Stability of self-consistent higher-dimensional cosmological solutions

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The problem of self-consistent solutions in higher-dimensional spacetimes is considered. The importance of quantum effects in this analysis is emphasized. It is shown how the one-loop effective action for conformally invariant matter fields on a time-dependent Kaluza-Klein background may be obtained from previously known results on static backgrounds. The field equations are obtained and solved for self-consistent solutions in cases for which the four-dimensional part of the space is a Friedmann-Robertson-Walker space. Stability of the solutions to small perturbations is studied. All solutions, including the product of flat four-dimensional space with a sphere, are shown to be unstable.

I. INTRODUCTION

One of the possibilities for obtaining a theory that treats gravity and non-Abelian gauge fields in a similar manner is the generalization of Kaluza and Klein's[1,2] five-dimensional theory of gravity to higher spacetime dimensions.[3] These theories are based on a spacetime of the form $M^4 \times D^N$, where M^4 is some four-dimensional space, and D^N is a compact N-dimensional space with an isometry group. Non-Abelian gauge theories are included in the metric for this higher-dimensional spacetime with the gauge transformations generated by the isometries of D^N. (For a detailed review see Refs. 4 and 5.)

Since Witten's[6] observation that eleven spacetime dimensions is the maximum allowed for supergravity, and the minimum for a Kaluza-Klein theory with a gauge group containing $SU(3)\times SU(2)\times U(1)$, a great deal of effort has gone into studying various aspects of higher-dimensional theories. In particular, many authors have looked at the cosmological implications of higher-dimensional theories. (See Refs. 7–16, for example.) By associating time-dependent radii with the extra compact spatial dimensions it is possible to generalize the standard Friedmann-Robertson-Walker cosmological solutions to Kaluza-Klein theory. At late times the radii of the extra dimensions must be extremely small in order to explain why we appear to live in a four-dimensional world. In addition, the rate of change of the size of the extra dimensions must be sufficiently small so that any time variation in the Newtonian gravitational constant and the gauge coupling constants are consistent with observation.[17]

The simplest models occur when the extra dimensions are static. Because of the extremely small size of the extra dimensions, quantum corrections to the classical action are very important. Not only are large vacuum energies (which have been considered by many people[18–25]) generated, but there is also a term in the four-dimensional gravitational action involving the scalar curvature of M^4 which is equally important.[26–32] Candelas and Weinberg[29] were able to find self-consistent static solutions to the quantum-corrected equations of motion for which M^4 was flat Minkowski spacetime, and D^N was the N-dimensional sphere S^N.

In order to study the stability of the static self-consistent solutions, it is necessary to evaluate the low-curvature limit of the effective action in a varying background. Previous work on this includes Refs. 33–35. If the quantum matter fields are conformally invariant in the higher-dimensional spacetime, then it may be possible to obtain the result by a simple conformal rescaling of the known results in the static case. (See Refs. 28–30 for the one-loop effective action in the static case.) We discuss this method below and use the resulting effective action to find the quantum-corrected field equations. We will then discuss the stability of the static solutions with respect to time-dependent perturbations, and also the stability of nonstatic cosmological solutions.

II. THE TIME-DEPENDENT EFFECTIVE ACTION

We will adopt the curvature conventions of Ref. 36. Consider a $(4+N)$-dimensional space $M^4\times S^N$ with metric

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,y)=\begin{bmatrix} g_{\mu\nu}(x) & 0 \\ 0 & a^2\Omega^2(x)\gamma_{ij}(y)\end{bmatrix} . \tag{2.1}$$

Here $g_{\mu\nu}(x)$ is the metric tensor for M^4, and $\gamma_{ij}(y)$ denotes the standard metric on the unit N-sphere. S^N is taken to have a varying radius $a\Omega(x)$ for constant a. Our index conventions are that careted greek indices ($\hat{\mu}$,$\hat{\nu}$, etc.) are indices for the higher-dimensional space, with $\mu,\nu,\ldots$ indices for M^4 and $i,j,\ldots$ indices for S^N. Local coordinates on M^4 are denoted by x^μ, and y^i gives local coordinates on the unit N-sphere.

In order to obtain the field equations for a time-varying radius, it is first necessary to calculate the one-loop effective action in the background spacetime (2.1). One method of doing this is to use our knowledge of the effective action in cases where the extra dimensions are static, and then by making the appropriate conformal transformation obtain the required action. (See also Ref. 33.)

