Dynamical structure of the quantum relativistic rotator and analogy to Rohrlich's version of the quantum relativistic string

R. R. Aldinger
Department of Physics, Eastern Illinois University, Charleston, Illinois 61920
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This is a continuation of three recent papers in which the quantum relativistic rotator was defined and analyzed as an extended model for hadrons. In this paper, an analogy is established between the quantum relativistic rotator and the covariant noncanonical treatment of Rohrlich's version of the quantum relativistic string (when the latter is restricted to its lowest mode of excitation). It is shown that the models utilize identical techniques in their basic construction which result in many similarities between the two, regardless of the fact that they ultimately account for the hadron spectrum in two distinct manners.

I. INTRODUCTION

The interpretation of the excitation spectrum of the dual resonance models[1] as a vibrating relativistic string has led to an attractive phenomenological model for the internal structure of hadrons.[2] The geometrical description of the string was originated by Nambu[3] who generalized the action of a relativistic point particle to a system of one-dimensional spacelike extension. As a consequence of reparametrization invariance of the string action, quantization can be carried out only after a specific choice of gauge has been made. It is standard practice to utilize this freedom of reparametrization invariance by imposing the orthogonal gauge which, although it linearizes the Euler-Lagrange equations of motion, leaves an arbitrariness in the string-time (evolution) parameter. It therefore becomes necessary to impose a further condition in order to specify the system uniquely. For the conventional treatment of the relativistic string in the orthogonal gauge,[4] quantization is carried out by specifying the evolution parameter through the application of the null-plane coordinates (null gauge) which results in a positive-definite Hilbert space in 26 space-time dimensions with a tachyonic ground state.

Various authors[5–7] have proposed that it might be possible to arrive at a more physically attractive result by application of an alternate quantization procedure, i.e., characterization of the evolution parameter through a gauge other than the conventional null-like choice. Among these alternate procedures, Rohrlich's[7] center-of-mass approach is quite appealing.

Rohrlich introduced a technique which quantizes the relativistic string while circumventing the unphysical results of the conventional string model by fixing the evolution parameter through the application of a timelike gauge in the center-of-mass frame (Rohrlich's string model). The basic results of this quantization procedure are (a) Lorentz invariance in four-dimensional space-time; (b) the Hilbert space is made positive-definite (no ghosts) by imposing a constraint on the indefinite Hilbert space which holds either as a condition on the physical state vectors[7(b)] or as an operator expression[7(c)] (which depends on whether the commutation relations are consistent with them); (c) linearly rising Regge trajectories; and (d) a mass spectrum that contains an arbitrary additive constant which can be made strictly positive (no tachyons). The resulting spectrum of the string is explained in terms of arbitrary-dimensional harmonic-oscillator states (where, for the special case of the lowest mode, the spectrum is a result of the excitations of a three-dimensional oscillator at rest).

On the other hand, the model of the quantum relativistic rotator (QRR) [defined and analyzed in a series of three recent papers, Refs. 8(a), 8(b), and 8(c)] explains the hadron spectrum in terms of infinite "towers" which consist of an arbitrary number of rotational states. The model is specified by a Hamiltonian which is obtained by imposing a (first-class) constraint on the second-order Casimir operator of a de Sitter SO(4,1). The constraint relation leads to an experimentally verifiable[8(c)] rotatorlike mass-spin spectrum while the relativistic Hamiltonian is used to determine the dynamical structure of the QRR. Theoretical justification[8(c)] for the QRR is established through the methods of group contraction[9] where, in the elementary contraction limit (length parameter $R\rightarrow\infty$), the model reduces to the description of a structureless relativistic mass point and, in the nonrelativistic contraction limit (velocity of light $c\rightarrow\infty$), it reduces to the description of the nonrelativistic rotator. The resulting theory for the QRR is relativistically covariant in four-dimensional space-time without the appearance of ghost states or tachyons.

It is the purpose of this paper to show that the basic construction of the QRR bears a remarkable resemblance to that of the covariant noncanonical treatment of the Rohrlich string[7(c)] which results in many similarities between the two hadron models (when the latter is restricted to its lowest mode of excitation). A brief review of the QRR is given in Sec. II where the basic quantum-mechanical observables of the model are introduced as well as their defining algebraic relations. After the relativistic symmetry of the QRR is established, an SO(4,1) substructure is introduced which supplies a relation that breaks the symmetry thereby providing the model with a nontrivial (rotatorlike) mass-spin spectrum. A first-class constraint is then imposed upon the substructure which supplies the model with a relativistic Hamiltonian which is used in Sec. III to obtain the QRR equations of motion. Here a solution to the QRR's particle position equation of motion is obtained which contains a τ-independent constant of integration which plays the role of an arbitrary "center operator" (which is not equivalent to the center-of-mass position). It is shown in Sec. IV that if this integration constant is identified to be the Finkelstein center operator, then we are automatically placed into a timelike center-of-mass gauge equivalent to that used by Rohrlich in his treatment of the relativistic string. The similarity is investigated further in Sec. V where the algebraic structure of the two models is compared on both the "external" and "internal" levels. In Sec. VI the structure of the QRR is recast in terms of its fundamental mode operators and the resulting relations are compared with those of the Rohrlich string. It is also shown that a ghost-elimination condition exists for the QRR (identical to that for the Rohrlich string) which is a direct consequence of the specific definition of its "relative" position operator; i.e, the operator which is responsible for the spatial extension of the QRR. A summary of the basic results is given in Sec. VII.

