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Relativistic radiation transport in dispersive media

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A general-relativistic radiative transfer equation in an isotropic, weakly absorbing, nonmagnetized dispersive medium is derived using the kinetic-theoretical approach and the relativistic Hamiltonian theory of geometrical optics in those media. It yields the generally accepted classical equation in the special-relativistic approximation and in stationary conditions. The influence of the gravitational field and of space-time variations of the refractive index n on the radiation distribution is made explicit in the case of spherical symmetry.

I. INTRODUCTION

The importance of radiative transfer in radio astrophysics and radio emission in plasmas led many authors to discuss the transfer equation in dispersive media. The first derivation[1–3] valid for the geometrical-optics approximation and for stationary intensity modifies the well-known consequence of Snell's law $(d/dl)(I/n^2)=0$ (Ref. 4) into

$$n^2\frac{d}{dl}(I/n^2)=-kI+\epsilon \;, \tag{1.1}$$

so as to take into account the phenomenological change of the specific intensity I along an element dl of ray path, due to opacity k and emissivity ϵ in an isotropic medium of refractive index n. Different equations are due to Oster[5] using a semiclassical transport theory for a photon gas and Harris[6] using a nonrelativistic kinetic-theoretical approach, the photon propagation being described by the classical Hamiltonian;[7,8] these equations do not reduce to Eq. (1.1) in the stationary case. However, Zheleznyakov,[9] arguing that the equation of continuity for the energy density should be written in the phase space (**r**, photon position vector; ***l***, unit vector tangent to the ray), derived practically the same equation as Harris, showed its equivalence with Eq. (1.1) when dl verifies the classical Fermat principle, and invalidated Oster's equation. For the sake of completeness, we note attempts[10–12] made to extend Eq. (1.1) for radiative transfer in nonuniform magnetoactive plasmas.

It is the purpose of this paper to implement these heuristic derivatives with a general-relativistic transport theory using a kinetic-theoretical approach[13–15] and the invariant Hamiltonian theory of geometrical optics in an isotropic dispersive medium.[16] Indeed, kinetic theory relies heavily upon the phenomenological concept of the photon which is not always well defined.[17] However, when the eikonal approximation can associate to wave fields, rays along which the amplitude is propagated, and when the square of amplitude can be related to average electromagnetic energy, we can recover a quasiphoton number;[18] then, we may use a heuristic theory of photons whose trajectories are the rays and extend its validity in a weakly absorbing, normally dispersive medium where opacity k and emissivity ϵ are phenomenologically introduced. A more general treatment could proceed along the path paved by Wolf.[19]

After giving a few definitions in Sec. II we recall in Sec. III the relativistic Hamiltonian theory of geometrical optics in a nonmagnetized, normally dispersive medium. In the corresponding phase space, the invariant distribution function of photons $F(x,p)$ is defined and its equivalence to the classical one is shown in Sec. IV. The radiation transport equation is derived, and its form in the rest frame of the medium is made explicit in Sec. V. The special-relativistic approximation in stationary conditions determines by comparison with Eq. (1.1) the relativistic transformation laws for opacity and emissivity (Sec. VI). A spherically symmetric space-time is considered in Sec. VII and the influence of the gravitational field and gradient of n on $F(x,p)$ is determined. A brief conclusion is the content of the final section.

II. NOTATION AND DEFINITIONS

Space-time is a Lorentzian manifold; in local coordinates x^i $(i=0,1,2,3)$ its metric has the components g_{ij} and its connection Γ^i_{jk}; they are related by

$$g_{ij;k}=\partial_k g_{ij}-g_{hj}\Gamma^h_{ik}-g_{ih}\Gamma^h_{jk}=0,\quad \partial_k=\frac{\partial}{\partial x^k}\;. \tag{2.1}$$

For physical interpretation, a tetrad basis $\mathbf{e}_a$ may be introduced by

$$\mathbf{e}_a=e_a{}^k\mathbf{e}_k,\quad \mathbf{e}_k=\{\partial_k\}\quad (a,b,\ldots=0,1,2,3)\;. \tag{2.2}$$

Indices $a,b,\ldots$ are used for the tetrad basis and $i,j,\ldots$ for

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the coordinate basis. The tetrad components of a vector A^i are

$$A^a = A^j e_j{}^a, \quad e_a{}^k e_k{}^b = \delta_a{}^b, \quad e_i{}^c e_c{}^j = \delta_i{}^j . \tag{2.3}$$