It may be noted immediately from Eq. (2.1) that the line element may be written as

$$ds^2=\Omega^2(x)d\bar{s}^2 , \tag{2.2}$$

where

$$d\tilde{s}^2=\tilde{g}_{\mu\nu}(x)dx^\mu dx^\nu+a^2\gamma_{ij}(y)dy^i dy^j \quad (2.3)$$

with

$$\tilde{g}_{\mu\nu}(x)=\Omega^{-2}(x)g_{\mu\nu}(x)\ . \quad (2.4)$$

$d\tilde{s}^2$ is seen to be the line element on a space $\tilde{M}^4\times S^N$ for which $\tilde{g}_{\mu\nu}(x)$ is the metric on $\tilde{M}^4$ (which is conformally related to M^4), with the radius of S^N constant and given by a.

Consider the action for a real scalar field on $M^4\times S^N$ defined by

$$I_\phi=-\tfrac{1}{2}\int d^4x\int d^Ny(-\hat{g})^{1/2}(\hat{g}^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\phi\partial_{\hat{\nu}}\phi+m^2\phi^2+\xi\hat{R}\phi^2)\ . \quad (2.5)$$

Perform the conformal transformation $\hat{g}_{\hat{\mu}\hat{\nu}}=\Omega^2(x)\hat{\tilde{g}}_{\hat{\mu}\hat{\nu}}$ in Eq. (2.5), where $\hat{\tilde{g}}_{\hat{\mu}\hat{\nu}}$ is the metric corresponding to the line element $d\tilde{s}^2$ in Eq. (2.3). The fields ϕ may be transformed by $\phi=\Omega^{-2-N/2}\tilde{\phi}$. If we take $m^2=0$, and the constant ξ to be given by its conformal value of $\xi=\xi_c$ where

$$\xi_c=\frac{1}{4}\left[\frac{N+2}{N+3}\right], \quad (2.6)$$

then the transformed action becomes

$$\tilde{I}_\phi=-\tfrac{1}{2}\int d^4x\int d^Ny(-\hat{\tilde{g}})^{1/2}(\hat{\tilde{g}}^{\hat{\mu}\hat{\nu}}\partial_{\hat{\mu}}\tilde{\phi}\partial_{\hat{\nu}}\tilde{\phi}+\xi_c\hat{\tilde{R}}\tilde{\phi}^2)\ , \quad (2.7)$$

with $\hat{\tilde{R}}$ the transformed scalar curvature.

The important point is that $\tilde{I}_\phi$ is now the action for a scalar field on the space $\tilde{M}^4\times S^N$ for which the radius of the extra dimensions is constant. Furthermore, for odd-dimensional spheres, the conformal transformations described above may be done with impunity as there is no conformal anomaly.[37,38] The one-loop effective action takes the general form

$$\tilde{\Gamma}^{(1)}=\int d^4x(-\tilde{g})^{1/2}(Aa^{-4}+Ba^{-2}\tilde{R}+\cdots)\ , \quad (2.8)$$

where A,B are calculable numbers, $\tilde{g}=\det(\tilde{g}_{\mu\nu})$, and $\tilde{R}$ is the scalar curvature for $\tilde{g}_{\mu\nu}(x)$. Terms of order $\tilde{R}^2$ and higher have been dropped in Eq. (2.8).

In order to calculate the one-loop effective action in terms of the original four-dimensional metric $g_{\mu\nu}$ it is simply necessary to use the relation given in Eq. (2.4). This leads to

$$\Gamma^{(1)}=\int d^4x(-g)^{1/2}[Aa^{-4}\Omega^{-4}+Ba^{-2}(\Omega^{-2}R+6\Omega^{-4}g^{\mu\nu}\partial_\mu\Omega\partial_\nu\Omega)+\cdots]\ . \quad (2.9)$$

The terms dropped in Eq. (2.9) are of two types. First, there will be higher-order terms in the four-dimensional curvature which will be negligible in comparison with R, provided that $|R|\ll a^{-2}$. The second type of term dropped in Eq. (2.9) involves higher derivatives of Ω. If we specialize to the case where Ω is a function of time only, then the neglect of these terms corresponds to a higher-dimensional generalization of the adiabatic expansion method of Parker and Fulling.[39] (See also Ref. 33.) Terms that are of higher order in the adiabatic expansion will be negligible if the radius of the extra dimensions is slowly varying. It would be possible to extend the result in Eq. (2.9) to higher order by a straightforward extension of the methods presented in Refs. 26, 28, and 30.