II. QRR HAMILTONIAN VIA CONSTRAINED MECHANICS

The model of the QRR was defined in Ref. 8(a) as a one-dimensionally extended object capable of performing rotations and global translations in Minkowski space whose dynamics is obtained through the quantum analog of constrained Hamiltonian mechanics.[10] The relativistic symmetry of the extended object is expressed by the Poincaré group, $\mathscr{P}_{P_\mu,J_{\mu\nu}}$ (which determines the "external" space-time symmetry) together with the spectrum-generating[11] group, $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$ (which is the "internal" symmetry group). The Poincaré group is generated by the observables momentum P_μ and angular momentum $J_{\mu\nu}$ (with $\mu,\nu=0,1,2,3$) which obey the following commutation relations $[g_{\mu\nu}=\text{diag}(1,-1,-1,-1)]$:

$$[J_{\mu\nu},J_{\rho\sigma}]=-i(g_{\mu\rho}J_{\nu\sigma}+g_{\nu\sigma}J_{\mu\rho}-g_{\mu\sigma}J_{\nu\rho}-g_{\nu\rho}J_{\mu\sigma}) , \tag{2.1a}$$

$$[P_\mu,J_{\rho\sigma}]=-i(g_{\mu\sigma}P_\rho-g_{\mu\rho}P_\sigma) , \tag{2.1b}$$

$$[P_\mu,P_\nu]=0 , \tag{2.1c}$$

where the total angular momentum decomposes into orbital $M_{\mu\nu}$ and spin $S_{\mu\nu}$ parts:

$$J_{\mu\nu}=Q_\mu P_\nu-Q_\nu P_\mu+S_{\mu\nu}\equiv M_{\mu\nu}+S_{\mu\nu} . \tag{2.2}$$

The properties of the position operators Q_μ are defined by correspondence to the nonrelativistic theory and therefore satisfy

$$[Q_\mu,P_\nu]=-ig_{\mu\nu}1 , \tag{2.3a}$$

$$[Q_\mu,Q_\nu]=0 . \tag{2.3b}$$

Since the Q_μ and P_μ are assumed to commute with the $S_{\mu\nu}$, we also have

$$[J_{\mu\nu},Q_\rho]=i(g_{\nu\rho}Q_\mu-g_{\mu\rho}Q_\nu) . \tag{2.4}$$

The spectrum-generating group $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$ is generated by the spin part of angular momentum $S_{\mu\nu}$ and the Hermitian vector operator Γ_μ which obey the following set of algebraic relations:

$$[S_{\mu\nu},S_{\rho\sigma}]=-i(g_{\mu\rho}S_{\nu\sigma}+g_{\nu\sigma}S_{\mu\rho}-g_{\mu\sigma}S_{\nu\rho}-g_{\nu\rho}S_{\mu\sigma}) , \tag{2.5a}$$

$$[S_{\mu\nu},\Gamma_\rho]=i(g_{\nu\rho}\Gamma_\mu-g_{\mu\rho}\Gamma_\nu) , \tag{2.5b}$$

$$[\Gamma_\mu,\Gamma_\nu]=-iS_{\mu\nu} . \tag{2.5c}$$

The relativistic symmetry described by Eqs. (2.1)–(2.5) contains a substructure which plays the central role for the QRR. By defining

$$B_\mu\equiv P_\mu-\lambda b_\mu M=P_\mu+\frac{\lambda}{2}\{J_{\rho\mu},\hat{P}^\rho\} \tag{2.6}$$

(where b_μ is the Finkelstein center operator,[12] $\lambda=1/R$ is the inverse radius of a micro—de Sitter space and $\hat{P}_\mu=P_\mu/M$ with $P_\mu P^\mu=M^2$) it can be shown that the physical angular momentum $J_{\mu\nu}$ and the B_μ form a de Sitter $SO(4,1)_{B_\mu,J_{\mu\nu}}$. The central role that this $SO(4,1)_{B_\mu,J_{\mu\nu}}$ plays comes from the fact that its second-order Casimir operator:

$$\lambda^2 Q=B_\mu B^\mu-\frac{\lambda^2}{2}J_{\mu\nu}J^{\mu\nu} \tag{2.7}$$

commutes with every element of the relativistic symmetry and is therefore an invariant of the extended object whose irreducible representations are characterized by the eigenvalues α^2 of Q. [In the following we shall restrict ourselves to the principal series representation of the group SO(4,1) (Ref. 13) which requires, $\alpha^2\geq\frac{9}{4}-s(s+1)$.] Substituting the definition of B_μ into Eq. (2.7) gives

$$\lambda^2 Q=P_\mu P^\mu+\lambda^2\tfrac{9}{4}-\lambda^2\hat{W}\overset{\text{irrep}}{=}\lambda^2\alpha^2 \tag{2.8}$$

[where $\hat{W}=(P_\mu P^\mu)^{-1}W$ with $W\equiv-w_\mu w^\mu$ and $w_\mu=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu J^{\rho\sigma}$].