An orthonormal basis is defined when

$$g_{ab} = e_a{}^i e_b{}^j g_{ij} = \eta_{ab} , \qquad \eta_{0a} = -\delta_{0a}, \quad \eta_{\mu\nu} = \delta_{\mu\nu} \quad (\mu,\nu,\ldots = 1,2,3) . \tag{2.4}$$

We define the Pfaffian derivative ∂_a and the Ricci rotation coefficients[20]

$$\Gamma_{abc} = e_a{}^j e_{bj;k} e_c{}^k, \quad \partial_a = e_a{}^k \partial_k . \tag{2.5}$$

We have

$$\Gamma_{abc} = g_{ah}\Gamma^h_{bc} = \tfrac{1}{2}(\partial_b g_{ac} + \partial_c g_{ba} - \partial_a g_{bc}) + \tfrac{1}{2}(b_{abc} - b_{cab} - b_{bac}) \tag{2.6}$$

with $b^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb}$.

The physical interpretation of the theory is based on (3 + 1) decomposition of mathematical quantities with respect to an observer's frame of four-velocity u^i $(u^k u_k = -1)$. The projection tensor γ_{ij} into the rest space of this frame is

$$\gamma_{ij} = g_{ij} + u_i u_j, \quad \gamma_{ik} u^k = 0, \quad \gamma_i{}^k \gamma_k{}^j = \delta_i{}^j . \tag{2.7}$$

The corresponding tensor in the rest frame of a medium of velocity V^i will be denoted h_{ij}.

So, the frequency four-vector p^i of light waves (or eventually, the four-momentum of the corresponding photon) may be written

$$p^i = \chi u^i + p s^i , \tag{2.8}$$

with $\chi = -p^k u_k$, $p = (\gamma_{ij} p^i p^j)^{1/2}$, $s^k s_k = 1$, $s^k u_k = 0$; χ is the energy of the photon for the observer of velocity u^i, s^i is the direction of the wave propagation in this observers rest space. With respect to an orthonormal basis with $e_0{}^i = u^i$, we have $s^a = (0, s^u)$.

III. GEOMETRICAL OPTICS

Consider a beam of radiation propagating through a weakly absorbing, isotropic, normally dispersive medium specified by its four-velocity V^i, refractive index n, opacity k, and emissivity ϵ in a given Lorentzian manifold; we may consider that g_{ij} describes the gravitational field due either to the medium itself or to other bodies.

In the geometrical-optics approximation, we derive the equations of the rays along which the amplitudes of wave fields are transported, from the Hamiltonian $H(x^i, p_i)$, where $p_i = \partial_i S$, S being an eikonal; $H(x^i, p_i)$ is in general any function of (x^i, p_i) such that $H=0$ (and $\partial H/\partial p_a \neq 0$ on $H=0$) gives the dispersion relation for the medium. In an unmagnetized plasma, we may choose the Hamiltonian

$$H(x^i, p_i) = \tfrac{1}{2}(g^{ij} p_i p_j + \omega_0{}^2) , \tag{3.1a}$$

$$\omega_0{}^2 + (n^2 - 1)E^2 = 0 , \tag{3.1b}$$

where ω_0 is the plasma frequency, the refractive index n the reciprocal of phase speed, and $E = -p_k V^k$. This Hamiltonian has the advantage of including the vacuum case as well.

The ray equations are

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = p^i + B^i, \quad B^i = \left[(n^2-1)E + n\frac{\partial n}{\partial E} E^2 \right] V^i , \tag{3.2a}$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\tfrac{1}{2}\partial_i g^{jk} p_j p_k + A_i , \qquad A_i = \tfrac{1}{2}\partial_i[(n^2-1)E^2] , \tag{3.2b}$$

where the overdot denotes derivation with respect to a parameter λ. From Eqs. (3.2a) and (3.2b) we get

$$\dot{p}^i = -\Gamma^i_{jk} p^j p^k + B^k p_j \partial_k g^{ij} + A^i . \tag{3.2c}$$

With respect to an orthonormal frame with $e_0{}^i = V^i$, let

$$p^i = E V^i + p l^i, \quad p = (h_{ij} p^i p^j)^{1/2} , \tag{3.3}$$

with $p = nE$ as a consequence of the dispersion relation $H=0$.