The action for a massless Dirac spinor field ψ is

$$I_\psi=\int d^4x\int d^Ny(-\hat{g})^{1/2}\bar{\psi}\hat{\nabla}\psi\ , \quad (2.10)$$

where $\hat{\nabla}$ is computed using the appropriate higher-dimensional spin connection. This action is invariant under $\hat{g}_{\hat{\mu}\hat{\nu}}\to\Omega^2\hat{g}_{\hat{\mu}\hat{\nu}}$ and $\psi\to\Omega^{-(N+3)/2}\psi$. Therefore the result obtained in Eq. (2.9) also holds for the massless Dirac spinor (although of course the constants A and B will differ from the scalar case).

The complete form for the one-loop effective action is

$$\Gamma=I_G+\Gamma^{(1)}\ , \quad (2.11)$$

where

$$I_G=(16\pi G_0)^{-1}\int d^4x\,d^Ny(-\hat{g})^{1/2}(\hat{R}-2\Lambda_0) \quad (2.12)$$

is the Einstein-Hilbert action. Here G_0 and Λ_0 are constants. We have set the background matter fields to zero. Substitution of the metric given in Eq. (2.1) leads to

$$\hat{R}=R+N(N-1)a^{-2}\Omega^{-2}-2N\Omega^{-1}\Box\Omega-N(N-1)\Omega^{-2}\partial^\mu\Omega\partial_\mu\Omega\ , \quad (2.13)$$

where $\Box$ is the four-dimensional d'Alembertian operator. The effective action then becomes

$$\Gamma=(16\pi\bar{G}_0)^{-1}\int dv_x\Omega^N[R+N(N-1)a^{-2}\Omega^{-2}-2\Lambda_0+N(N-1)\Omega^{-2}\partial^\mu\Omega\partial_\mu\Omega]+\int dv_x[Aa^{-4}\Omega^{-4}+Ba^{-2}(\Omega^{-2}R+6\Omega^{-4}\partial^\mu\Omega\partial_\mu\Omega)]\ , \quad (2.14)$$

where $dv_x=(-g)^{1/2}d^4x$ is the four-dimensional invariant volume element, and

$$\bar{G}_0=G_0V_N{}^{-1}\ . \quad (2.15)$$

Here $V_N=2\pi^{(N+1)/2}a^N/\Gamma((N+1)/2)$ is the volume of the static N-sphere with radius a.

III. THE FIELD EQUATIONS

The field equations are obtained by varying the effective action (2.14) with respect to the higher-dimensional metric and then setting the variation equal to zero. The variation with respect to the metric on S^N must be proportional to the metric because S^N is a maximally symmetric space. This is easily seen to give only one independent equation which may be obtained by varying the scale factor $\Omega(x)$ for the radius of the sphere. The resulting equation is

$$0=N[R-2\Lambda_0+(N-1)(N-2)\Omega^{-2}a^{-2}]-N(N-1)(N-2)\Omega^{-2}\partial^\mu\Omega\partial_\mu\Omega-2N(N-1)\Omega^{-1}\Box\Omega$$
$$-(16\pi\overline{G}_0)(4Aa^{-4}\Omega^{-4-N}+2Ba^{-2}\Omega^{-2-N}R-24Ba^{-2}\Omega^{-4-N}\partial^\mu\Omega\partial_\mu\Omega+12Ba^{-2}\Omega^{-3-N}\Box\Omega)\ . \quad (3.1)$$

The equations obtained by setting the variation with respect to the four-dimensional metric $g_{\mu\nu}(x)$ equal to zero are

$$0=R_{\mu\nu}-\tfrac{1}{2}[R-2\Lambda_0+N(N-1)a^{-2}\Omega^{-2}]g_{\mu\nu}+\tfrac{1}{2}N(N-1)\Omega^{-2}\partial^\lambda\Omega\partial_\lambda\Omega g_{\mu\nu}+N\Omega^{-1}\Box\Omega g_{\mu\nu}$$
$$-N\Omega^{-1}\nabla_\mu\nabla_\nu\Omega+(16\pi\overline{G}_0)[Ba^{-2}\Omega^{-2-N}R_{\mu\nu}-\tfrac{1}{2}(Aa^{-4}\Omega^{-4-N}+Ba^{-2}\Omega^{-2-N}R)g_{\mu\nu}$$
$$-2Ba^{-2}\Omega^{-3-N}\Box\Omega g_{\mu\nu}+2Ba^{-2}\Omega^{-3-N}\nabla_\mu\nabla_\nu\Omega+3Ba^{-2}\Omega^{-N-4}\partial^\lambda\Omega\partial_\lambda\Omega]\ . \quad (3.2)$$

Here ∇_μ denotes the usual covariant derivative and $\Box=\nabla^\mu\nabla_\mu$.

