PHYSICAL REVIEW D VOLUME 32, NUMBER 6 15 SEPTEMBER 1985

Boson subalgebras and classification of boson state vectors

M. Bolsterli
*T Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545**
(Received 29 April 1985)

In the construction and classification of the possible state vectors of a limited number of boson modes, the use of subalgebras of invariant operators can simplify the procedure. The subalgebra of all the invariant operators (invariant subalgebra) and the subalgebra generated by the invariant-pair operators (invariant-pair subalgebra) are both considered. The invariant-pair subalgebra has the decisive advantage of allowing the easy evaluation of matrix elements. The construction problem is reduced to the problem of constructing the invariant-pair-free states, and a general procedure for determining these states is presented.

I. INTRODUCTION

In the treatment of systems of interacting fermions and bosons, a "static" model of the system has often served as a useful guide to some properties of the full dynamical system. For example, the static model of the pion-nucleon system has the Hamiltonian

$$H=\int \omega(k)\sum_{1}^{3}a_{\lambda}^{\dagger}(\mathbf{k})a_{\lambda}(\mathbf{k})d\mathbf{k}$$
$$-g\int \sigma\cdot\mathbf{k}v(k)\sum_{1}^{3}\tau_{\lambda}[a_{\lambda}(\mathbf{k})+a_{\lambda}^{\dagger}(\mathbf{k})]d\mathbf{k} \tag{1.1}$$

that describes the interaction of the isospin-1 pion field whose annihilation operator is $a_{\lambda}(\mathbf{k})$ with a static source that represents the nucleon. The source has spin and isospin $\frac{1}{2}$, but does not recoil, that is to say, the static source has just the four states with spin and isospin projections $\pm\frac{1}{2}$, rather than the three-dimensional continuum of states for each spin-isospin projection that a dynamical source would have. Similar static models have been used to treat the gluon part of hadronic state vectors in quantum chromodynamics[1] and the interaction of two sources of meson field.[2] The strong-coupling polaron[3] and the interaction of a nonlinear scalar field with a static source[4] have also been treated in corresponding approximations; in both of these cases the interaction with the source is Abelian [no noncommuting operators like the σ and τ in (1.1)].

In general, the low-lying spectrum of static models involves only one or more modes of the boson field, that is, if the boson field is decomposed[5] into an "external" part $a_{\rm ext}(\mathbf{k})$ and an "internal" part $a_{\rm int}(\mathbf{k})$ that consists of one or two orthonormal modes,

$$a(\mathbf{k})=a_{\rm int}(\mathbf{k})+a_{\rm ext}(\mathbf{k}) ,$$
$$a_{\rm int}(\mathbf{k})=\sum_{i=1}^{1 \text{ or } 2}A_i\phi_i(\mathbf{k}) , \tag{1.2}$$

then the low-lying states lie mainly in the internal subspace of the full Hilbert space, the internal subspace being defined as the subspace that is generated by just the internal-mode creation operators $A_i^{\dagger}$ and the source operators like σ and τ acting on the bare source state. Within the internal subspace, the Hamiltonian is equivalent to an internal Hamiltonian. For example, in the static model of the nucleon, where a single p-wave pion mode is predominant, the internal Hamiltonian is

$$H_{\rm int}=\sum_{\lambda=1}^{3}\sum_{i=1}^{3}[WA_{\lambda,i}^{\dagger}A_{\lambda,i}-V\tau_{\lambda}\sigma_i(A_{\lambda,i}+A_{\lambda,i}^{\dagger})] . \tag{1.3}$$

This paper is devoted to the problem of constructing useful state vectors in the internal subspace. Of course, the specific state vectors depend on the particular system under consideration, but some general principles can be used to simplify and systematize the process.

The complications arise because each state vector should evidently be chosen to belong to a particular irreducible representation (irrep) of the invariance group (IG) of the Hamiltonian, since the Hamiltonian does not connect states belonging to different irreps of its invariance group. In the pion-nucleon static model of (1.3), referred to in the following as the PNSM, the IG is the group of rotations in space and rotations in isospin space, and a state vector that belongs to an irrep is a state vector that is an eigenvector of the total angular momentum of the source and pion field and of the total isospin of the source and pion field. An irrep I contains $d(I)$ states that transform among themselves under the action of the operators of the IG; it is useful to consider the $d(I)$ states in the irrep I as a single "irvec" v^I. An irvec belonging to the I irrep will also be called an I state. In the PNSM, an irvec contains the $(2J+1)(2T+1)$ substates belonging to its total angular momentum J and total isospin T. The IG as an abstract group determines the rules for coupling irvecs to form states that belong to a definite irrep. When irvecs v^I and w^J are coupled to form the states belonging to the irrep K, the resulting irvec will be written $\{v^I,w^J\}^K$. In the PNSM, the irvec coupling coefficients are products of two Clebsch-Gordan coefficients, one for angular momentum and one for isospin.

The essential feature of the PNSM that allows further progress is the fact that both the angular momentum and the isospin are each the sum of a source part and a pion part. Formally, in the static models enumerated above, each generator of the IG is the sum of a source term and a boson term. In such cases, it follows directly that a general I state of a system of bosons interacting with a static source can be (but need not be) written as a sum of terms, each of which belongs to definite irrep of the source part of the IG and to a definite irrep of the boson part of the IG, that is, each term is of the form $\{\Phi^J,\Sigma^K\}^I$, where Σ^K is a K irvec of the source and Φ^J is a J irvec involving only boson creation operators $A^\dagger$. While the Hamiltonian is diagonal in the irrep index I, it is, of course, not necessarily diagonal in either J or K. In the PNSM, there is only one irrep for the source, so that the Hamiltonian is in this case trivially diagonal in the source irrep; it is not diagonal in the pion irrep J. It is assumed that the states Σ^K of the source are known. If boson operators belonging to different modes of the Bose field (that is, to distinct irvecs) are present, the states Φ^J can be further reduced to terms that involve the coupling of states, each of which involves $A^\dagger$ operators for a single mode of the field. Hence, the problem of cataloging the states of the system reduces to the problem of cataloging the states Φ^J of a single mode of the Bose field.

Besides just a list of the states, any computation involving the boson states will require some matrix elements. One set of required matrix elements is the matrix of the unit operator, or, equivalently, an orthonormal choice of the boson states. The second set of required elements is the matrix of the operator $A^\dagger$. From these two sets, other needed matrix elements can be derived. This cataloging problem is the analog for Bose systems of the familiar problem of classifying the states of N fermions in a single shell. In the fermion case, the number of particles, that is, the number of particle creation operators, is fixed, so that only states with a fixed number N of fermions are required.

