Angular momentum and wave functions in monopole and related potentials

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In a three-dimensional context, it is shown how wave functions may be chosen from among the solutions of the Schrödinger equation in a consistent way without demanding single valuedness, but by imposing an invariance condition on the domain of certain given linear operators. This leads to multiple-valued wave functions in certain situations, but disallows unquantized monopoles and "unusual" angular momentum eigenvalues.

This work is an application of Sturm-Liouville theory to the problem of solving the Schrödinger equation in a certain family of vector potentials. We will demonstrate a consistent precedure for selecting wave functions from solutions of the Schrödinger equation without imposing single valuedness. This leads to an angular momentum eigenvalue spectrum in which "anyonic" eigenvalues (that is, ones which are discrete but neither integral nor half-integral) do not occur. It also yields the Dirac quantization condition for monopoles. The wave functions are, in general, multiple valued, but in "familiar" situations such as that of an arbitrary central (scalar) potential they remain single valued.

The procedure for obtaining wave functions arises in the following way: We assume that a dynamical variable $\mathscr{L}$ defines a mapping of the space of physical states on itself —what is to be emphasized is that $\mathscr{L}$ is defined on the entire domain of physical states, and that its range space is contained within that domain. More picturesquely,

$$\mathscr{L}|\text{any state}\rangle = |\text{a state}\rangle \quad . \tag{1a}$$

Assuming that physically acceptable wave functions ψ are a subset of square-integrable functions, we must have

$$\mathscr{L}^n\psi \text{ square integrable} \quad . \tag{1b}$$

Given an algebraic operator $\mathscr{L}$ (corresponding to some dynamical variable), (1b) is taken as a necessary condition for the physical acceptability of any ψ.

Note that the operator $\mathscr{L}$ need not be bounded—$||\mathscr{L}^n\psi||$ can be arbitrarily large, but it must be finite.

For any three-dimensional angular momentum $\mathbf{M}$ it is condition (1a) (with $M_\pm \equiv M_x \pm iM_y$ for $\mathscr{L}$) which forces quantization into an integral or half-integral ladder.[1] We will show that, for the particular (algebraic) angular momentum operators we study, condition (1b) also decides whether the ladder is integral or half-integral. (Since an argument based on $M_\pm$ has meaning only in three dimensions, one cannot make from it any assertions about strictly two-dimensional systems—the essential distinction has been carefully considered by Goldin and Sharp.[2])

The question of angular momentum quantization (both in two[3,4] and in three dimensions[5,6]) is at present a very controversial one. At issue is whether physical significance is to be attached (in the light of the correspondence principle[4]) to the anyonic angular momentum eigenvalues (first given for two dimensions by Wilczek[3]) and which choice of angular momentum operator is the correct rotation generator (in two and in three dimensions). In this context, the results obtained by imposing (1b) are of some interest.

Our position is the following: The kinetic angular momentum operator, with a monopole g present,

$$\mathbf{L} = \mathbf{r}\times(-i\nabla - e\mathbf{A}) - \mu\hat{\mathbf{r}} \tag{2}$$

(where $\mu = eg$), is manifestly gauge invariant, and L^2 commutes with the Hamiltonian. [Neither of these is true for the canonical angular momentum operator $\mathbf{L}_{\text{can}} = \mathbf{r}\times(-i\nabla)$.] Furthermore, $\mathbf{L}$ is clearly analogous to the classical conserved vector $\mathbf{r}\times m\mathbf{v} - \mu\hat{\mathbf{r}}$. We therefore assert that it is meaningful to write the solution in terms of simultaneous eigenfunctions of H, L^2, and L_z. It does not, however, follow that $\mathbf{L}$ must be the generator of rotations; in fact (in agreement with Jackiw and Redlich[4]), the wave functions which we calculate are consistent only with the interpretation of $\mathbf{L}_{\text{can}}$ as rotation generator.

