)}\right]dq . \tag{41}$$

If $L\gg t/ma$ we have in (39) $X_+\gg|Y|\gg X_-$, i.e., a nearly pure spin state. Notice that this is possible only if $V\gg 1/ma$. The velocity $V$, which plays the role of a coupling constant, must have a minimum value to allow a reliable measurement.

Then, as $T$ is finite and $t$ is unbounded, the result of the measurement will finally be washed out. It becomes useless for $t\gtrsim maVT/\hbar$, where we have restored $\hbar$, for clarity.

## V. SYNOPSIS

We finally consider the full Hamiltonian (1), with interplay of $H_s$, $H_a$, and $H_i$. A complete calculation as above is quite difficult, but the following semiquantitative argument will give the flavor of the problem.

We have found in Sec. III that a good measurement can be performed if the width of the pointer wave packet is small. We need $\hbar\omega\ll Vp$ (for typical $p$), therefore $a\ll V/\omega$. On the other hand, the results of Sec. IV imply that $a$ should not be too small. A good observation requires $a\gg\hbar/mV$.

Comparing these results, we obtain $V\gg(\hbar\omega/m)^{1/2}$. This is, in the model which we considered, the minimum value of the coupling constant allowing a reliable measurement. If $\omega$ is too large, or if $m$ is too small, the required value of $V$ (which also is the velocity of the pointer during the measurement) may not be attainable. For example, the required $V$ may be larger than the velocity of light, which is, of course, impossible. In fact, it should not even approach the velocity of sound in the material from which the pointer is made, because the latter would not behave as a rigid body. Its motion would be very complex and could not be approximated by a single (center-of-mass) coordinate $q$.

It is obvious that the condition we found, $V\gg(\hbar\omega/m)^{1/2}$, is specific to the model we investigated. Other more realistic models may be subject to different limitations. It seems, however, that if one puts realistic limits on coupling constants, masses, etc., there are fundamental limits on our ability to measure some dynamical variables, even if these variables are mathematically well defined.

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## APPENDIX

Here we explicitly evaluate the three functions $A$, $B$, and $C$ which appear in the kernel defined by Eq. (27). From Eqs. (26)–(28) we obtain immediately the following expressions for these functions:

$$A(x)=\frac{1}{\pi}\int_0^\infty \cos[L(b^2+k^2)^{1/2}]\cos kx\,dk , \tag{A1}$$

$$B(x)=-\frac{i}{\pi}\int_0^\infty \frac{b\sin[L(b^2+k^2)^{1/2}]}{(b^2+k^2)^{1/2}}\cos kx\,dk , \tag{A2}$$

$$C(x)=\frac{1}{\pi}\int_0^\infty \frac{k\sin[L(b^2+k^2)^{1/2}]}{(b^2+k^2)^{1/2}}\sin kx\,dk . \tag{A3}$$

Here $b=\theta/L$.

Of these three integrals, only the one for $B(x)$ defines a genuine function rather than a more general distribution. $B(x)$ turns out to be[20]

$$B(x)=-\frac{ib}{2}J_0(b(L^2-x^2)^{1/2})\ \text{ if }\ |x|<L \tag{A4}$$

$$=-\frac{ib}{4}\ \text{ if }\ |x|=L \tag{A5}$$

$$=0\ \text{ if }\ |x|>L . \tag{A6}$$

We are free to change the value of $B(x)$ at the special points $x=\pm L$ without changing the distribution it defines. For simplicity we will from now on let $B(\pm L)=-ib/2$ so that Eq. (A5) can be absorbed into Eq. (A4).

$C(x)$ can be obtained from $B(x)$ by differentiating with respect to $x$:

$$C(x)=\frac{1}{ib}\frac{dB}{dx}=\tfrac{1}{2}[\delta(x-L)-\delta(x+L)]+\tilde{C}(x) , \tag{A7}$$

where

$$\tilde{C}(x)=-\frac{1}{2}\frac{bx}{(L^2-x^2)^{1/2}}J_1(b(L^2-x^2)^{1/2})\ \text{ if }\ |x|\leq L \tag{A8}$$

$$=0\ \text{ if }\ |x|>L . \tag{A9}$$

$A(x)$ can be obtained from $B(x)$ by integrating over the parameter $b$:

