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Kaluza-Klein mixmaster universes

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We analyze the behavior of some Bianchi type-IX, mixmaster cosmological models possessing extra dimensions. We find that, unlike the three-dimensional case, they do not exhibit chaotic behavior on approach to their initial singularity. A finite sequence of stochastic mixmaster oscillations is, in general, replaced by monotonic contraction of the three-space scale factors on approach to a singularity whenever additional spatial dimensions exist with the metric form we consider.

In this Rapid Communication we would like to report the results of an analysis of a Kaluza-Klein "mixmaster" universe. Recent interest in the behavior of cosmological models possessing extra dimensions has been confined to the simple cases in which the three-space expands isotropically.[1] This behavior is known to be unstable[2] in three-dimensional spaces and the most general dynamical behavior exhibited by spatially homogeneous cosmological models on approach to an initial singularity is formally chaotic.[3] Viewed as a dynamical system it possesses a nonzero metric entropy. It has been studied in particular detail by various authors[4] in the case of the Bianchi type-IX (mixmaster) universe. We consider the effect of one extra dimension with expansion scale factor $R(t)$, added to the Bianchi type-IX metric, so its form becomes

$$ds^2 = dt^2 - \sum_{i,j=1}^{3} \gamma_{ij}(t)\sigma^i(x)\sigma^j(x) - R^2(t)dx^4dx^4 \ , \tag{1}$$

where $\sigma^i(x)$ are the differential forms for the SO(3)-invariant Bianchi type-IX homogeneous space satisfying $d\sigma^i = \epsilon^i_{jk}\sigma^j \wedge \sigma^k$; ϵ^i_{jk} is the completely antisymmetric rank-3 tensor and

$$\gamma_{ij}(t) = \mathrm{diag}(a^2(t), b^2(t), c^2(t)) \ . \tag{2}$$

The vacuum Einstein equations for the metric (1) yield the field equations

$$\left(\frac{a'}{a}\right)' + \frac{R^2}{2}[a^4 - (b^2 - c^2)^2] = 0 \ , \tag{3}$$

$$\left(\frac{b'}{b}\right)' + \frac{R^2}{2}[b^4 - (c^2 - a^2)^2] = 0 \ , \tag{4}$$

$$\left(\frac{c'}{c}\right)' + \frac{R^2}{2}[c^4 - (a^2 - b^2)^2] = 0 \ , \tag{5}$$

$$\left(\frac{R'}{R}\right)' = 0 \ , \tag{6}$$

where a prime denotes $d/d\tau$ and $abcR d\tau = dt$. When the x^4 dimension scale factor is static (R = constant), the remaining equations are identical to those for the three-dimensional mixmaster universe[5] and describe an infinite sequence of chaotically unpredictable oscillations of the scale factors a,b,c as the singularity at $t=0$ (which corresponds to $\tau = -\infty$) is approached. In the four-dimensional case (6) has the exact solution $R(\tau) \propto e^{p_4\tau}$ and so the particular solution with static additional dimensions is unstable. If we define new variables $\alpha(\tau)$, $\beta(\tau)$, and $\gamma(\tau)$ by

$$(\alpha(t), \beta(t), \gamma(t)) = R^{1/2}(\tau)(a(\tau), b(\tau), c(\tau)) \ , \tag{7}$$

then the system (3)–(5) is transformed into a set of evolution equations that are identical to those satisfied by a,b,c in the three-dimensional mixmaster model (which arises when R is constant), that is,

$$\left(\frac{\alpha'}{\alpha}\right)' + \tfrac{1}{2}[\alpha^4 - (\beta^2 - \gamma^2)^2] = 0 \ , \tag{8}$$

and the two equations obtained from (8) by cyclically permuting α, β, and γ. Therefore, we can deduce that the solution of the system (8) will evolve by permuting α, β, and γ through a series of pseudo-Kasner cycles in which α, β, and γ are well approximated by

$$(\alpha(t), \beta(t), \gamma(t)) = (t^{s_1}, t^{s_2}, t^{s_3}) \ , \tag{9}$$

which arises when the terms enclosed by square brackets in (8) are negligible. However, the constants s_i will not satisfy the usual Kasner relations.[5] We will have $\sum_{i=1}^{3} s_i \neq 1 \neq \sum_{i=1}^{3} s_i^2$.

If we begin the evolution of the original system (3)–(6) towards the singularity from a four-dimensional Kasner state, then we have, approximately,

$$(a(t), b(t), c(t), R(t)) = (t^{p_1}, t^{p_2}, t^{p_3}, t^{p_4}) \ , \tag{10}$$

where

$$\sum_{i=1}^{4} p_i = \sum_{i=1}^{4} p_i^2 = 1 \quad . \tag{11}$$

To determine the algebraic relations satisfied by the pseudo-Kasner indices, s_i, we use (9) and (10) to obtain $s_i = p_i + p_4/2$, $i = 1, 2, 3$. Hence, we have that the pseudo-Kasner indices satisfy

$$\sum_{i=1}^{3} s_i = 1 + \frac{p_4}{2}; \quad \sum_{i=1}^{3} s_i^2 = 1 + p_4 - \frac{5p_4^2}{2} \quad . \tag{12}$$