II. SIMPLEST EXAMPLES

The creation operator for neutral scalar s-state bosons is invariant under the IG of the Hamiltonian; it belongs to the 0 irrep of dimension $d(0)=1$. (In general, an invariant operator is one that belongs to the 0 irrep of the IG.) The normalization of the creation operator $A^\dagger$ is specified by

$$[A,A^\dagger]=1 \,. \tag{2.1}$$

Then in this well-known case, the orthonormal states are given by

$$|\,n\,\rangle=\frac{1}{\sqrt{n!}}(A^\dagger)^n\Omega \,, \tag{2.2}$$

where n is any non-negative integer and Ω will always be used to denote the boson vacuum,

$$A\Omega=0 \,. \tag{2.3}$$

The only nonzero matrix elements of the operator $A^\dagger$ are

$$\langle\, n+1\,|\,A^\dagger\,|\,n\,\rangle=n^{1/2} \,. \tag{2.4}$$

The case of a vector mode is nearly as simple. In this case the operator $A^\dagger$ is a three-vector, corresponding to an isovector scalar s-state boson or a neutral scalar p-state boson. It belongs to the 1 irrep with dimension $d(1)=3$ of the IG, so it is convenient to use the notation

$$A^\dagger=C^1 \,, \tag{2.5}$$

with normalization specified so that

$$[A_i,A_j^\dagger]=\delta_{i,j} \,. \tag{2.6}$$

Call the quantum number J-spin, which can be angular momentum or isospin. The states of this mode can be classified by total J-spin, which takes on all non-negative integer values. The dimension of the J irrep is $d(J)=2J+1$. For fixed value K of the J-spin there is a minimal irvec $|\,K,0\,\rangle$ with K creation operators C^1; the normalized irvec $|\,K,0\,\rangle$ can be defined recursively by

$$\begin{aligned} &|\,0,0\,\rangle=\Omega \,, \\ &|\,K,0\,\rangle=\frac{1}{\sqrt{K}}\{C^1,|\,K-1,0\,\rangle^{K-1}\}^K \,. \end{aligned} \tag{2.7}$$

Every other K irvec is obtained by applying the invariant-pair creation operator $P^\dagger$,

$$\begin{aligned} &P^\dagger\equiv\tfrac{1}{2}C\cdot C=\tfrac{1}{2}\sum_1^3 C_i^1C_i^1 \,, \\ &[P,P^\dagger]=\tfrac{1}{4}[A\cdot A,A^\dagger\cdot A^\dagger]=A^\dagger\cdot A+\tfrac{3}{2}=\hat{N}+\tfrac{3}{2} \,, \end{aligned} \tag{2.8}$$

to the state $|\,K,0\,\rangle$. Since the state $|\,K,0\,\rangle$ has each of its pairs maximally coupled to J-spin 2, it follows that

$$P\,|\,K,0\,\rangle=0 \,. \tag{2.9}$$

With the commutation relation just given, it is easy to see that the complete orthonormal set of irvecs of the three-vector mode is given by

$$|\,K,q\,\rangle=\left[\frac{(K+\frac{1}{2})!}{q!(K+\frac{1}{2}+q)!}\right]^{1/2}(P^\dagger)^q\,|\,K,0\,\rangle \,, \tag{2.10}$$

where K and q both run over the non-negative integers. The state $|\,K,q\,\rangle$ has J-spin K and is an eigenstate of the boson number operator $\hat{N}=A^\dagger\cdot A$ with eigenvalue $K+2q$. The matrix elements of the operator $A^\dagger$ in this basis will be evaluated below in a more general context.

III. WHY SUBALGEBRAS?

So far, everything has been simple, straightforward, and well known. The first interesting case is that of p-wave pions, whose creation operator belongs to the (1,1) irrep of the IG; that is, the creation operator $A^\dagger=C^{11}$ is a three-vector in isospin space and a p-wave three-vector in coordinate space, with

$$[A_{\lambda i},A_{\mu j}^\dagger]=\delta_{\lambda,\mu}\delta_{i,j} \,. \tag{3.1}$$

Now the states are eigenstates of the isospin T and the angular momentum L (TL irvecs). For classification purposes, they will also be chosen to be eigenstates of the IG-invariant boson number operator $\hat{N}$,

$$\hat{N}=A^{\dagger}\cdot A=\sum_{\lambda=1}^{3}\sum_{i=1}^{3}A_{\lambda i}^{\dagger}A_{\lambda i}\ , \tag{3.2}$$

so that a state will be written $|N,T,L,\alpha\rangle$ where α represents other indices necessary to completely characterize the state. The state $|N,T,L,\alpha\rangle$ belongs to the TL irrep of the IG. The orthonormal states with $N=0$, $N=1$, and $N=2$ are simple enough; the full list is

$$|0,0,0\rangle=\Omega\ ,$$
$$|1,1,1\rangle=C^{11}\Omega\ , \tag{3.3}$$
$$|2,T,L\rangle=\frac{1}{\sqrt{2}}\{C^{11},C^{11}\}^{TL}\Omega\ ,\quad TL=00,11,02,20,22\ ,$$

where the restriction on the TL values or $N=2$ follows from the fact that the two identical C^{11} operators must be coupled symmetrically. Note that while the state $|2,0,0\rangle$ is related to the state $|0,0,0\rangle$ in the same way that $P^{\dagger}\Omega$ is related to Ω in the simple isovector case, the state $|2,1,1\rangle$ is not simply related to the state $|1,1,1\rangle$.

The key properties here are certain features of the *algebra* generated by the operators $A^{\dagger}$ and A. Some aspects of this algebra are treated elegantly in a paper by Friedman, Lee, and Christian,[6] referred to as FLC in the following. Let an invariant operator constructed from $A^{\dagger}$ operators alone (without A operators) be called an IC-operator (invariant C-operator). FLC shows that in the (1,1) case there are three independent IC-operators; let a set of three independent IC-operators be $[R_2,R_3,R_4]$, where R_i is of degree i in C^{11} (that these degrees are appropriate follows from FLC). It follows that a general 0 state (or 00 state) can be written as $f(R_2,R_3,R_4)\Omega$, where f is a polynomial in its variables. It can also be shown that a general 11 state (FLC showed that there are no 01 states or 10 states) can be written in the form

$$\sum_{N=1}^{3}f_N(R_2,R_3,R_4)\,|N,1,1\rangle\ ,$$

where the states $|N,1,1\rangle$ for $N=1$ and $N=2$ are given in (3.3), the state $|3,1,1\rangle$ is an appropriately chosen 11 state with three creation operators, and the f_N are again polynomials. The states $|N,1,1\rangle$ in this resolution satisfy the conditions

$$R_j^{\dagger}\,|N,1,1\rangle=0\ ,\quad N=1,2,3,\ j=2,3,4 \tag{3.4}$$

so that they are "invariant free." These resolutions lead to the idea that a general TL state can be written in terms of a finite number of invariant-free TL states $|N_a,T,L,a\rangle$ in the form

$$\sum_a f_a(R_2,R_3,R_4)\,|N_a,T,L,a\rangle\ , \tag{3.5}$$

where the f_α are polynomials in the R_i. The number of independent invariant-free states is one and three for the cases 00 and 11, respectively. The problem of listing all the states can thus be reduced to the problem of listing just the invariant-free states.