We will assume that the four-dimensional line element $ds_4{}^2$ has the usual Friedmann-Robertson-Walker form

$$ds_4{}^2=-dt^2+s^2(t)[d\chi^2+f^2(\chi)(d\theta^2+\sin^2\theta\, d\phi^2)]\ , \quad (3.3)$$

where

$$f(\chi)=\begin{cases}\sin\chi, & 0\le\chi\le 2\pi,\quad k=+1\\ \chi, & 0\le\chi<\infty,\quad k=0\\ \sinh\chi, & 0\le\chi<\infty,\quad k=-1\ .\end{cases} \quad (3.4)$$

In addition, we will only consider the case when Ω is a function of time. The explicit field equations are given in Eqs. (3.16)—(3.18) below.

A. Flat four-dimensional solutions

This case may be obtained by taking $k=0$ and $s(t)=1$ in Eqs. (3.3) and (3.4). Solving Eqs. (3.1) and (3.2) for Ω leads to

$$0=\ddot{\Omega}[(32\pi\overline{G}_0)Ba^{-2}\Omega^{-3-N}-N\Omega^{-1}]\ , \quad (3.5)$$

where $\dot{\Omega}=d\Omega/dt$. The two solutions to (3.5) correspond either to a static sphere (i.e., $\Omega=$ const), or else a sphere with the radius growing linearly with time. The latter case we reject as unphysical. Without loss of generality we may set $\Omega=1$ in Eqs. (3.1) and (3.2) and obtain an equation for the radius a of the static sphere.

We may identify the physical value for Newton's gravitational constant G from Eq. (2.14) to be given by

$$(16\pi G)^{-1}=(16\pi\overline{G}_0)^{-1}+Ba^{-2}\ . \quad (3.6)$$

This follows simply by associating the overall coefficient of R in the effective action with $(16\pi G)^{-1}$. Equation (3.6) shows that the higher-dimensional gravitational constant $\overline{G}_0$ is not simply related to the four-dimensional one as in classical Kaluza-Klein theory. In addition, the physical cosmological constant Λ is found from Eq. (2.14) to be given by

$$-2(16\pi G)^{-1}\Lambda$$
$$=(16\pi\overline{G}_0)^{-1}[N(N-1)a^{-2}-2\Lambda_0]+Aa^{-4}\ . \quad (3.7)$$

Substitution of (3.6) and (3.7) into Eqs. (3.1) and (3.2) shows that we must have $(N\neq 1)$

$$\Lambda=0\ , \quad (3.8)$$

$$\overline{G}_0=\left[1-\frac{2N(N-1)}{(N+4)}\frac{B}{A}\right]G\ , \quad (3.9)$$

$$a^2=\left[B-\frac{(N+4)}{2N(N-1)}A\right](16\pi G)\ . \quad (3.10)$$

This case is just the one considered by Candelas and Weinberg.[29] For self-consistent solutions we require

$$B>\frac{(N+4)}{2N(N-1)}A\ . \quad (3.11)$$

This shows the importance of the induced gravity term involving B since if it is ignored we find only the condition $A<0$.

The results for the coefficients A and B are given in Table I for both conformally coupled scalars and for massless Dirac spinors. We have considered cases where the extra dimensions are odd-dimensional spheres with dimension less than or equal to seven. It is easily seen that the consistency condition in Eq. (3.11) is only satisfied for S^3 and S^7 in both cases. The self-consistent radius may be found from Eq. (3.10).

It is worth noting that if the extra dimensions are not static, then there is another possible interpretation for the four-dimensional theory arising from a different parametrization of the higher-dimensional metric. It is observed from (2.14) that the coefficient of R in the part of the effective action coming from the Einstein-Hilbert action I_G in the higher-dimensional space is multiplied by Ω^N. Ignoring for a moment the quantum contribution,

TABLE I. The values of the constants A and B that enter the one-loop part of the effective action in Eq. (2.8). Results are given for conformally coupled scalar fields and massless Dirac spinor fields on S^3, S^5, and S^7.

