Rephasing-invariant formulation of *CP* violation in the Kobayashi-Maskawa framework

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A formulation of *CP* violation is given which is manifestly invariant under rephasing the quark fields (or, equivalently, under rephasing the states) in the Kobayashi-Maskawa framework.

Recently, there has been a good deal of interest in the notion of "maximal" *CP* violation in the framework of the Kobayashi-Maskawa (KM) matrix.[1,2] Work on this question has been carried out using various parametrizations of the KM matrix. What is "maximal" in one parametrization need not be "maximal" in a parametrization which differs from the first by rephasing the quark fields. Further, the notion of "maximal" *CP* violation is not uniquely defined. It seems clear that we need a reparametrization- (or rephasing-) invariant formulation of *CP* violation in the KM framework to properly assess the notion of "maximal" *CP* violation. Surprisingly, such a formulation does not appear in the literature. My purpose in this Brief Report is to give such a formulation.

The KM matrix appears in the Lagrangian of the standard model in the terms

$$-(g/\sqrt{2})(\bar{U}_L\gamma^\mu V D_L W_\mu^+ + \bar{D}_L\gamma^\mu V^\dagger U_L W_\mu^-) \; , \tag{1}$$

which give the coupling of the left-handed quark currents to the W's. Here the subscript L stands for left-handed projection, $D_L^T=(d,s,b)_L$, $U_L^T=(u,c,t)_L$, and the superscript T is just used to convert column vectors to row vectors in generation space.

Since changing the phase of the quark field of each flavor preserves the anticommutation relations, observables must be independent of such rephasing. If

$$D_L \to T^L D_L \text{ and } U_L \to T^R U_L \; ,$$

then

$$V \to T^{R\dagger} V T^L \; .$$

Here the T's are diagonal matrices whose elements have modulus one. For the case of N generations, $2N-1$ independent parameters are contained in the two T's. It is clear that we can restrict V to SU(N), rather than U(N); then the T's will have $2N-2$ independent parameters.

Since the same Fermi field operators which appear in the Lagrangian also make the quark states, rephasing the field operators in both the Lagrangian and the states will leave all matrix elements unchanged; however, the fact that rays, not vectors, correspond to physical states in quantum mechanics allows the states to be rephased independently of the rephasing of the operators, and observable quantities must be invariant under this rephasing. Thus, we can require observables to be invariant under either rephasing of the fields in the Lagrangian or under rephasing of the quark fields in the states.

The determination of the K_S and K_L states as eigenstates of the decay and mass operator $M-i\Gamma/2$ was first given by Lee, Oehme, and Yang.[3] The result is

$$|K_{S,L}\rangle = (|p|^2+|q|^2)^{-1/2}(p|K^0\rangle \pm q|\bar{K}^0\rangle) \; , \tag{2}$$

where

$$p^2 = M_{12} - i\Gamma_{12}/2$$

and

$$q^2 = M_{12}^* - i\Gamma_{12}^*/2 \; .$$

Here 1 and 2 stand for K^0 and $\bar{K}^0$, respectively. (Note that K_S and K_L are not eigenstates of M and of Γ separately.) Under the rephasing

$$|K^0\rangle \to (\exp i\phi)|K^0\rangle, \quad |\bar{K}^0\rangle \to [\exp(-i\phi)]|\bar{K}^0\rangle \; ,$$

$$p \to [\exp(-i\phi)]p, \quad q \to (\exp i\phi)q \; ,$$

so that K_S and K_L remain invariant. Thus, the amplitudes

$$\langle \pi^+\pi^- | K_{S,L}\rangle \text{ and } \langle \pi^0\pi^0 | K_{S,L}\rangle$$

are independent of the choice of phase of K^0 and $\bar{K}^0$ (and also of the choice of phase of the d and s quarks), and the measured quantities

$$\eta_{+-} = \frac{\langle \pi^+\pi^- | K_L\rangle}{\langle \pi^+\pi^- | K_S\rangle}, \quad \eta_{00} = \frac{\langle \pi^0\pi^0 | K_L\rangle}{\langle \pi^0\pi^0 | K_S\rangle}$$