In more general terms, some of the operators in the algebra generated by the $A^{\dagger}$ and A operators belonging to a single irrep are invariant operators; the algebra of the invariant operators is evidently a subalgebra, the "invariant subalgebra," of the full algebra, and the classifications of the type of (3.4) and (3.5) are based on the use of those elements of this invariant subalgebra that are composed of $A^{\dagger}$ operators alone, together with finite sets of invariant-free irvecs.

In the case of a single vector mode, the invariant subalgebra is generated by the operators P and $P^{\dagger}$ of (2.8). In the present (1,1) case, an invariant-pair creation operator can be defined again by

$$P^{\dagger}\simeq\tfrac{1}{2}C\cdot C=\tfrac{1}{2}\sum_{\lambda=1}^{3}\sum_{i=1}^{3}C_{\lambda i}^{11}C_{\lambda i}^{11}\ ,$$
$$[P,P^{\dagger}]=\tfrac{1}{4}[A\cdot A,A^{\dagger}\cdot A^{\dagger}]=A^{\dagger}\cdot A+\tfrac{9}{2}=\hat{N}+\tfrac{9}{2}\ , \tag{3.6}$$

but in this (1,1) case, the algebra generated by the operators P and $P^{\dagger}$ is distinct from the invariant subalgebra; clearly, it is a subalgebra of the invariant subalgebra. Call the subalgebra generated by P and $P^{\dagger}$ the "invariant-pair" subalgebra or IP subalgebra. The simplicity of the vector or 1 case is due to the identity of the invariant subalgebra and the IP subalgebra.

IV. CLASSIFICATION ACCORDING TO THE INVARIANT-PAIR SUBALGEBRA

For any irvec of creation operators, there is a corresponding IP subalgebra. Let the irrep of the IG to which the irvec of the creation operator $A^{\dagger}$ belongs to denoted F (for fundamental), and, for ease of notation, write $A^{\dagger}=C^F$; the irvec C^F consists of $\nu=d(F)$ operators. Choose the operators A_α so that

$$[A_\alpha,A_\beta^{\dagger}]=\delta_{\alpha,\beta}\ . \tag{4.1}$$

The IP subalgebra is generated by the operators P and $P^{\dagger}$, where

$$P^{\dagger}=\tfrac{1}{2}C\cdot C=\tfrac{1}{2}\sum_{1}^{\nu}C_\alpha^F C_\alpha^F\ ,$$
$$[P,P^{\dagger}]=\tfrac{1}{4}[A\cdot A,A^{\dagger}\cdot A^{\dagger}]=A^{\dagger}\cdot A+\frac{\nu}{2}=\hat{N}+\frac{\nu}{2}\ , \tag{4.2}$$

and where the boson number operator is now

$$\hat{N}=A^{\dagger}\cdot A=\sum_{1}^{\nu}A_\alpha^{\dagger}A_\alpha\ . \tag{4.3}$$

Let an IPF irvec (IPF for invariant-pair-free) with N mesons be defined to be an irvec $|N,a\rangle$ that has no invariant pairs and therefore satisfies

$$P\,|N,a\rangle=0\ ,$$
$$\hat{N}\,|N,a\rangle=N\,|N,a\rangle\ . \tag{4.4}$$

Then it follows that, just as in the 1 case, the set of irvecs formed by adding invariant pairs to a given normalized IPF state $|N,a\rangle$,

$$|N,a,q\rangle=\left[\frac{\left[N+\frac{\nu}{2}-1\right]!}{q!\left[N+\frac{\nu}{2}-1+q\right]!}\right]^{1/2}(P^\dagger)^q|N,a\rangle\ ,\qquad(4.5)$$

forms an orthonormal "ladder" based on the IPF state $|N,a\rangle$. Moreover, it is very easy to see that

$$\langle M,a,p\,|N,b,q\rangle=\delta_{p,q}\delta_{M,N}\langle M,a\,|N,b\rangle\ ,\qquad(4.6)$$

so that a complete orthonormal set of states can be obtained from a complete orthonormal set of IPF states; the listing problem has been reduced to the problem of listing just the IPF states. In the matrix element of (4.6), as in all matrix elements considered in this work, it is implicit that the same irvec substate [any one of the $d(M,a)=d(N,b)$ substates] is used on both sides of the matrix element; the IG ensures that matrix elements between substates with differing "m values" vanish, and therefore only matrix elements between substates with identical m values are ever considered. This convention avoids an extra trivial set of Kronecker deltas throughout.

Of course, there are in general many more IPF states than there are invariant-free states. On the other hand, the IPF states have the overwhelming advantage that construction of the pair ladders (4.5) is straightforward and the simple relation (4.6) holds for them; *no* similar simple construction exists for the states in the ladders based on invariant-free states, nor is there any simple way of deriving orthonormality relations. The main general result of all the preceding discussion is, therefore, that while the set of ladders based on invariant-free states has the possibility of shortening the classification process, the ladders based on the invariant-pair-free states are to be preferred because they allow matrix elements to be evaluated easily.

As the main necessary example, consider the matrix elements of the creation operator $A^\dagger$ or C^F. Let $|M,K\rangle$ and $|N,L\rangle$ be IPF states belonging to irreps K and L, respectively, of the IG, and $|M,K,p\rangle$ and $|N,L,q\rangle$ be formed from $|M,K\rangle$ and $|N,L\rangle$ by the process of (4.5). From the commutation relation of P and $A^\dagger$,

$$[P,A^\dagger]=A\ ,\qquad(4.7)$$

it follows that

$$\langle M,K\,|P^pA^\dagger|N,L\rangle=\begin{cases}0,\ p>1\\ \langle M,K\,|A\,|N,L\rangle\delta_{M,N-1}\ ,\ p=1\\ \langle M,K\,|A^\dagger|N,L\rangle\delta_{M,N+1}\ ,\ p=0\ .\end{cases}\qquad(4.8)$$

It is also straightforward to see that

$$P^qP^{\dagger p}|M,K\rangle=\frac{p!\left[M+\frac{\nu}{2}-1+p\right]!}{(p-q)!\left[M+\frac{\nu}{2}-1+p-q\right]!}(P^\dagger)^{p-q}|M,K\rangle\ ,\ p\ge q$$
$$=0\ ,\ p<q\ .\qquad(4.9)$$

From these it follows easily that the only nonzero matrix elements of the operator $A^\dagger$ are

$$\langle M,K,p\,|A^\dagger|M-1,L,p\rangle=\left[\frac{M+\frac{\nu}{2}-1+p}{M+\frac{\nu}{2}-1}\right]^{1/2}\langle M,K\,|A^\dagger|M-1,L\rangle\ ,$$
$$\langle M,K,p\,|A^\dagger|M+1,L,p-1\rangle=\left[\frac{p}{M+\frac{\nu}{2}}\right]^{1/2}\langle M,K\,|A\,|M+1,L\rangle\ ,\qquad(4.10)$$

so that all matrix elements of $A^\dagger$ are simply related to the IPF-state matrix elements of $A^\dagger$. The ease of evaluation of this matrix element demonstrates the usefulness of the IP subalgebra.