N		Scalar	Spinor
3	A	-7.1×10^{-6}	-1.9×10^{-4}
	B	2.7×10^{-6}	5.6×10^{-5}
5	A	7.9×10^{-7}	1.1×10^{-4}
	B	-7.4×10^{-7}	-2.6×10^{-4}
7	A	-7.0×10^{-8}	-7.2×10^{-4}
	B	7.3×10^{-8}	1.8×10^{-4}

this may be interpreted as a time-dependent gravitational constant if Ω is a function of time. This suggests that in order to obtain a constant coefficient multiplying R, the original parametrization of the higher-dimensional metric should be

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,y)=\begin{bmatrix}\Omega^{-N}(x)g_{\mu\nu}(x) & 0\\ 0 & a^2\Omega^2(x)\gamma_{ij}(y)\end{bmatrix} . \qquad (3.12)$$

This leads to the effective action being given by

$$\Gamma=(16\pi\overline{G}_0)^{-1}\int d^4x(-g)^{1/2}[R-2\Lambda_0\Omega^{-N}+N(N-1)\Omega^{-2-N}a^{-2}-\tfrac{1}{2}N(N+2)\Omega^{-2}\partial^\mu\Omega\partial_\mu\Omega]$$
$$+\int d^4x(-g)^{1/2}\{Aa^{-4}\Omega^{-4-2N}+Ba^{-2}[\Omega^{-2-N}R+\tfrac{3}{2}(N+2)\Omega^{-4-N}\partial^\mu\Omega\partial_\mu\Omega]\} . \qquad (3.13)$$

[The result in (3.13) is most easily obtained by performing a conformal transformation of the four-dimensional part of the metric in (2.14).] Note that the coefficient of R in the quantum contribution to Γ involves Ω which still means that the effective gravitational constant is time dependent. It is not possible to find a parametrization of the higher-dimensional metric which makes both coefficients of R constant.

Proceding as before, we are led to field equations which in the case $g_{\mu\nu}=\eta_{\mu\nu}$ give

$$\ddot{\Omega}=-\tfrac{1}{2}N[\Omega^{N+1}-(16\pi\overline{G}_0)Ba^{-2}\Omega^{-1}]\dot{\Omega}^2 . \qquad (3.14)$$

This equation does not lead to any solutions for which $\Omega(t)$ approaches a constant at late times. The only physically interesting case is when Ω is a constant for all time. In this situation it does not matter whether the parametrization in Eq. (3.12) or Eq. (2.1) is chosen.

B. Four-dimensional Friedmann-Robertson-Walker solutions

Solutions to the equations given in (3.1)—(3.4) are not easy to obtain for nonconstant $s(t)$ or $k\neq 0$. However, it is possible to find late-time solutions assuming that

$$\lim_{t\to\infty}\Omega(t)=\Omega_0 , \qquad (3.15)$$

where Ω_0 is a constant. This guarantees that the effective gravitational constant really does become constant at late times. Without loss of generality we will normalize the radius of the extra dimensions so that $\Omega_0=1$.

The field equations are

$$0=N[6(s^{-1}\ddot{s}+s^{-2}\dot{s}^2+ks^{-2})-2\Lambda_0]+N(N-1)(N-2)a^{-2}\Omega^{-2}+N(N-1)(N-2)\Omega^{-2}\dot{\Omega}^2$$
$$+2N(N-1)\Omega^{-1}(\ddot{\Omega}+3s^{-1}\dot{s}\dot{\Omega})-4C\Omega^{-4-N}-12D\Omega^{-2-N}(s^{-1}\ddot{s}+s^{-2}\dot{s}^2+ks^{-2})$$
$$-24D\Omega^{-4-N}\dot{\Omega}^2+12D\Omega^{-3-N}(\ddot{\Omega}+3s^{-1}\dot{s}\dot{\Omega}) , \qquad (3.16)$$

$$0=3(s^{-1}\ddot{s}+s^{-2}\dot{s}^2+ks^{-2})-\Lambda_0+\tfrac{1}{2}N(N-1)a^{-2}\Omega^{-2}+3N\Omega^{-1}s^{-1}\dot{s}\dot{\Omega}+\tfrac{1}{2}C\Omega^{-4-N}$$
$$+3D\Omega^{-2-N}(s^{-1}\ddot{s}+s^{-2}\dot{s}^2+ks^{-2})+3D\Omega^{-2-N}\dot{\Omega}^2-3s^{-1}\ddot{s}-2D\Omega^{-3-N}(\ddot{\Omega}+3s^{-1}\dot{s}\dot{\Omega})$$
$$+2D\Omega^{-3-N}\ddot{\Omega}-3D\Omega^{-2-N}s^{-1}\ddot{s}+\frac{N}{2}(N-1)\dot{\Omega}^2\Omega^{-2} , \qquad (3.17)$$