are also independent of these phase choices. The amplitudes $a_{0,2}$ (from which the final-state strong-interaction phase shifts have been removed) for K^0 decay into two π states of isospin 0 or 2 do depend on the choice of phase for $|K^0\rangle$. Following Wolfenstein,[4] we choose $a_{0,2}$ to be real if *CP* is conserved. Changing the phase of $|K^0\rangle$ from this convention results in

$$a_{0,2} \to (\exp i\phi) a_{0,2} \; .$$

It is straightforward to calculate the η parameters in terms of p, q, a_0, and a_2. The result is

$$\eta_{+-} = \frac{p(\sqrt{2}a_0+a_2) - q(\sqrt{2}a_0^* + a_2^*)}{p(\sqrt{2}a_0+a_2) + q(\sqrt{2}a_0^* + a_2^*)} = \frac{\epsilon_m \mathrm{Re}(\sqrt{2}a_0+a_2) + i\,\mathrm{Im}(\sqrt{2}a_0+a_2)}{\mathrm{Re}(\sqrt{2}a_0+a_2) + i\epsilon_m \mathrm{Im}(\sqrt{2}a_0+a_2)} \; , \tag{3}$$

and

$$\eta_{00} = \frac{p(a_0-\sqrt{2}a_2) - q(a_0^* - \sqrt{2}a_2^*)}{p(a_0-\sqrt{2}a_2) + q(a_0^* - \sqrt{2}a_2^*)} = \frac{\epsilon_m \mathrm{Re}(a_0-\sqrt{2}a_2) + i\,\mathrm{Im}(a_0-\sqrt{2}a_2)}{\mathrm{Re}(a_0-\sqrt{2}a_2) + i\epsilon_m \mathrm{Im}(a_0-\sqrt{2}a_2)} \; , \tag{4}$$

where

$$\epsilon_m = \frac{p-q}{p+q} \tag{5}$$

is the parameter which gives the mixing of the CP eigenstates

$$|K_{1,2}\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle \pm |\bar{K}^0\rangle) \tag{6}$$

in $|K_{S,L}\rangle$,

$$\begin{aligned} |K_S\rangle &= (1+|\epsilon_m|)^{-1/2}(|K_1\rangle + \epsilon_m|K_2\rangle) \ , \\ |K_L\rangle &= (1+|\epsilon_m|)^{-1/2}(\epsilon_m|K_1\rangle + K_2\rangle) \ . \end{aligned} \tag{7}$$

When CP is violated, $|K_{S,L}\rangle$ need not be orthogonal,

$$\langle K_L|K_S\rangle = \frac{2\,\mathrm{Re}\epsilon_m}{1+|\epsilon_m|^2} \ . \tag{8}$$

We now introduce the usual ϵ parameter,

$$\begin{aligned} \epsilon &= \frac{\langle \pi\pi, I=0|K_L\rangle}{\langle \pi\pi, I=0|K_S\rangle} \\ &= \frac{pa_0 - qa_0^*}{pa_0 + qa_0^*} \\ &= \frac{\sqrt{M_{12} - i\Gamma_{12}/2}\, a_0 - \sqrt{M_{12}^* - i\Gamma_{12}^*/2}\, a_0^*}{\sqrt{M_{12} - i\Gamma_{12}/2}\, a_0 + \sqrt{M_{12}^* - i\Gamma_{12}^*/2}\, a_0^*} \ . \end{aligned} \tag{9}$$

To good approximation,

$$\Gamma_{12} = a_0^{*2} \ , \tag{10}$$

and

$$\begin{aligned} &|\mathrm{Im}M_{12}| << |\mathrm{Re}M_{12}|, \quad |\epsilon| << 1 \ , \\ &|\epsilon_m| << 1, \quad \Delta\Gamma \approx -2\Delta m \ , \end{aligned} \tag{11}$$

where $\Delta X = X_S - X_L$. With these approximations,

$$\epsilon = \frac{1+i}{2}\,\mathrm{Im}\ln M_{12}a_0^2 \ . \tag{12}$$

The crucial quantity on which ϵ depends in both the exact and approximate invariant formulas is $M_{12}a_0^2$. If this is real, then ϵ vanishes.