There is also a group-theoretic language[7] that describes the processes that give Eqs. (4.5) and (4.10); the group is SU(1,1). I feel that the use of SU(1,1) complicates the description of the action of the invariant-pair operators unnecessarily; however, this is a matter of taste to be left to the reader.

Note that the IG determines the coefficients $U_j(K,L)$ in the relation

$$\langle K \mid \{T^J, \mid L\rangle\}^K = U_J(K,L)(\langle L \mid \{T^{\dagger\tilde{J}}, \mid K\rangle\}^L)^* , \quad (4.11)$$

where $\tilde{J}$ is the adjoint irrep to J and $T^{\dagger\tilde{J}}$ is the adjoint of T^J. When irreps J and L can combine to give irrep K (and therefore $\tilde{J}$ and K can combine to form L), the $U_J(K,L)$ satisfy

$$U_J(K,L)U_J^*(L,K)=1 . \quad (4.12)$$

It follows from (4.11) that the matrix elements of the operator A can also be derived from (4.10).

For the irreps available for bosons (but not for fermion irreps), it is always the case that $\tilde{J}=J$; in the following, $\tilde{J}$ and J will not always be distinguished. The identity of $\tilde{J}$ and J is responsible for the invariance of $C^F \cdot C^F$ in the boson case, since the invariant-operator combination is always really $T^{\tilde{J}} \cdot V^J$.

In the case of a single vector mode (1 case), the value of $U_J(K,L)$ is

$$U_J(K,L)=s_{JKL}\left[\frac{d(L)}{d(K)}\right]^{1/2} , \quad (4.13)$$

where J, K, and L are J-spins, and

$$s_{JKL}=(-1)^{J+K-L} . \quad (4.14)$$

In the (1-1) case, the symbol J in $U_J(K,L)$ stands for two spins (isospin and spin), and $U_J(K,L)$ is the product of two factors of the form of (4.13).

V. PROCEDURE FOR DETERMINING THE IPF STATES

In the general F case, the orthonormal IPF states for $N=0$ and $N=1$ are

$$|0,0\rangle=\Omega , \qquad |1,F\rangle=C^F\Omega , \quad (5.1)$$

and the matrix elements of the creation operator that involve these states are

$$\langle 1,F \mid \{C^F, \mid 0,0\rangle\}^F=1 . \quad (5.2)$$

Now suppose that all the IPF L states $|N,L,\alpha\rangle$ and the matrix elements of the creation operator,

$$D^{N-1}_{L\beta,K\alpha}=\langle N,K,\alpha \mid \{C^F, \mid N-1,L,\beta\rangle\}^K , \quad (5.3)$$

are known for $N=1,2,\ldots,M$; the following procedure determines the IPF K states for $N=M+1$ and the M to $M+1$ matrix elements of the creation operator.

Consider the states $\{C^F, |M,L,\alpha\rangle\}^K$. Clearly these K states have $M+1$ bosons, but they are not IPF states. In fact,

$$P\{C^F, \mid M,L,\alpha\rangle\}^K=\{A, \mid M,L,\alpha\rangle\}^K . \quad (5.4)$$

The right-hand side of (5.4) is obviously an IPF state with $M-1$ bosons and can therefore be written as a superposition of such states

$$\{A, \mid M,L,\alpha\rangle\}^K=\sum_\beta X^M_{L\alpha,K\beta} \mid M-1,K,\beta\rangle , \quad (5.5)$$

where the coefficients $X^M_{L\alpha,K\beta}$ are

$$X^M_{L\alpha,K\beta}=\langle M-1,K,\beta \mid \{A, \mid M,L,\alpha\rangle\}^K = U_F(K,L)(D^{M-1}_{K\beta,L\alpha})^* , \quad (5.6)$$

and the recoupling coefficient $U_F(L,K)$ is given in (4.11). The second line of (5.6) shows that the X coefficients depend only on the known matrix elements. Since

$$PP^\dagger \mid M-1,K,\beta\rangle=\left[M+\frac{\nu}{2}-1\right] \mid M-1,K,\beta\rangle , \quad (5.7)$$

it follows that for fixed irrep K the states

$$|M+1,K,(L,\alpha)\rangle=\{C^F, \mid M,L,\alpha\rangle\}^K - \frac{P^\dagger}{M+\frac{\nu}{2}-1}\sum_\beta X^M_{L\alpha,K\beta} \mid M-1,K,\beta\rangle \quad (5.8)$$

are $(M+1)$-boson IPF states.

All that remains is to choose an orthonormal basis that spans the space generated by the states $|M+1,K,(L,\alpha)\rangle$; the states $|M+1,K,(L,\alpha)\rangle$ can then be expanded in terms of these basis states, which are the states $|M+1,K,\beta\rangle$:

$$|M+1,K,(L,\alpha)\rangle=\sum_\beta D^M_{L\alpha,K\beta} \mid M+1,K,\beta\rangle , \quad (5.9)$$

where the appearance of D here follows from the relation

$$\langle M+1,K,\beta \mid M+1,K(L,\alpha)\rangle = \langle M+1,K,\beta \mid \{C^F, \mid M,L,\alpha\rangle\}^K . \quad (5.10)$$

The two ways of evaluating

$$\langle M+1,K,(L,\alpha) \mid M+1,K,(J,\gamma)\rangle$$

give the equation to be satisfied by the D parentage coefficients:

$$\sum_\beta D^{M*}_{L\alpha,K\beta}D^M_{J\gamma,K\beta}=Q^{MK}_{L\alpha,J\gamma} = \{C^F, \mid ML\alpha\rangle\}^{K\dagger}\{C^F, \mid MJ\gamma\rangle\}^K - \frac{U_F^*(L,K)U_F(J,K)}{M+\frac{\nu}{2}-1}\sum_\beta D^{M-1*}_{K\beta,L\alpha}D^{M-1}_{K\beta,J\gamma} . \quad (5.11)$$

TABLE I. Independent IPF states.

	0	1	2	3	4	5	6
0	1 001 101		10 111	0001	101	00	1
1		111 122	01 111	1122	011	11	0
2			11 224	0122	113	01	1
3				1123	012	11	0
4					112	01	1
5						11	0
6							1

TABLE II. $1 \rightarrow 2$ parentages.