$$0=-3(s^{-1}\ddot{s}+s^{-2}\dot{s}^2+ks^{-2})+\Lambda_0-\tfrac{1}{2}N(N-1)a^{-2}\Omega^{-2}-\tfrac{1}{2}N(N-1)\Omega^{-2}\dot{\Omega}^2+s^{-1}\ddot{s}+2s^{-2}\dot{s}^2$$
$$+2ks^{-2}-N\Omega^{-1}(\ddot{\Omega}+3s^{-1}\dot{s}\dot{\Omega})+Ns^{-1}\dot{s}\dot{\Omega}\Omega^{-1}-\tfrac{1}{2}C\Omega^{-4-N}-3D\Omega^{-2-N}(s^{-1}\ddot{s}+s^{-2}\dot{s}^2+ks^{-2})$$
$$-3D\Omega^{-2-N}\dot{\Omega}^2+(2\dot{s}^2s^{-2}+s^{-1}\ddot{s}+2ks^{-2})D\Omega^{-N-2}+2\Omega^{-3-N}D(\ddot{\Omega}+3s^{-1}\dot{s}\dot{\Omega})-2s^{-1}\dot{s}D\Omega^{-3-N}\dot{\Omega} , \qquad (3.18)$$

where

$$D=(16\pi\overline{G}_0)Ba^{-2} , \qquad (3.19)$$

$$C=(16\pi\overline{G}_0)Aa^{-4} . \qquad (3.20)$$

If we now look at sufficiently late times such that (3.15) is satisfied, then from Eqs. (3.17) and (3.18) we obtain

$$s\ddot{s}=k+\dot{s}^2 , \qquad (3.21)$$

assuming that $D\neq 1$. This result may be substituted back into Eq. (3.17) to obtain

$$\dot{s}^2=\omega^2s^2-k , \qquad (3.22)$$

where

$$\omega^2=\tfrac{1}{6}(1+D)^{-1}[2\Lambda_0-C-N(N-1)a^{-2}] . \qquad (3.23)$$

Different classes of solutions to (3.22) exist depending on the sign of ω^2 and the value of k. A complete list is given in Table II. It is found that physically acceptable solutions exist for the cases $\Lambda>0$ ($k=0,\pm1$), $\Lambda=0$ ($k=-1,0$), or $\Lambda<0$ ($k=-1$).

A second equation for ω^2 may be obtained by substituting Eq. (3.21) into (3.16)

$$\omega^2=\tfrac{1}{12}(N-2D)^{-1}[2N\Lambda_0-N(N-1)(N-2)a^{-2}+4C] . \qquad (3.24)$$

TABLE II. The different types of solutions for $M^4 \times S^N$ in Sec. III B. ω^2 is defined in Eq. (3.23).

$3\omega^2 = \Lambda$	k	$s(t)$
>0	1	$\frac{1}{2\omega}(e^{\omega t}+e^{-\omega t})$
	0	$pe^{\pm\omega t}$
	-1	$\frac{1}{2\omega}(e^{\omega t}+e^{-\omega t})$
$=0$	1	No solution
	0	Constant
	-1	$t+q$
<0	1	No solution
	0	No solution
	-1	$\pm\lvert\omega\rvert^{-1}\sin(\lvert\omega\rvert t)$

Using Eqs. (3.6), (3.7), and (3.23) leads to

$$\omega^2 = \tfrac{1}{3}\Lambda \ . \tag{3.25}$$

Equating Eq. (3.24) with Eq. (3.25) gives the following quadratic equation for a^{-2}:

$$(16\pi G)[(N+4)A - 2N(N-1)B]a^{-4} + [2N(N-1) + 64\pi G(N+2)\Lambda B]a^{-2} - 2N\Lambda = 0 \ . \tag{3.26}$$

Using the results from Table I is seen to lead to two real values for a^{-2} for both scalars and spinors on S^3, S^5, and S^7. If we use the present experimental bounds on the dimensionless quantity $\lvert\Lambda\rvert G$, we have

$$\lvert\Lambda\rvert G \lesssim 10^{-120} \ . \tag{3.27}$$

This gives one of the roots for a^{-2} in Eq. (3.26) corresponding to macroscopic dimensions, and hence may be rejected as unphysical. The remaining solution for a^{-2} is identical to (3.10) as expected for $\Lambda \simeq 0$. The condition for self-consistent solutions for scalars and spinors are possible only on S^3 and S^7. Table III shows the value for the static radius a in these cases, where we have used $G = L_P{}^2$. (L_P is the Planck length.)