On the other hand, in a normally dispersive medium we may define[21] the group speed w by

$$w = \frac{\partial E}{\partial p} < 1 . \tag{3.4}$$

It gives an estimate of the velocity of the peak of a pulse.[22] The definition (3.4) is not applicable when anomalous dispersion is present, as the velocity of a pulse has then a nonzero imaginary part.[21] Then

$$\dot{x}^i = nE(w^{-1} V^i + l^i) \tag{3.5}$$

is timelike and the corresponding unit four-vector is the ray velocity.[16]

IV. INVARIANT DISTRIBUTION FUNCTION

The Hamiltonian extremals defined by Eqs. (3.2a) and (3.2b) are light rays. These equations guarantee the Liouville theorem, i.e., the conservation of the eight-volume $d^2 v$ by Lie dragging along the phase trajectories

$$\frac{d}{d\lambda}(d^8 v) = 0 , \tag{4.1a}$$

$$d^8 v = \epsilon_{ijkl} d_0 x^i d_1 x^j d_2 x^k d_3 x^l \times \epsilon^{hmnq} d_0 p_h d_1 p_m d_2 p_n d_3 p_q , \tag{4.1b}$$

$\epsilon_{ijkl}, \epsilon^{hmnq}$ are the Levi-Civita permutation symbols. It then follows that $d^4x\, d^4p$ is also conserved by the same Lie dragging with

$$d^4 x = (-g)^{1/2} \epsilon_{ijkl} d_0 x^i d_1 x^j d_2 x^k d_3 x^l , \tag{4.2a}$$

$$d^4 p = (-g)^{1/2} \epsilon_{ijkl} d_0 p^i d_1 p^j d_2 p^k d_3 p^l , \tag{4.2b}$$

as $d^4 p = -(-g)^{-1/2} \epsilon^{ijkl} d_0 p_i d_1 p_j d_2 p_k d_3 p_l$. Moreover,

$$d^4 p = b\, db\, d\pi , \tag{4.3}$$

where $d\pi$ is the element of three-area of the "mass shell" $H = -\frac{1}{2} b^2$ in momentum space; as a consequence of Eqs. (3.2), b is a constant of motion; hence the conservation of

$d^4x\, d\pi$ during Lie transport along rays,

$$\frac{d}{d\lambda}(d^4x\, d\pi)=0 \ . \tag{4.4}$$

With respect to the frame of velocity u^i, we have also

$$d^3p=(-g)^{1/2}\epsilon_{ijkl}u^i d_1p^j d_2p^k d_3p^l = |u_i\dot{x}^i|\, d\pi \ , \tag{4.5}$$

when (d_1p^i, d_xp^j, d_3p^k) span a three-surface element on the "mass shell" whose normal is $\dot{x}^i$.

Now, if dN is the number of phenomenological photons crossing along the rays tangent to $\dot{x}^i$, the three-surface element $d\Sigma_i = d\Sigma\, u_i$ in the positive sense of the normal and having the frequency vector p^i in the range $d\pi$, i.e, if dN is the "flux" of $\dot{x}^i$ across $d\Sigma\, d\pi$, the invariant distribution function $F(x,p)$ is defined by

$$dN=F(x,p)\,|\dot{x}^k u_k|\, d\Sigma\, d\pi \ . \tag{4.6}$$

To relate F to the nonrelativistic distribution function, consider at the event x, the four-volume d^4x constructed on $d\Sigma$ and a displacement $dx^i=\dot{x}^i\Delta\lambda$ along the rays that intersect target $d\Sigma$

$$\begin{aligned} d^4x &= (-g)^{1/2}\epsilon_{ijkl}\dot{x}^i d_1x^j d_2x^k d_3x^l \Delta\lambda \\ &= |\dot{x}^k u_k|\, d\Sigma\, \Delta\lambda \ . \end{aligned} \tag{4.7}$$

From Eqs. (4.5), (4.6), and (4.7) we get

$$dN=F(x,p)d^3x\, d^3p, \quad d^3x=d\Sigma \ . \tag{4.8}$$

This shows that $F(x,p)$ reduces to the nonrelativistic distribution function.[23]

However, with respect to an arbitrary frame, we may define another function $f(x,\mathcal{X},\mathrm{s})$ by

$$dN=f(x,\mathcal{X},\mathrm{s})d\mathcal{X}\, d\Omega d^3x \ , \tag{4.9}$$

where $d\Omega$ is the element of solid angle in the direction s; f is related to F by

$$F(x,p)=p^{-2}f(x,\mathcal{X},\mathrm{s})\frac{\partial\mathcal{X}}{\partial p} \ . \tag{4.10}$$

With respect to the rest frame of the medium, Eq. (4.10) reads

$$F(x,p)=(nE)^{-2}f(x,E,\mathbf{l})w \ , \tag{4.11}$$

where $f(x,E,\mathbf{l})$ is the distribution function used generally in astrophysics. As a consequence, we have in terms of the specific intensity $I=wEf$ the invariance of $In^{-2}E^{-3}$ $(=F)$.