	11	22	02
11	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$

TABLE III. $2 \rightarrow 3$ parentages with $[a] \equiv a^{1/2}$.

	00	11	22	33	12	13
11	$[3]$	$5[\frac{1}{22}]$	$[\frac{3}{2}]$	0	$[\frac{3}{2}]$	0
02	0	$*[\frac{10}{11}]$	0	0	$[\frac{2}{3}]$	$[\frac{5}{3}]$
22	0	$[\frac{1}{22}]$	$[\frac{3}{2}]$	$[3]$	$[\frac{5}{6}]$	$2[\frac{1}{3}]$

TABLE IV. $3 \rightarrow 4$ parentages. (a) $T_4 = L_4$, (b) $T_4 < L_4$ with $[a] \equiv a^{1/2}$.

(a)

	00	11	22	22	33	44
00	0	$5[\frac{2}{39}]$	0	0	0	0
11	2	$\frac{1}{2}[\frac{11}{13}]$	$\frac{1}{5}[11]$	$*\frac{11}{10}[\frac{11}{13}]$	0	0
12	0	$\frac{3}{2}[\frac{5}{13}]$	1	$\frac{3}{2}[\frac{1}{13}]$	0	0
22	0	$\frac{11}{2}[\frac{1}{39}]$	$*[\frac{1}{3}]$	$*\frac{17}{2}[\frac{1}{39}]$	$2[\frac{2}{3}]$	0
13	0	0	$*\frac{1}{5}[14]$	$\frac{3}{5}[\frac{14}{13}]$	0	0
33	0	0	$\frac{2}{5}[\frac{2}{3}]$	$*\frac{1}{5}[\frac{2}{39}]$	$2[\frac{1}{3}]$	2

(b)

	02	12	13	23	04	24
11	$*[\frac{77}{65}]$	$\frac{1}{2}[\frac{33}{5}]$	0	0	0	0
12	$[\frac{35}{13}]$	$\frac{1}{2}[\frac{5}{3}]$	$*[\frac{5}{3}]$	$[2]$	0	0
21	0	$*\frac{1}{2}[3]$	0	0	0	0
22	0	$*\frac{1}{2}$	1	$[\frac{2}{3}]$	0	0
13	$2[\frac{2}{65}]$	$[\frac{14}{15}]$	$2[\frac{1}{3}]$	$[\frac{2}{5}]$	2	$[\frac{14}{5}]$
33	0	0	0	$*[\frac{14}{15}]$	0	$[\frac{6}{5}]$

By using the commutation relation of the operators C^F and A, as well as the adjoint relation (4.11), the matrix $Q^{MK}_{L\alpha,J\gamma}$ can be evaluated in terms of the known matrix elements. The end of this section gives a few details.

With the usual normalization conventions, it is always true that

$$\langle K,\alpha \,|\, U^F \cdot V^{\tilde F} \,|\, K,\beta\rangle = \sum_{J\gamma} \langle K,\alpha \,|\, \{U^F, |J,\gamma\rangle\}^K \times (\langle K,\beta \,|\, \{V^{\dagger F}, |J,\gamma\rangle\}^K)^* \,. \tag{5.12}$$

Then the expectation value of $\hat N$ gives an orthonormality relation:

$$\sum_{L,\alpha} D^M_{L\alpha,K\beta} D^{M*}_{L\alpha,K\gamma} = (M+1)\delta_{\beta,\gamma} \,. \tag{5.13}$$

Thus, the trace of (5.11) gives

$$\sum_{\beta}(M+1) = \sum_{L\alpha} Q^{MK}_{L\alpha,L\alpha} \,, \tag{5.14}$$

and therefore the trace of Q divided by $M+1$ gives the number of independent basis states $|M+1,K,\beta\rangle$.

The evaluation of

$$\{C^F, |ML\alpha\rangle\}^{K\dagger}\{C^F, |MJ\gamma\rangle\}^K$$

proceeds by first using (4.11) to obtain

$$\{C^F, |ML\alpha\rangle\}^{K\dagger}\{C^F, |MJ\gamma\rangle\}^K = U_F(K,J)(\langle MJ\gamma \,|\, \{A^{\tilde F}, \{C^F, |ML\alpha\rangle\}^K\}^J)^* \,; \tag{5.15}$$

then

$$\{A^{\tilde F}, \{C^F, |ML\alpha\rangle\}^K\}^J = \sum_I U^J_{I,K}(\tilde F,F,L)\{\{A^{\tilde F}, C^F\}^I, |ML\alpha\rangle\}^J \,, \tag{5.16}$$

where the six-argument U coefficient is a recoupling coefficient determined by the IG. In the 1 case, it is just the usual $6J$ recoupling coefficient $U(\tilde F,F,J,L;I,K)$, and in the (1-1) case it is the product of two such coefficients. Now the relation

$$\{A^{\tilde F}, C^F\}^I = d^{1/2}(F)\delta_{I,0} + s^*_{\tilde F FI}\{C^F, A^{\tilde F}\}^I \,, \tag{5.17}$$

which is valid in the 1 and 11 cases, is assumed; the following expressions are valid whenever the relation (5.17) holds. When (5.17) is substituted into (5.15), the first term gives a contribution to (5.15) that a little thought shows must be just an overall delta function; hence

$$U^J_{0,K}(\tilde F,F,L) = \frac{\delta_{L,J}}{U_F(K,J)d(F)^{1/2}} \,, \tag{5.18}$$

and

$$\{C^F, |ML\alpha\rangle\}^{K\dagger}\{C^F, |MJ\gamma\rangle\}^K = \delta_{J,L}\delta_{\gamma,\alpha} + \sum_H W(J,L;K,H)\{A^{\tilde F}, |ML\alpha\rangle\}^{H\dagger}\{A^{\tilde F}, |MJ\gamma\rangle\}^H \,,$$
$$W(J,L;K,H) = U_F(K,J)U^*_F(J,H)\sum_I s^*_{\tilde F FI} U^J_{I,K}(\tilde F,F,L)U^J_{I,H}(\tilde F,F,L) \,. \tag{5.19}$$

TABLE V. 4→5 parentages. (a) $T_5 = L_5$, (b) $T_5 < L_5$ with $[a] \equiv a^{1/2}$.