$$A_b(x)=A_{b=0}(x)+\frac{L}{i}\int_0^b B_{b'}(x)db' \tag{A10}$$

$$=\tfrac{1}{2}[\delta(x-L)+\delta(x+L)]+\widetilde{A}(x)\ , \tag{A11}$$

where

$$\widetilde{A}(x)=-\frac{1}{2}\frac{Lb}{(L^2-x^2)^{1/2}}J_1(b(L^2-x^2)^{1/2}) \ \text{ if } \ |x|\le L \tag{A12}$$

$$=0 \ \text{ if } \ |x|>L\ . \tag{A13}$$

Finally, we add together the three functions to obtain

$$A(x)+B(x)\pm C(x)=\delta(x\mp L)+K_\pm(x)\ , \tag{A14}$$

where

$$K_\pm=-\frac{b}{2}\left[\frac{L\pm x}{L\mp x}\right]^{1/2}J_1(b(L^2-x^2)^{1/2}) - \frac{ib}{2}J_0(b(L^2-x^2)^{1/2}) \ \text{ if } \ |x|\le L \tag{A15}$$

$$=0 \ \text{ if } \ |x|>L\ . \tag{A16}$$

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[1]J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955) [translated from *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932)].

[2]In von Neumann's formulation, the collapse or "projection" is regarded as an actual change in the wave function. In the Everett interpretation [Rev. Mod. Phys. **29**, 454 (1957)], it is regarded as a way to simplify the observer's description, when the various components of the wave function may no longer interfere. For our purposes here, it makes no difference which view one takes of the collapse.

[3]A. Sudbery, Ann. Phys. (N.Y.) **157**, 512 (1984), also distinguishes "measurements" from "observations" but he ascribes to these terms meanings different from those which we use.

[4]A. Peres, Phys. Rev. D **22**, 879 (1980)

[5]A. Daneri, A. Loinger, and G. M. Prosperi, Nucl. Phys. **33**, 297 (1962); Nuovo Cimento **B44**, 119 (1966); W. H. Zurek, Phys. Rev. D **24**, 1516 (1981); **26** 1862 (1982).

[6]A. Peres and W. H. Zurek, Am. J. Phys. **50**, 807 (1982).

[7]E. P. Wigner, Z. Phys. **133**, 101 (1952).

[8]H. Araki and M. M. Yanase, Phys. Rev. **120**, 622 (1960); M. M. Yanase, *ibid.* **123**, 666 (1961).

[9]G. C. Ghirardi, F. Miglietta, A. Rimini, and T. Weber, Phys. Rev. D **24**, 347 (1981); **24** 353 (1981).

[10]It is possible to eliminate the explicit dependence on time by a suitable preparation of the initial state $\phi_0$, as shown in the Appendix of Ref. 4. Here, we keep the explicit time dependence, for the sake of simplicity.

[11]C. M. Caves, K. S. Thorne, R. W. P. Drever, V. Sandberg, and M. Zimmerman, Rev. Mod. Phys. **52**, 341 (1980).

[12]A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento **72B**, 79 (1982); Found. Phys. **13**, 779 (1983); A. Barchielli, Nuovo Cimento **74B**, 113 (1983).

[13]P. A. Moldauer, Phys. Rev. D **5**, 1028 (1972).

[14]A. Peres, Am. J. Phys. **42**, 886 (1974).

[15]J. S. Bell, Helv. Phys. Acta **48**, 93 (1975).

[16]A. Peres, in *Mathematical Foundations of Quantum Theory,* edited by A. R. Marlow (Academic, New York, 1978).

[17]C. B. Chiu, E. C. G. Sudarshan, and B. Misra, Phys. Rev. D **16**, 520 (1977); B. Misra and E. C. G. Sudarshan, J. Math. Phys. **18**, 756 (1977).

[18]A. Peres, Am. J. Phys. **48**, 931 (1980).

[19]L. I. Schiff, *Quantum Mechanics,* 2nd ed. (McGraw-Hill, New York, 1955), p. 58.

[20]*Tables of Integral Transforms,* edited by A. Erdélyi (McGraw-Hill, New York, 1954), Vol. 1, p. 26.