V. THE RADIATION TRANSPORT EQUATION

This equation results when the change $(dF/d\lambda)_{\rm coll}d^4x\, d\pi$ of the number of photons in the range $d\Sigma\, d\pi$, due to absorption and emission is equated with the range $\delta(dN)$ corresponding to the variation of $F(x,p)$ along the rays,

$$\delta(dN)=\left[\frac{dF}{d\lambda}\right]_{\rm coll} d^4x\, d\pi \ . \tag{5.1}$$

Now, defining $(dF/d\lambda)_{\rm coll}$ by

$$\left[\frac{dF}{d\lambda}\right]_{\rm coll} = -\alpha(E)F(x,p)+\beta(E) \ , \tag{5.2}$$

where $\alpha(E)$ and $\beta(E)$ are, respectively, the phenomenological invariant absorption and emission coefficients, and using Eqs. (4.4), (4.6), and (4.7) we have

$$\left[\dot{x}^k\partial_k+\dot{p}^k\frac{\partial}{\partial p^k}\right]F(x,p)=-\alpha(E)F(x,p)+\beta(E) \ . \tag{5.3}$$

Substitution in Eq. (5.3) from Eqs. (3.2a), (3.2c), (5.2a), and (5.2b) gives the radiation transport equation

$$(p^k+B^k)\partial_kF+(-\Gamma^k_{ij}p^ip^j+B^ip_j\partial_ig^{kj}+A^k)\frac{\partial F}{\partial p^k} = -\alpha F+\beta \ , \tag{5.4}$$

which obviously reduces when $n=1$, to the corresponding equation.[13] All the physical quantities (photon flux vector, energy-momentum tensor, etc.) then can be defined as the moments of $F(x,p)$, solution of Eq. (5.4).

Now, Eq. (5.3) concerns a distribution function $F(x,p)$ defined over the entire momentum space, for all possible four-momenta. But actually, we are interested in the values of F on the "mass shell" $H=-\frac{1}{2}b^2$. Geometrically, as in the case of a nondispersive medium,[13,24] we are led to think of F not as a function of all four-momenta, but rather as a function of those momenta tangent to $\{H=-\frac{1}{2}b^2\}$. Then, only three of $\partial F/\partial p^i$ are independent; we may therefore choose as independent variables three of p^i's or any three combinations of them.

With respect to the rest frame of the medium where p^i admit of the decomposition (3.3), we may choose E and two of l^i's. Equation (5.4) reads

$$nE(w^{-1}V^i+l^i)\partial_iF+(\Gamma^i_{jk}p^jp^k-B^kp_j\partial_kg^{ij}-A^i)\left[V_i\frac{\partial}{\partial E}-\left[\frac{h_i{}^k}{nE}+\frac{l^kV_i}{nEw}\right]\frac{\partial}{\partial l^k}\right]F=-\alpha F+\beta \ , \tag{5.5}$$

where in $F=F(x,E,l^i)$ one should retain only two of l^i's.

Replacing in Eq. (5.5), (V^i, E, l^i, w, nE, and h_{ij}) by (u^i, $\mathcal{X}$, s^i, $\partial\mathcal{X}/\partial p$, p, and γ_{ij}), we get, naturally, the transport equation with respect to an arbitrary observer.

VI. SPECIAL-RELATIVISTIC TRANSFER

In this section we assume space-time to be Minkowskian with coordinates x^i such that

$$\begin{aligned} &g_{ij}=\eta_{ij}, \quad \Gamma^i_{jk}=0 \ , \\ &V^i=\delta_0{}^i \to l^0=0, \quad l^ul_u=1 \quad (u,v=1,2,3) \ . \end{aligned} \tag{6.1}$$

As in $F(x^i,E,l^u)$ only two of the l^u's should be retained, we must, on account of the constraint $l^ul_u=1$, replace $\partial F/\partial l^u$ by $(\delta_u{}^v-l_ul^v)\partial F/\partial l^v$. Then in the rest frame of the medium the transport equation becomes

$$nE(w^{-1}\partial_0+l^u\partial_u)F+\left[A^0\frac{\partial}{\partial E}+A^u(\delta_u{}^v-l_ul^v)\frac{\partial}{\partial l^v}\right]F=-\alpha F+\beta\ . \quad (6.2)$$