	(a)							
	11	11	22	22	33	33	44	55
00	1	$\frac{2}{3}[\frac{13}{15}]$	0	0	0	0	0	0
11	$[\frac{13}{11}]$	$*17[\frac{1}{165}]$	$\frac{1}{5}[13]$	$\frac{3}{5}[\frac{26}{5}]$	0	0	0	0
02	0	$*\frac{1}{3}[\frac{77}{15}]$	0	0	0	0	0	0
12	$3[\frac{1}{11}]$	$2[\frac{13}{165}]$	$2[\frac{1}{5}]$	$*2[\frac{1}{5}]$	0	0	0	0
22	0	$\frac{1}{3}[\frac{143}{15}]$	$\frac{1}{3}$	$\frac{2}{3}[\frac{2}{5}]$	$\frac{31}{21}$	$\frac{2}{7}[\frac{2}{15}]$	0	0
22	$*[\frac{13}{11}]$	$\frac{7}{3}[\frac{1}{165}]$	$\frac{1}{3}[13]$	$*\frac{1}{3}[\frac{26}{5}]$	$*\frac{2}{21}[13]$	$\frac{8}{7}[\frac{26}{15}]$	0	0
13	0	0	$\frac{1}{5}[7]$	$*\frac{2}{5}[\frac{14}{5}]$	0	0	0	0
23	0	0	$\frac{1}{3}[\frac{14}{5}]$	$\frac{4}{15}[7]$	$\frac{2}{3}$	$[\frac{2}{15}]$	0	0
33	0	0	$\frac{4}{15}[2]$	$\frac{19}{15}[\frac{1}{5}]$	$*\frac{1}{3}[2]$	$*\frac{11}{2}[\frac{1}{15}]$	$\frac{1}{2}[15]$	0
24	0	0	0	0	$*\frac{6}{7}$	$\frac{3}{7}[\frac{6}{5}]$	0	0
44	0	0	0	0	$\frac{1}{7}[6]$	$\frac{1}{14}[\frac{1}{5}]$	$\frac{1}{2}[5]$	$[5]$

TABLE V. (*Continued*).

(b)

	02	12	13	13	23	23	14	24	34	15	35
11	$[\frac{13}{5}]$	0	0	0	0	0	0	0	0	0	0
02	0	$\frac{1}{3}[13]$	$\frac{2}{3}[\frac{13}{7}]$	$[\frac{13}{15}]$	0	0	0	0	0	0	0
12	$*1$	$\frac{1}{3}[7]$	$*\frac{4}{3}$	$[\frac{7}{15}]$	$*\frac{1}{3}[14]$	$\frac{2}{3}$	0	0	0	0	0
21	0	$*[\frac{7}{5}]$	0	0	0	0	0	0	0	0	0
22	0	$\frac{1}{3}[7]$	$\frac{2}{3}$	$*8[\frac{1}{105}]$	$*4[\frac{2}{35}]$	$*2[\frac{1}{5}]$	0	0	0	0	0
22	0	0	0	$[\frac{39}{35}]$	$\frac{1}{3}[\frac{26}{35}]$	$*\frac{2}{3}[\frac{13}{5}]$	0	0	0	0	0
13	$[\frac{7}{5}]$	$\frac{1}{3}[5]$	$*\frac{1}{3}[\frac{1}{2}]$	$*[\frac{14}{15}]$	$\frac{4}{15}[7]$	$\frac{19}{15}[\frac{1}{2}]$	$[\frac{5}{2}]$	$[\frac{21}{10}]$	0	0	0
23	0	$*\frac{1}{3}[\frac{2}{5}]$	$\frac{7}{6}$	$[\frac{7}{15}]$	0	$\frac{1}{2}[5]$	$\frac{1}{2}[5]$	$*\frac{1}{2}[\frac{7}{3}]$	$[\frac{10}{3}]$	0	0
32	0	0	0	0	1	0	0	0	0	0	0
33	0	0	0	0	$\frac{2}{15}[2]$	$*\frac{4}{15}[7]$	0	$4[\frac{1}{15}]$	$\frac{1}{2}[\frac{5}{3}]$	0	0
04	0	0	$*[\frac{2}{7}]$	$2[\frac{2}{15}]$	0	0	$[\frac{2}{3}]$	0	0	$[\frac{7}{3}]$	0
24	0	0	$\frac{1}{2}$	$*[\frac{1}{105}]$	$*4[\frac{2}{35}]$	$\frac{1}{2}[\frac{1}{5}]$	$*\frac{1}{2}[\frac{7}{3}]$	$\frac{1}{2}[5]$	$[\frac{2}{7}]$	$2[\frac{2}{3}]$	$3[\frac{3}{7}]$
44	0	0	0	0	0	0	0	0	$*\frac{3}{2}[\frac{3}{7}]$	0	$2[\frac{2}{7}]$

This completes the evaluation of the matrix $Q^{MK}_{L\alpha,J\gamma}$.

VI. THE (1,1) CASE

The techniques described above have been used to derive the parentage tables of D coefficients of (5.3) for the case of p-wave pions, the (1-1) case. MACSYMA (MIT Mathlab group) was used to perform the necessary algebraic manipulations. Table I shows the number q of independent n-meson IPF states for various values of the isospin T and angular momentum L; this number is the same for TL and LT. The values for n less than the minimum of T and L are all zero and are omitted from the table. The digits at a particular value of TL are the values of q for n going from $\min(T,L)$ through 6. For example, for $(T,L)=(2,3)$ there are no IPF states with

TABLE VI. 5→6 parentages. (a) $T_6=L_6$, (b) $T_6<L_6$ with $[a]\equiv a^{1/2}$.

(a)

	00	11	11	22	22	22	22
11	$2[\frac{78}{187}]$	$[\frac{3}{11}]$	$*2[\frac{11}{17}]$	$\frac{18}{17}[\frac{5}{11}]$	0	$*\frac{3}{5}[\frac{14}{55}]$	$\frac{831}{85}[\frac{1}{55}]$
11	$*9[\frac{10}{187}]$	$\frac{1}{2}[\frac{65}{11}]$	0	$*\frac{46}{85}[\frac{39}{11}]$	0	$*\frac{1}{5}[\frac{273}{22}]$	$\frac{57}{170}[\frac{39}{11}]$
02	0	$[\frac{1}{3}]$	$4[\frac{1}{17}]$	0	0	0	0
12	0	$\frac{1}{2}[7]$	0	$*\frac{2}{17}[\frac{7}{3}]$	$*2[\frac{2}{5}]$	$\frac{11}{5}[\frac{1}{6}]$	$\frac{13}{170}[21]$
22	0	$*\frac{1}{2}[\frac{1}{3}]$	$4[\frac{1}{17}]$	$\frac{2}{17}[5]$	0	$[\frac{7}{10}]$	$\frac{63}{34}[\frac{1}{5}]$
22	0	0	$*[\frac{10}{17}]$	$[2]$	0	0	0
13	0	0	0	$\frac{4}{17}[\frac{7}{15}]$	$[\frac{1}{2}]$	$\frac{23}{5}[\frac{1}{30}]$	$*\frac{13}{85}[\frac{21}{5}]$
13	0	0	0	$\frac{64}{85}$	0	$*\frac{1}{5}[14]$	$*\frac{33}{85}$
23	0	0	0	$\frac{4}{51}[10]$	$*\frac{1}{5}[21]$	$\frac{1}{5}[\frac{7}{5}]$	$*\frac{116}{85}[\frac{2}{5}]$
23	0	0	0	$\frac{4}{51}[35]$	$*\frac{1}{5}[\frac{3}{2}]$	$\frac{1}{5}[\frac{1}{10}]$	$\frac{37}{85}[\frac{7}{5}]$
33	0	0	0	$*\frac{16}{17}[\frac{1}{5}]$	0	$*\frac{2}{5}[\frac{14}{5}]$	$\frac{54}{85}[\frac{1}{5}]$
33	0	0	0	$*\frac{22}{85}[6]$	0	0	$*\frac{4}{17}[6]$

TABLE VI. (*Continued*).