From these relations we see that

$$s_1 s_2 + s_2 s_3 + s_1 s_3 = \frac{3p_4^2}{4} \geq 0 \quad . \tag{13}$$

When $p_4 = 0$, we reduce to the description of the three-dimensional mixmaster universe ($s_i = p_i$), and then the relation (13) shows that one of the s_i must be negative. This is necessary[3,4] to produce an infinite sequence of chaotic oscillations of $a(t)$, $b(t)$, and $c(t)$ as $t \to 0$. However, when $p_4 \neq 0$ it can be seen from (12) and (13) that it is possible for all the s_i to be positive depending on the value of $|p_4|$. When $p_4 \neq 0$ the stochastic permutation of the pseudo-Kasner indices may run through a series of values in which s_1, say, is always negative; but eventually the s_i will be permuted into the regime where all $s_i > 0$. When this occurs there will be no further oscillations and the scale factors α, β, γ (or a,b,c) will evolve monotonically and anisotropically ($s_1 \neq s_2 \neq s_3$ in general) as $t \to 0$ towards the singularity. Clearly, the number of stochastic oscillations necessary to reach this configuration from a typical initial condition will be greater for large values of $|p_4|$. A similar breakdown of the mixmaster cycles was found in the three-dimensional mixmaster model containing a massless scalar field by Belinskii and Khalatnikov.[6]

By this argument we can conclude that four-dimensional mixmaster universes of the form (1) will not exhibit chaotic behavior on approach to a space-time singularity. Viewed as a dynamical system they possess zero metric entropy[3] and are deterministic. These conclusions will hold for the Bianchi type-VIII model also. In fact, it is obvious that these conclusions will also hold for a more general metric than (1) in which the one-dimensional component $R^2(t)dx^4dx^4$ is replaced by

$$\mathrm{diag}[R_4^2(t)(dx^4)^2, \ldots, R_D^2(dx^D)^2] \quad .$$

The resulting Einstein equations admit a transformation analogous to (7) and the associated three-dimensional pseudo-Kasner indices admit a range of values with $s_i > 0$. The scale factors $R_4, \ldots, R_D$ evolve monotonically.[7]

Our result implies that the chaotic unpredictability of mixmaster oscillations on approach to a space-time singularity may be a unique property of worlds with three spatial dimensions. However, we should point out that metrics of the form (1) couple extra cosmological dimensions to the three-space evolution in a very simple fashion. It is quite possible that chaotic behavior will reappear when anisotropic curvatures[8] are admitted that couple the behavior in the extra dimensions to the mixmaster dynamics of the three-space (see, for example, the vector field example in Ref. 6). Finally, we note that Eqs. (3)–(6) also admit an exact axisymmetric solution

$$a(\tau) = e^{-p_4\tau/2}\,\mathrm{sech}^{1/2}(\tau - \tau_0), \quad R = e^{p_4\tau} \quad ,$$
$$b(\tau) = c(\tau) = e^{-p_4\tau/2}\cosh^{1/2}(\tau - \tau_0)\,\mathrm{sech}(\tau - \tau_1) \quad , \tag{14}$$

where τ_0, τ_1, and p_4 are constants. This solution together with the absence of mixmaster oscillations may provide some insight into the possible structure of a general Kaluza-Klein cosmological model which evolves away from the initial quantum state of the universe.[9]

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[4] This model was first found to exhibit stochastic properties by C. Misner, Phys. Rev. Lett. **22**, 1071 (1969); and they were studied in detail by V. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Usp. Fiz. Nauk **102**, 463 (1970) [Sov. Phys. Usp. **13**, 745 (1971)]. For a different viewpoint and further results on the full mixmaster dynamical system, see Ref. 3; J. D. Barrow, Phys. Rep. **85**, 1 (1982); D. Chernoff and J. D. Barrow, Phys. Rev. Lett. **50**, 134 (1983).

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[7] While this paper was being prepared there appeared a paper by T. Furusawa and A. Hosoya, Prog. Theor. Phys. **73**, 467 (1985), which independently analyzes a similar problem to that considered here. These authors employ a completely different technique of analysis compared to ourselves, and examine the mixmaster behavior employing the Hamiltonian "qualitative cosmology" of Misner (Ref. 4); and M. Ryan, *Hamiltonian Cosmology* (Springer, Heidelberg, 1972), but our conclusions regarding the absence of chaos in higher-dimensional mixmaster models are in agreement.

[8] We have recently analyzed the N-dimensional mixmaster universes which possess SO(N)-invariant dynamics [J. D. Barrow and J. Stein-Schabes, Astronomy Centre, University of Sussex report, 1985 (unpublished)]. Again we find that they are only chaotic when $N = 3$.

[9] It is worth pointing out that chaotic behavior, in the dynamical sense, only occurs in three-dimensional Bianchi type-VIII and -IX spaces when the evolution is followed into the singularity at $t = 0$. There are an infinite number of space-time oscillations on any interval $t \in (0,T]$ for $T > 0$. However, as first stressed by A. Doroshkevich and I. D. Novikov, Astron. Zh. **47**, 948 (1970) [Sov. Astron. **14**, 763 (1971)], if the evolution ceases to be of mixmaster type at the Planck time, $t_P \sim 10^{-43}$ sec, then only a small number of oscillations could have occurred in the interval of time from t_P to the present. Thus, the truncated mixmaster behavior found in the presence of extra dimensions may be as "chaotic" as a physically realistic three-dimensional model.