If stationary conditions are moreover prevailing, i.e.,

$$\partial_0 F=0,\ \ \partial_0 n=0,\ \ E=\text{const}\ , \quad (6.3)$$

Eqs. (3.2a) and (3.2b) yield

$$dl=nE\,d\lambda,\ \ k_u=\frac{dl_u}{dl}=(\delta_u{}^v-l_ul^v)\partial_v \ln n\ , \quad (6.4)$$

which states that the spatial gradient $\partial_u n$ lies in the osculating plane to the curved light ray,[25] k^u being its principal normal. Equation (6.2) now reads

$$\left[l^u\partial_u+k^u\frac{\partial}{\partial l^u}\right]F=-\frac{\alpha}{nE}F+\frac{\beta}{nE}\ , \quad (6.5)$$

which agrees with Eq. (1.1) if we define phenomenological opacity k and emissivity ϵ by

$$k=\frac{\alpha}{nE},\ \ \epsilon=\beta E^2 n\ . \quad (6.6)$$

Equation (6.5) determines the relativistic transformation laws for opacity and emissivity; when $n=1$, we thus recover the known relativistic transformation properties of k and ϵ (Refs. 26 and 27). The proportionality to n^{-1} resulting from Eq. (6.6) for k is also well known.[11,28]

VII. SPHERICAL SYMMETRY

We assume the gravitational field to be described in comoving coordinates of the medium by the spherically symmetric metric

$$ds^2=-e^{2\Phi}dt^2+e^{2\Lambda}dr^2+R^2(d\theta^2+\sin^2\theta\, d\varphi^2)\ , \quad (7.1)$$

with Φ, Λ, and R functions of r and t. The matrix $e_a{}^i$ relating the coordinate basis $\mathbf{e}_i$ to the co-moving orthonormal basis $\mathbf{e}_a$ is determined by

$$V^i=e_a{}^i=e^{-\Phi}\delta_0{}^i,\ \ e_1{}^i=e^{-\Lambda}\delta_1{}^i\ ,$$
$$e_2{}^i=R^{-1}\delta_2{}^i,\ \ e_3{}^i=(R\sin\theta)^{-1}\delta_3{}^i\ . \quad (7.2)$$

The Ricci rotation coefficients can now be easily computed. Moreover, in the subspace spanned by $\mathbf{e}_u$ ($u=1,2,3$), we introduce spherical coordinates with $\mathbf{e}_1$ as the polar axis. The physical components of the photon four-momentum are

$$p^a=(E,nE\cos\bar\theta,nE\sin\bar\theta\cos\bar\varphi,nE\sin\bar\theta\sin\bar\varphi)\ . \quad (7.3)$$

On the other hand, owing to spherical symmetry, only two of the components of p^a are independent; we may choose them to be E and $l^1=\mu=\cos\bar\theta$ so that

$$F=F(r,t,E,\mu)\ . \quad (7.4)$$

Equation (5.5) becomes, with respect to the basis $\mathbf{e}_a$, after a long but straightforward calculation

$$nE(w^{-1}D_t+\mu D_r)F-E^2\left[n^2\mu^2 D_t\Lambda+n^2(1-\mu^2)\frac{U}{R}+n\mu D_r\Phi\right]\frac{\partial F}{\partial E}+\left[nE(1-\mu^2)\left[\frac{\Gamma}{R}+\frac{U}{R}\frac{\mu}{w}\right]\right.$$
$$\left.-\mu E\left[1-\frac{n\mu^2}{w}\right]D_t\Lambda-\frac{E}{n}\left[1-\frac{n\mu^2}{w}\right]D_r\Phi-2B^0\mu\left[D_t\Lambda-\frac{1}{nw}D_t\Phi\right]+\frac{A^0}{nE}\left[1-\frac{\mu}{w}\right]\right]\frac{\partial F}{\partial \mu}=-\alpha F+\beta\ , \quad (7.5)$$

where $D_t\equiv e^{-\Phi}\partial_t$, $D_r\equiv e^{-\Lambda}\partial_r$, $U\equiv D_tR$, and $\Gamma=D_rR$.