(a)

	33	33	33	44	44	55	66
22	$\frac{22}{21}[2]$	$*\frac{1}{3}[\frac{2}{11}]$	$*\frac{6}{7}[\frac{10}{187}]$	0	0	0	0
22	$\frac{20}{63}[5]$	$\frac{4}{9}[\frac{5}{11}]$	$\frac{150}{7}[\frac{1}{187}]$	0	0	0	0
23	$*\frac{59}{27}[\frac{2}{35}]$	$\frac{58}{27}[\frac{10}{77}]$	$3[\frac{2}{1309}]$	0	0	0	0
23	$\frac{13}{27}[\frac{1}{5}]$	$\frac{49}{27}[\frac{5}{11}]$	$*6[\frac{1}{187}]$	0	0	0	0
33	$\frac{2}{9}[2]$	$\frac{19}{9}[\frac{2}{11}]$	$3[\frac{5}{374}]$	$\frac{4}{3}[2]$	$[\frac{1}{102}]$	0	0
33	$\frac{5}{6}[\frac{5}{3}]$	$\frac{7}{6}[\frac{5}{33}]$	$*\frac{35}{4}[\frac{3}{187}]$	$*\frac{1}{6}[\frac{5}{3}]$	$\frac{41}{12}[\frac{5}{17}]$	0	0
24	$\frac{1}{7}[3]$	$*[\frac{3}{11}]$	$*\frac{18}{7}[\frac{15}{187}]$	0	0	0	0
34	$\frac{1}{2}[\frac{15}{7}]$	$*\frac{1}{2}[\frac{15}{77}]$	$\frac{45}{4}[\frac{3}{1309}]$	$\frac{1}{2}$	$\frac{3}{4}[\frac{3}{17}]$	0	0
44	$\frac{5}{14}[\frac{5}{3}]$	$\frac{1}{2}[\frac{5}{33}]$	$\frac{71}{28}[\frac{3}{187}]$	$*\frac{1}{2}[\frac{3}{5}]$	$*\frac{53}{4}[\frac{1}{85}]$	$2[\frac{6}{5}]$	0
35	0	0	0	$*\frac{1}{3}[\frac{22}{3}]$	$\frac{1}{3}[\frac{22}{17}]$	0	0
55	0	0	0	$\frac{4}{3}[\frac{1}{5}]$	$\frac{2}{3}[\frac{1}{85}]$	$[\frac{6}{5}]$	$[6]$

(b)

	02	12	03	13	13	23	23
11	$[\frac{102}{55}]$	$*13[\frac{3}{935}]$	0	0	0	0	0
11	$3[\frac{26}{187}]$	$\frac{9}{2}[\frac{13}{187}]$	0	0	0	0	0
02	0	$-7[\frac{1}{51}]$	0	$[\frac{7}{6}]$	$3[\frac{3}{34}]$	0	0
12	$*[\frac{14}{17}]$	$\frac{1}{2}[\frac{7}{17}]$	$2[\frac{5}{7}]$	$[\frac{1}{2}]$	$*15[\frac{1}{238}]$	$\frac{4}{3}$	$\frac{2}{3}[\frac{2}{85}]$
21	0	$*\frac{3}{2}[\frac{7}{85}]$	0	0	0	0	0
22	0	$\frac{11}{2}[\frac{1}{51}]$	0	$*[\frac{1}{42}]$	$3[\frac{3}{34}]$	0	$*6[\frac{6}{119}]$
22	0	$2[\frac{10}{51}]$	0	$\frac{5}{2}[\frac{5}{21}]$	$\frac{1}{2}[\frac{15}{17}]$	0	$2[\frac{15}{119}]$
13	$2[\frac{14}{85}]$	$2[\frac{7}{85}]$	$[\frac{5}{2}]$	$*\frac{1}{8}[7]$	$\frac{39}{8}[\frac{1}{17}]$	$\frac{1}{6}[\frac{7}{2}]$	$\frac{7}{3}[\frac{7}{85}]$
13	$2[\frac{6}{17}]$	$[\frac{3}{17}]$	0	$*\frac{1}{4}[15]$	$\frac{1}{4}[\frac{105}{17}]$	$[\frac{5}{6}]$	$*4[\frac{1}{51}]$
23	0	$*\frac{3}{5}[\frac{6}{17}]$	0	$[\frac{3}{10}]$	$[\frac{35}{102}]$	$*\frac{4}{3}[\frac{1}{3}]$	$*\frac{1}{3}[\frac{34}{15}]$
32	0	0	0	0	0	$\frac{19}{3}[\frac{2}{105}]$	$*\frac{176}{15}[\frac{1}{357}]$
23	0	$*\frac{4}{5}[\frac{21}{17}]$	0	$*\frac{3}{8}[\frac{21}{5}]$	$\frac{19}{8}[\frac{5}{51}]$	$\frac{1}{6}[\frac{7}{6}]$	$\frac{19}{3}[\frac{7}{255}]$
32	0	0	0	0	0	$*\frac{8}{3}[\frac{1}{15}]$	$*\frac{2}{15}[\frac{34}{3}]$
33	0	0	0	0	0	$[\frac{2}{3}]$	$*2[\frac{1}{255}]$
33	0	0	0	0	0	0	$3[\frac{1}{34}]$
14	0	0	$*3[\frac{1}{14}]$	$\frac{3}{8}[5]$	$\frac{11}{8}[\frac{5}{119}]$	$\frac{3}{2}[\frac{1}{10}]$	$*3[\frac{1}{17}]$
24	0	0	0	$\frac{9}{8}[\frac{1}{7}]$	$\frac{23}{8}[\frac{1}{17}]$	$\frac{1}{6}[\frac{35}{2}]$	$\frac{13}{3}[\frac{1}{119}]$
34	0	0	0	0	0	$*\frac{4}{3}[\frac{1}{35}]$	$\frac{1}{3}[\frac{1}{238}]$

three mesons, one IPF state with four mesons, two independent IPF states with five mesons, and two independent IPF states with six mesons.