It is worth remarking that the inclusion of n extra fields simply rescales the values of A and B from the single field result by a factor of n. In order to obtain a on the order of the Planck length, we require about 10^4 scalar fields on S^3, although 100 spinor fields on S^7 would suffice. We have investigated the effects of looking at quantized antisymmetric tensor fields in order to try and reduce the number of fields that are needed.[31]

TABLE III. The radius of the extra dimensions in terms of the Planck length L_P in cases where self-consistent solutions exist.

N	a for scalars	a for spinors
3	$1.85\times10^{-2}L_P$	$9.16\times10^{-2}L_P$
7	$2.03\times10^{-3}L_P$	$0.117L_P$

IV. STABILITY ANALYSIS OF SOLUTIONS

Having obtained solutions (3.10), (3.21), and (3.26) to the higher-dimensional field equations, we now want to see how stable they are to small variations in $s(t)$ and $\Omega(t)$.

Consider the perturbations

$$\Omega(t) = 1 + \delta\Omega(t) \ , \tag{4.1}$$

$$s(t) = s_0(t) + \delta s(t) \ . \tag{4.2}$$

Perturbing the field equations (3.1) and (3.2) to linear order in $\delta\Omega(t)$ and $\delta s(t)$ gives for the case $M^4 \times S^N$ (where $s_0 = 1$, $\Lambda = 0$)

$$0 = \alpha\delta\ddot{s} + \beta\delta\ddot{\Omega} + \gamma\delta\Omega \ , \tag{4.3}$$

$$0 = \rho\delta\ddot{s} + \alpha\delta\ddot{\Omega} \ . \tag{4.4}$$

Here

$$\alpha = N - 2D \ , \tag{4.5a}$$

$$\beta = 2D + \tfrac{1}{3}N(N-1) \ , \tag{4.5b}$$

$$\gamma = \tfrac{2}{3}(N+4)C - \tfrac{1}{3}N(N-1)(N-2)a^{-2} \ , \tag{4.5c}$$

$$\rho = 2(1+D) \ . \tag{4.5d}$$

Using results from Eqs. (3.6), (3.19), and (3.20), we obtain

$$D = -\frac{2N(N-1)B}{(N+4)A} \ , \tag{4.6}$$

$$C = -\frac{2N(N-1)}{(N+4)a^2} \ . \tag{4.7}$$

From Eqs. (4.3) and (4.4) we find

$$0 = (\rho\beta - \alpha^2)\delta\ddot{\Omega} + \rho\gamma\delta\Omega \ . \tag{4.8}$$

Using the results in Tables I and III, the constants α,β,γ,ρ may be found. It is seen that the physically acceptable solutions for scalars and spinors on S^3 and S^7 all lead to exponentially expanding solutions for $\delta\Omega$ and δs. This indicates that the static solutions are unstable.

For the Friedmann-Robertson-Walker—type solutions, the perturbed field equations are given in the Appendix. The late-time solutions $s(t)$ for the case $\Lambda > 0$ are given in Table II and are of the form

$$\lim_{t\to\infty} s(t) \propto e^{\omega t} \ , \tag{4.9}$$

with $\omega^2 = \Lambda/3$. When this is substituted into Eqs. (A1)—(A3), and Eq. (3.27) is used, we obtain

$$0 = \delta\ddot{\Omega} + 3\omega\delta\dot{\Omega} + pa^{-2} \tag{4.10}$$

as the equation governing the behavior of $\delta\Omega$ at late times. Here p is a constant given by

$$p = 2N(N-1)(1+D)[N - 2(N+3)D]^{-1} \ . \tag{4.11}$$

Self-consistent solutions exist for the cases of scalars and spinors on S^3 and S^7. These results yield [again using (3.27)]

$$\lim_{t\to\infty} \delta\Omega(t) \propto e^{\lvert p\rvert^{1/2}a^{-1}t} \ , \tag{4.12}$$

$$\lim_{t\to\infty} \delta s(t) \propto e^{(\omega+|p|^{1/2}a^{-1})t} . \quad (4.13)$$

The late-time behavior of the perturbations again indicates that the solutions are unstable.