For $n=1$, we recover the radiation transfer equation in a nonrefractive spherically symmetric medium.[13] In the static case, E is a constant of motion, and we can further choose as radial variable $r=R(r)$. We get

$$e^{-\Lambda}\left[\left[\mu\partial_r+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}\right]F-\left[\frac{1}{n^2}(1-n\mu^2w^{-1})\frac{\partial}{\partial\mu}+\mu E\frac{\partial}{\partial E}\right]F\frac{d\Phi}{dr}-\partial_r \ln n\frac{\partial F}{\partial\mu}\right]=-kF+n^{-2}E^{-3}\epsilon\ . \quad (7.6)$$

The terms in the first parentheses are the classical ones, those in the second parentheses describe the variation of F due to gravitation light deflection and red-shift; the third term on the left-hand side of Eq. (7.6) exhibits a transverse effect on F due to the radial gradient of $\ln n$.

VIII. CONCLUSION

It is remarkable that the radiative transfer equation derived by standard relativistic methods confirm the classical Eq. (1.1) in a stationary situation. Indeed, in more general physical situations, Eq. (5.4) and its transforms provide a natural tool to investigate the influence of the gravitational field and the effect of space-time variations of n on radiation distribution function, in an isotropic, weakly absorbing, normally dispersive medium.

[1]S. F. Smerd, Aust. J. Sci. Res. **3A**, 34 (1950).
[2]R. v. d. R. Wooley and D. N. W. Stibbs, *The Outer Layers of a Star* (Clarendon, Oxford, 1953), p. 240.
[3]V. V. Zheleznyakov, *Radio Emission of the Sun and Planets* (Nauka, Moscow, 1964), Sec. 26.
[4]E. A. Milne, in *Handbuch der Astrophysik* [Reprinted in *Selected Papers on the Transfer of Radiation,* edited by Donald H. Menzel (Dover, New York, 1966), pp. 88—93] (Springer, Ber-

lin, 1930), Vol. 3, pp. 74—79.
[5]L. Oster, Astrophys. J. **138**, 761 (1963).
[6]E. G. Harris, Phys. Rev. **138**, B479 (1965).
[7]T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill, New York, 1962), Secs. 3-3 and 3-6.
[8]S. Weinberg, Phys. Rev. **126**, 1899 (1962).
[9]V. V. Zheleznyakov, Astrophys. J. **148**, 849 (1967).
[10]G. Bekefi, *Radiation Processes in Plasma* (Wiley, New York, 1966), Chap. 1.
[11]V. V. Zheleznyakov, Astrophys. J. **155**, 1129 (1969).
[12]V. V. Zheleznyakov, Astrophys. J. **168**, 281 (1971).
[13]R. W. Lindquist, Ann. Phys. (N.Y.) **37**, 487 (1966).
[14]J. Ehlers, in *General Relativity and Cosmology,* proceedings of the International School of Physics, "Enrico Fermi" Course XLVII, edited by R. K. Sachs (Academic, New York, 1971), p. 1.
[15]W. Israël, in *General Relativity,* edited by L. O'Raifeartaigh (Clarendon, Oxford, 1972), p. 201.
[16]J. L. Synge, *Relativity, The General Theory* (North-Holland, Amsterdam, 1960), Chap. 11.
[17]E. C. G. Sudarshan. Phys. Rev. A **23**, 2802 (1981).
[18]R. A. Breuer and J. Ehlers, Proc. R. Soc. London **A374**, 65 (1981).
[19]E. Wolf, Phys. Rev. D **13**, 869 (1976).
[20]L. P. Eisenhart, *Riemannian Geometry* (Princeton University, Princeton, New Jersey, 1949), p. 97.
[21]L. Robinett, Phys. Rev. D **18**, 3610 (1978).
[22]L. Brillouin, *Wave Propagation and Group Velocity* (Academic, New York, 1960), Chaps. 1—3.
[23]J. L. Synge, *The Relativistic Gas* (North-Holland, Amsterdam, 1957), Chap. 2.
[24]N. A. Chernikov, Acta. Phys. Pol. **23**, 629 (1963).
[25]A. Sommerfeld, *Optics, Lectures on Theoretical Physics* (Academic, New York, 1964), Vol. IV, Sec. 48.
[26]L. H. Thomas, Q. J. Math. **1**, 239 (1930).
[27]S. Kichenassamy and R. A. Krikorian, C. R. Acad. Sci. **286B**, 127 (1978).
[28]V. L. Ginzburg, *The Propagation of Electromagnetic Waves in Plasma* (Pergamon, Oxford, 1970).