Let us study $M_{12}a_0^2$. The matrix element M_{12} comes from the box graph (see Fig. 1). It has the form

$$M_{12} = (\mathrm{const})\, V_{xd} V_{xs}^* A_{xy} V_{yd} V_{ys}^* \ , \tag{13}$$

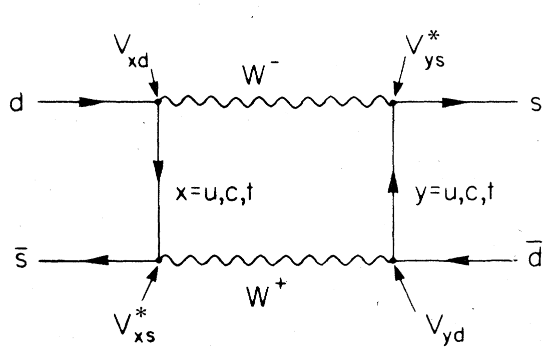

FIG. 1. Box graph for M_{12}.

where x and y run over the up quark generations u,c,t. The explicit form of A and of the constant are given in the review of Chau.[5] We introduce the real eigenvalues c^α and real eigenvectors h^α of A_{xy}:

$$A_{xy} h_y^\alpha = c^\alpha h_x^\alpha \ , \tag{14}$$

$$A_{xy} = \sum_\alpha c^\alpha h_x^\alpha h_y^\alpha \ . \tag{15}$$

The decay amplitudes have the form

$$a_{0,2} = (\mathrm{const}')\, V_{xs} f_x^{0,2} V_{xd}^* \ , \tag{16}$$

where, again, the explicit formulas for f and for the constant are given in Chau. When we put these formulas together, we can express the result as a sum of squares of traces:

$$M_{12}a_0^2 = (\mathrm{const}'') \sum_\alpha c^\alpha [\mathrm{tr}(V^\dagger H^\alpha V \Lambda_d V^\dagger F^0 V \Lambda_s)]^2 \ , \tag{17}$$

where the $\Lambda_{d,s}$ are projection operators onto the d or s generations of D quarks, and H and F are h and f elevated to diagonal matrices.

Since we know that there is no CP violation in the KM framework for two generations, we must show that $M_{12}a_0^2$ is real in that case for any V. (Of course, use of the usual rephasing argument shows that there is no CP violation in the two-generation case.) Direct substitution of an arbitrary SU(2) matrix in our formula (17) indeed yields a real quantity; however, we would like to be able to show this without direct calculation. Since our formula involves the square of a trace, it suffices to show that the trace is real. Let

$$T_{ds} = \mathrm{tr}(V^\dagger H^\alpha V \Lambda_d V^\dagger F^0 V \Lambda_s) \ . \tag{18}$$

Complex conjugation yields

$$T_{ds}^* = T_{sd} \ ,$$

which holds for any number of generations. For the case of two generations, the projection operators Λ_d and Λ_s have the special forms

$$\Lambda_{d,s} = \tfrac{1}{2}(1 \pm \sigma^3) \ . \tag{19}$$

Introducing these expressions into T gives four terms: the terms with both 1's or both σ^3's are symmetric under d,s interchange, and the terms with one 1 and one σ^3 cancel in each T. Thus, T is symmetric and real in this case. For three or more generations, the projection operators will have more than two terms when expressed in terms of the unit matrix and the diagonal matrices which represent the commuting generators in the fundamental representation. Those terms in which the same matrix occurs for both projection operators will be real; those terms in which the unit matrix occurs for one projection operator and a generator occurs for the other will either be real or will cancel; the only possible complex terms occur when different generators occur for the two projection operators.