When $\Lambda=0$, Table II shows that a solution of the form

$$s(t)=t+q \quad (4.14)$$

exists for some constant q when $k=-1$. Substitution of (4.14) into Eqs. (A1)—(A3) leads to a solution for $\delta\Omega(t)$ of the form

$$\delta\Omega(t)=mt^{-1}Y_{-1}(ipt)+nt^{-1}J_{-1}(ipt) , \quad (4.15)$$

where m,n are constants, and

$$p^2=-\tfrac{2}{3}N(N-1)(N+2)(1+D)a^{-2}\{(N-2D)^2-\tfrac{2}{3}(1+D)[6D+N(N-1)]\}^{-1} . \quad (4.16)$$

(Using the values in Table I shows that $p^2>0$.) The large-t expansion of (4.15) shows that $\delta\Omega(t)$ again diverges exponentially.

A similar situation arises for the final physically acceptable scenario $\Lambda<0$ and $k=-1$. This time the solution for $\delta\Omega(t)$ may be given in terms of a hypergeometric function, whose late-time behavior leads to exponential growth.

V. DISCUSSION AND CONCLUSIONS

We have obtained solutions to the higher-dimensional field equations using the one-loop effective action of a quantum field in a time-dependent Kaluza-Klein background. The stability of these solutions to small perturbations was then analyzed. In general, the solutions, including those of the Candelas-Weinberg[29] type, have proved unstable. This seems to be rather an unsatisfactory feature of the self-consistent solutions that we have examined. However, it was shown in Ref. 35 that combinations of minimally coupled scalar fields and massless spinor fields could lead to stable perturbations. It would be interesting to repeat this analysis for antisymmetric tensor fields[31] which are also not conformally invariant. This merits further attention.

The importance of the induced gravity term, both for obtaining self-consistent solutions as well as for the stability analysis is apparent. In another study,[13] only the induced Λ term was used, and self-consistent solutions were obtained by the introduction of a radiation term into the stress-energy tensor. It is questionable whether this is physically meaningful since the universe is matter dominated at late times. However, one possible reason for including such a term is if gravitons or other heavy particles created in the early universe interact, creating radiation that floods the universe.

Finally, we wish to point out that there will be a further condition for self-consistent solutions if a true Kaluza-Klein ansatz (which includes gauge fields) is adopted. This comes about because there will be two terms in the Yang-Mills action—one arising from the classical Einstein-Hilbert action in the higher-dimensional space, and one induced from quantum corrections.[26,28–31] The overall sign of the total Yang-Mills action must be correct, which places a further constraint on the theory. Using the general result in Ref. 30, it may be shown that the wrong sign is obtained for the conformal scalars on S^3 and S^7, so that although the gravitational part of the action comes out correctly, the Yang-Mills part does not.

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APPENDIX

In this appendix we write down the general perturbed field equations for an $M^4\times S^N$ manifold where the line element for M^4 is given by Eq. (3.3):

$$0=6(N-2D)[\delta\ddot{s}-3\ddot{s}s^{-1}\delta s+2s^{-1}\dot{s}\delta\dot{s}]-[2N(N-1)(N-2)a^{-2}-4(N+4)C-24(N+2)D\ddot{s}s^{-1}]s\delta\Omega$$
$$+2[\delta\ddot{\Omega}+3\dot{s}s^{-1}\delta\dot{\Omega}][N(N-1)+6D]s , \quad (A1)$$

$$0=6(1+D)(\dot{s}s^{-1}\delta\dot{s}-\ddot{s}s^{-1}\delta s)+3\dot{s}(N-2D)\delta\dot{\Omega}-[3(N+2)D\ddot{s}s^{-1}+N(N-1)a^{-2}+\tfrac{1}{2}(N+4)C]s\delta\Omega , \quad (A2)$$

$$0=2(1+D)\delta\ddot{s}+2(1+D)s^{-1}\dot{s}\delta\dot{s}+2(1+D)(s^{-1}\dddot{s}-\Lambda)\delta s+(N-2D)(\delta\ddot{\Omega}s+2\dot{s}\delta\dot{\Omega})$$
$$-[N(N-1)a^{-2}+\tfrac{1}{2}(N+4)C+3(N+2)Ds^{-1}\dddot{s}]s\delta\Omega . \quad (A3)$$

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