Tables II–VI show the D parentage coefficients in condensed form; all D coefficients not explicitly listed can be obtained by using the relation

$$D^N_{TL,tl} = D^N_{LT,lt} \ . \tag{6.1}$$

In Tables II–VI, the TL values down the left side are the values for $M-1$ mesons, and the values across the top are for M mesons. Part (a) of Tables IV–VI gives the coefficients for $T_M = L_M$, while part (b) is for unequal values of T_M and L_M. For states with identical values of TL, the order is significant. To enhance readability, the asterisk has been used to denote negative parentage coefficients. The values in these tables extend and supersede the ones given previously in Ref. 8; the tables of Ref. 8 include meson numbers up to and including 4 and do not use orthonormal meson states.

TABLE VI. (*Continued*).

	(b)						
	04	14	24	24	24	34	34
13	$[\frac{77}{102}]$	$\frac{1}{2}[\frac{35}{6}]$	$\frac{5}{2}[\frac{7}{33}]$	$\frac{4}{11}[\frac{7}{5}]$	$\frac{65}{22}[\frac{5}{34}]$	0	0
13	$\frac{2}{3}[\frac{110}{17}]$	$*\frac{5}{3}[\frac{1}{2}]$	$\frac{1}{3}[\frac{5}{11}]$	$\frac{28}{11}[\frac{1}{3}]$	$*\frac{19}{11}[\frac{7}{102}]$	0	0
23	0	$*\frac{4}{3}$	$*\frac{1}{3}[\frac{2}{11}]$	$\frac{4}{33}[\frac{2}{15}]$	$\frac{74}{11}[\frac{7}{255}]$	$\frac{13}{3}[\frac{2}{21}]$	$\frac{11}{3}[\frac{2}{21}]$
23	0	$\frac{1}{6}[\frac{7}{2}]$	$*\frac{1}{6}[77]$	$\frac{2}{3}[\frac{7}{15}]$	$\frac{23}{2}[\frac{1}{510}]$	$\frac{4}{3}[\frac{1}{3}]$	$*\frac{7}{3}[\frac{1}{3}]$
33	0	0	0	$*2[\frac{1}{15}]$	$[\frac{70}{51}]$	$*[\frac{5}{6}]$	0
33	0	0	$*[\frac{5}{66}]$	$\frac{8}{11}[8]$	$\frac{1}{33}[\frac{7}{17}]$	$*\frac{5}{12}$	$\frac{5}{6}$
14	$[\frac{77}{34}]$	$\frac{1}{2}[\frac{7}{10}]$	$*\frac{1}{10}[77]$	0	$\frac{11}{2}[\frac{3}{170}]$	0	0
24	0	$*\frac{1}{2}[\frac{1}{2}]$	$\frac{3}{2}[\frac{1}{11}]$	$*\frac{10}{11}[\frac{5}{3}]$	$\frac{7}{22}[\frac{105}{34}]$	$4[\frac{1}{21}]$	$*[\frac{1}{21}]$
34	0	0	$\frac{27}{5}[\frac{1}{22}]$	$\frac{4}{11}[\frac{10}{3}]$	$\frac{7}{11}[\frac{21}{85}]$	$\frac{11}{4}[\frac{1}{15}]$	$*\frac{7}{2}[\frac{1}{15}]$
43	0	0	0	0	0	$\frac{9}{4}[\frac{1}{7}]$	$\frac{3}{2}[\frac{1}{7}]$
44	0	0	0	0	0	$*\frac{1}{4}[\frac{15}{7}]$	$\frac{1}{2}[\frac{15}{7}]$
15	$\frac{4}{3}[\frac{1}{17}]$	$\frac{2}{3}[\frac{11}{5}]$	$*\frac{2}{15}[2]$	$[\frac{10}{33}]$	$*14[\frac{7}{2805}]$	0	0
35	0	0	$\frac{2}{5}[\frac{1}{3}]$	0	$\frac{1}{3}[\frac{154}{85}]$	$*\frac{1}{3}[\frac{22}{5}]$	$*\frac{1}{3}[\frac{22}{5}]$

	(b)						
	15	25	35	45	06	26	46
14	$*\frac{1}{5}[14]$	$\frac{3}{7}[7]$	0	0	0	0	0
24	[2]	1	$2[\frac{6}{7}]$	0	0	0	0
34	0	$\frac{4}{5}[2]$	$*[\frac{3}{10}]$	$3[\frac{1}{2}]$	0	0	0
44	0	0	$[\frac{15}{14}]$	$[\frac{3}{10}]$	0	0	0
15	$[\frac{6}{5}]$	$\frac{2}{5}[3]$	0	0	[6]	$3[\frac{2}{5}]$	0
35	0	$*\frac{3}{5}[2]$	$[\frac{6}{5}]$	$\frac{1}{3}[2]$	0	$2[\frac{3}{5}]$	$\frac{2}{3}[11]$
55	0	0	0	$*\frac{2}{3}[\frac{11}{5}]$	0	0	$\frac{1}{3}[10]$

VII. SUMMARY

The use of the invariant-pair subalgebra has been shown to provide some simplification of the process of constructing and classifying the possible state vectors of a limited number of boson modes. In particular, only invariant-pair-free (IPF) states actually require classification, since all states with invariant pairs can be systematically cataloged once the IPF states are listed. The invariant subalgebra was also considered and found to be less suitable because of the difficulty of evaluating matrix elements between invariant-free states.

A procedure for determining the IPF states has been described and used to catalog the IPF states for up to six *p*-wave pions.

ACKNOWLEDGMENT

This work was performed under the auspices of the U. S. Department of Energy.

*Mailing address: Group T-9 MS B279, Los Alamos National Laboratory, Los Alamos, New Mexico 87545.

[1]D. E. L. Pottinger and R. C. Warner, Nucl. Phys. **B216**, 336 (1983).

[2]M. Bolsterli, Phys. Rev. D **22**, 437 (1980); **31**, 603 (1985); in *Quarks and Nuclear Structure*, edited by K. Bleuler (Springer, Berlin, 1984), p. 405.

[3]S. Pekar, J. Phys. (Moscow) **10**, 341 (1946); G. Höhler, Z. Phys. **140**, 192 (1955); **146**, 372 (1956).

[4]M. Bolsterli, Phys. Rev. D **27**, 349 (1983).

[5]M. Bolsterli, Phys. Rev. D **25**, 1095 (1982); Phys. Rev. C **25**, 2728 (1982); **27**, 2725 (1983); Phys. Rev. D **27**, 2940 (1983).

[6]M. H. Friedman, T. D. Lee, and R. Christian, Phys. Rev. **100**, 1494 (1955).

[7]H. Ui, Ann. Phys. (N.Y.) **49**, 69 (1968).

[8]M. Bolsterli, Phys. Rev. D **24**, 400 (1981).