The parameter ϵ'/ϵ can be expressed in terms of traces in an analogous way. I find

$$\begin{aligned} \frac{\epsilon'}{\epsilon} &= \frac{1}{\sqrt{2}}\left[\frac{\langle 2\pi, I=2|K_L\rangle}{\langle 2\pi, I=0|K_L\rangle} - \frac{\langle 2\pi, I=2|K_S\rangle}{\langle 2\pi, I=0|K_S\rangle}\right] \\ &= \frac{1}{\sqrt{2}}\left[\frac{pa_2 - qa_2^*}{pa_0 - qa_0^*} - \frac{pa_2 + qa_2^*}{pa_0 + qa_0^*}\right] \ . \end{aligned} \tag{20}$$

Again using $\Gamma_{12}=a_0^{*2}$, I must express $M_{12}a_2^2$ and $a_0^*a_2$ as traces. The result for $M_{12}a_2^2$ is the same as (17), except that 2 replaces 0 on F. The result for $a_0^*a_2$ is

$$a_0^*a_2=(\text{const}''')\,\text{tr}(V^\dagger F^0 V\Lambda_d V^\dagger F^2 V\Lambda_s) \quad . \tag{21}$$

I have written the discussion in terms of the K^0-$\bar{K}^0$ system. Analogous considerations hold for the D^0-$\bar{D}^0$, B_s^0-$\bar{B}_s^0$, B_d^0-$\bar{B}_d^0$, and T^0-$\bar{T}^0$ systems.

Weak-interaction processes which do not involve *CP* violation can also be expressed in terms of traces. For example, the decay rate of the neutron involves

$$|V_{ud}|^2=\text{tr}(V^\dagger\Lambda_u V\Lambda_d) \quad . \tag{22}$$

Since the diagonal matrices H and F can be calculated in terms of the diagonal matrices 1, λ^3, and λ^8 for the case of three generations, or in terms of the diagonal matrices 1 and λ_{n^2-1}, $n\le N$, for the case of N generations, all observable quantities can be expressed in terms of traces of strings of V's, $V^\dagger$'s, and λ's in which the same number of V's and $V^\dagger$'s occurs, the V's and $V^\dagger$'s are interleaved with λ's, and there are an even number of λ's. For the case of two generations, there is only one independent trace:

$$\text{tr}(V^\dagger\sigma^3V\sigma^3)=2(2\cos^2\theta_C-1) \quad , \tag{23}$$

where θ_C is the Cabibbo angle. The quartic trace can be expressed in terms of the quadratic one as follows:

$$\text{tr}(V^\dagger\sigma^3V\sigma^3V^\dagger\sigma^3V\sigma^3)=[\text{tr}(V^\dagger\sigma^3V\sigma^3)]^2-2 \quad . \tag{24}$$

For three generations, the four independent invariants can, in principle, be expressed in terms of

$$T^{ij}=\text{tr}(V^\dagger\lambda^iV\lambda^j),\quad i,j=3,8 \quad . \tag{25}$$

For N generations, the $(N-1)^2$ real T^{ij}'s of (25), with $i,j=3,8,\ldots,N^2-1$, are the independent rephasing-invariant parameters. However, in practice, using these quantities is not the best way to determine the *CP*-violating parameter. For that purpose, it would be better to use the quantity which occurs in ϵ. I conjecture that the $(N-1)(N-2)/2$ complex T^{iijj}'s,

$$T^{iijj}=T^{jjii}=\text{tr}(V^\dagger\lambda^iV\lambda^iV^\dagger\lambda^jV\lambda^j) \quad ,\quad i,j=3,8,\ldots,N^2-1 \quad , \tag{26}$$

with $i<j$, are related to the $(N-1)(N-2)/2$ invariant KM phases.

This suggests a program of parametrization of all data in terms of such traces. I plan to carry out this program in a later article. I also plan to analyze the notion of "maximal" *CP* violation using the insight gained from the rephasing-invariant formulation given here. Where an explicit parametrization of the KM matrix may be useful I suggest an SU(N) parametrization, rather than the customary form using products of SU(2) matrices.

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