The vector potential involved is

$$A_r = 0, \quad A_\theta = 0, \quad eA_\phi = \frac{\chi - \mu\cos\theta}{r\sin\theta} \quad , \tag{3}$$

where μ is a constant pseudoscalar and χ is piecewise constant in θ. In particular, a Dirac monopole would be given by

$$A_\phi = \frac{-g[\cos\theta + \text{sgn}(\theta-\theta_0)]}{r\sin\theta} \quad . \tag{4}$$

$\theta_0 = 0$ or π represents a string monopole; other values represent the monopoles used by Wu and Yang.[7] The Aharonov-Bohm flux[8] (a δ-function flux tube along the z axis) corresponds to the case ($\mu = 0$, $\chi = \text{const}$).

An essentially equivalent potential has been studied by Roy and Singh,[6] but (as will be seen presently) their assumption of single valuedness leads to results very different from ours.

The operator $\mathbf{L}$ of (2) has

$$L_z = -i\frac{\partial}{\partial\phi} - \chi \quad , \tag{5a}$$

$$L_\pm = e^{\pm i\phi}\left[\pm\frac{\partial}{\partial\theta} - \cot\theta L_z - \frac{\mu}{\sin\theta}\right] \quad , \tag{5b}$$

which satisfy $[L_z, L_\pm] = \pm L_\pm$ and $[L_+, L_-] = 2L_z$, and

$$L^2 = \frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\right] + \frac{L_z{}^2 + \mu^2 + 2\mu L_z\cos\theta}{\sin^2\theta} \quad . \tag{6}$$

The Hamiltonian [with an arbitrary $V(r)$] is

$$H=\frac{1}{2m}\left[-\nabla^2+\frac{2ie}{r\sin\theta}A_\phi+e^2A_\phi{}^2\right]+V(r)$$

$$=\frac{1}{2m}\left[\frac{1}{r}P_r{}^2r+\frac{1}{r^2}(L^2-\mu^2)\right]+V(r)\ . \quad (7)$$

Clearly, H, L^2, and L_z all commute, and we look for simultaneous eigenfunctions ψ such that $L_z\psi=m\psi$, $L^2\psi=l(l+1)\psi$, $H\psi=E\psi$. In view of (1a) and (1b) we will assume that m is allowed eigenvalues from $-l$ to l in integer steps. (Here $\mathbf{L}$ is not necessarily the total angular momentum; but since it is quantized and conserved independently of any other angular momentum, our discussion is decoupled from considerations of the "return flux".[5])

ψ is now separable into $R(r)Y(\cos\theta)\exp[i(m+\chi)\phi]$. We will continue to look upon the wave function as an ordinary function, rather than as a section. If we do treat it as a section (as in Wu and Yang[7]), some of our statements about the wave function will have to be modified, but the difference is not important for the present discussion.

For the r and θ dependence

$$r\frac{d^2}{dr^2}(rR)+[2m(E-V)r^2+\mu^2-l(l+1)]R=0\ , \quad (8)$$

$$(1-x^2)Y''-2xY'+\left[l(l+1)-\frac{m^2+\mu^2+2m\mu x}{1-x^2}\right]Y=0\ , \quad (9)$$

with $x=\cos\theta$ in $(-1,1)$.

The θ part of $L_\pm$ is (with $D\equiv d/dx$)

$$(1-x^2)^{1/2}\left[\pm D+\frac{\mu+mx}{1-x^2}\right]\ . \quad (10)$$

Solutions for (9) can be written down conveniently using Rodrigues's formula. A general Rodrigues formula is

$$F_n=e^{-w/2}D^n(S^ne^w)\ , \quad (11)$$

with S quadratic or less, $(S'+w'S)$ linear, e^w positive and integrable in the (finite or infinite) interval $[a,b]$, and $e^w=0$ at a and b.

The F_n then satisfy a Sturm-Liouville eigenvalue equation

$$F''+S'F'-(\tfrac{1}{4}w'^2S+\lambda)F=0\ , \quad (12)$$

where

$$\lambda_n=\tfrac{1}{2}n(n+1)S''+(n+\tfrac{1}{2})(w''S+w's')\ . \quad (13)$$

The F_n are orthogonal and complete in $[a,b]$ and $e^{-w/2}F_n$ is a polynomial of degree n.[9]

Because we already have $(l+m)$ and $(l-m)$ = integers, all the $Y(l,m,\mu)$ may be written as

$$(1-x^2)^{\pm m/2}\left[\frac{1-x}{1+x}\right]^{\pm\mu/2}D^{l\pm m}\left[(1-x^2)^l\left[\frac{1+x}{1-x}\right]^{\pm\mu}\right]\ , \quad (14)$$

which is square integrable if and only if

$$(l\pm m)=\text{integer}>0;\ \ l\geqslant|\mu|\ , \quad (15a)$$

$$\pm(m+\mu)\text{ and }\pm(m-\mu)\text{ integral if }>0 \quad (15b)$$

[upper and lower signs in (15a) and (15b) relate to the ones in (14)].

[If (1a) and (1b) were not imposed, and m were anyonic with ψ single valued, two cases would arise—(i) for $|m|>|\mu|$ the upper and lower signs would yield two sets of solutions with the eigenvalues l being m dependent, and (ii) for $|m|<|\mu|$, m and μ would interchange roles in (14), (15a), and (15b) and l would be μ dependent. Note that μ need not be quantized; because of this, Roy and Singh have suggested the possibility of unquantized monopoles within flux tubes.[6]]

Now we are ready to impose (1b). It is easily verified that, with the upper sign, L_+ augments m by 1, and with the lower sign L_- diminishes m by 1. If $(m+\mu)$ or $(m-\mu)\neq$integers, then repeated application of L_+ or L_- will eventually yield an m which violates the conditions (15b), which in turn means that (1b) is violated. Thus, both $(m+\mu)$ and $(m-\mu)$ must be integers, so that l,m,μ must be *either all integral or all half-integral.*

To normalize the $Y(l,m,\mu)$ the necessary constant factor is

$$\frac{1}{2^l}\left[\frac{(l+\frac{1}{2})(l\mp m)!}{(l-\mu)!(l+\mu)!(l\pm m)!}\right]^{1/2}\ , \quad (16)$$

and it can be evaluated by first taking the $m=l$ case (which is simple) and then using induction with L_-.

These eigenfunctions Y (now identical for upper and lower signs) are exactly those obtained by Fierz[10] for the pure monopole $(\chi=\mu)$ case; he derived them from the physical requirement that the $L_\pm$ should act as raising and lowering operators.

The radial equation (8) is easily dealt with. At least for $V(r)$ going as $1/r$, r^2, or 0, one can obtain both energy eigenvalues and radial eigenfunctions from the $\mu=0$ case, simply by replacing l by $[(l+\frac{1}{2})^2-\mu^2]^{1/2}-\frac{1}{2}$. It is worth mentioning that since $l\geqslant|\mu|$, if $\mu\neq0$ there is no s-wave solution. It is known from very general considerations that there can be an s-wave solution only if $j=s$.[5]

The simultaneous quantization of l,m,μ (as either all integral or all half-integral) was first obtained by Fierz[10] (for the $\chi=\mu$ case) by imposing a rotational-invariance condition due to Pauli.[11] [This remarkable connection between the quantization of angular momentum and of Dirac monopoles has a curious semiclassical parallel: If the classical-field angular momentum is required to be quantized, then the Dirac quantization condition (2μ = integer) follows.[12]] The wave functions Fierz calculates (and with which the ones here agree), turn out to be single valued, although this is not put as a requirement. But rotational invariance is not applicable for more general χ, whereas (1b) can still be used; and in general the ψ calculated here are multiple valued. However, the quantization of l,m,μ remains the same. Incidentally, Dirac's original argument for the quantization condition[13] requires the single valuedness of the wave function; his later work includes a derivation from the correspondence principle.[14]

This brings us to the question of the observability of the Aharonov-Bohm effect. With our assumptions, there is no effect on the quantities with classical-mechanical analogs (kinetic angular momentum and energy). But the wave function itself *is* changed. An Aharonov-Bohm flux given by a constant χ [so that formally $\mathbf{A}=\nabla(\chi\phi)$] simply multiplies the wave function by $\exp(i\chi\phi)$, just as if it were an ordinary gauge transformation; this can produce a shift in interference fringes, for example, and hence be observable.

Before concluding, we mention that an interesting analogy exists between Dirac monopoles and quantized vortices in a superfluid.[15] So even if discussions of Dirac monopoles are of questionable value, the results may be applicable elsewhere.

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