To obtain our quantum relativistic Hamiltonian, the methods of constrained Hamiltonian mechanics are applied. The constraint for a quantum relativistic mass point,

$$\Phi\equiv P_\mu P^\mu-m^2\approx 0 \tag{2.9}$$

(where the eigenvalue m^2 characterizes the mass point), shall be replaced with the constraint imposed on the second-order Casimir operator of $SO(4,1)_{B_\mu,J_{\mu\nu}}$, Eq. (2.8):

$$\Phi\equiv P_\mu P^\mu-\lambda^2\hat{W}+\lambda^2(\tfrac{9}{4}-\alpha^2)\approx 0 \tag{2.10a}$$

(where α^2 is the eigenvalue which characterizes the QRR). The symbol ≈ 0 signifies "set weakly to zero" since the constraint has nonvanishing commutators and one must evaluate all commutation relations prior to imposing the constraint. The constraint relation $\Phi=0$ taken between the canonical basis vectors $|pss_3\rangle$ leads to the mass formula [see Ref. 8(b)]

$$m^2=\lambda^2(\alpha^2-\tfrac{9}{4})+\lambda^2 s(s+1) \tag{2.10b}$$

(where the spin s has either the spectrum $s=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots$ or $s=0,1,2,\ldots$).

Following the rules of constrained mechanics, the QRR Hamiltonian may be expressed as

$$\mathscr{H}=\phi\Phi\equiv\phi[P_\mu P^\mu-\lambda^2\hat{W}+\lambda^2(\tfrac{9}{4}-\alpha^2)] \tag{2.11}$$

[where ϕ is an unknown velocity parameter which is to be determined by specifying the evolution parameter τ through a specific choice of gauge (see Sec. IV)].

We now restrict ourselves to those irreducible representations of $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$ which contain only a discrete sum of irreducible representations of the $SO(3,1)_{S_{\mu\nu}}$ subgroup. These are the Majorana[14] representations for which

$$\hat{W}\equiv-\hat{\omega}_\mu\hat{\omega}^\mu=(\hat{P}\cdot\Gamma)^2-\tfrac{1}{4}\ , \tag{2.12a}$$

which gives the following in an irreducible representation [see Ref. 8(b)]:

$$\hat{W}\overset{\text{irrep}}{=}(s+\tfrac{1}{2})^2-\tfrac{1}{4}=s(s+1) \tag{2.12b}$$

(where $s\equiv\text{spin}=0,\frac{1}{2},1,\frac{3}{2},\ldots$). Using Eq. (2.12a), we may write the QRR Hamiltonian, Eq. (2.11), in the following alternate form:

$$\mathscr{H}^{\text{Maj}}=\phi[P_\mu P^\mu-\lambda^2(\hat{P}\cdot\Gamma)^2+\lambda^2(\tfrac{5}{2}-\alpha^2)]\ , \tag{2.13}$$

which is valid only in the Majorana representation and is the form of the QRR Hamiltonian which shall be used in Sec. III to determine the equations of motion.

We assume that the system center of mass and particle position are separated in space thereby providing the QRR with a spatial extension. Therefore, the center-of-mass operator is expressed as

$$Y_\mu=Q_\mu+d_\mu\ , \tag{2.14}$$

where d_μ is the relative position operator which is directed from the particle position Q_μ to the center of mass. It is formally defined as[15]

$$d_\mu=S_{\mu\nu}\frac{P^\nu}{M^2}=S_{\mu\nu}\frac{\hat{P}^\nu}{M}\ , \tag{2.15a}$$

and is responsible for the extension of the QRR. We may write Eq. (2.15a) in an alternate form by introducing the dimensionless relative position operator $\hat{d}_\mu=d_\mu M$ which gives

$$\hat{d}_\mu=S_{\mu\nu}\hat{P}^\nu\ . \tag{2.15b}$$

The commutator of two $\hat{d}_\mu$'s is

$$[\hat{d}_\mu,\hat{d}_\nu]=-i(S_{\mu\nu}+\hat{d}_\nu\hat{P}_\mu-\hat{d}_\mu\hat{P}_\nu)\ , \tag{2.16}$$

which suggests the definition of a new operator:

$$\Sigma_{\mu\nu}=S_{\mu\nu}+\hat{d}_\nu\hat{P}_\mu-\hat{d}_\mu\hat{P}_\nu\ . \tag{2.17}$$

The form of Eq. (2.17) implies that we must distinguish between two types of rest frames: (i) the ordinary rest frame in which the center of mass is at rest, i.e., $\hat{P}_\mu=(1,0,0,0)$; and (ii) the rest frame in which the particle position is at rest, i.e., $\dot{Q}_\mu=(1,0,0,0)$. As a consequence of the forms for d_μ and $\Sigma_{\mu\nu}$, we have the following operator identities:

$$P_\mu\Sigma^{\mu\nu}=0=\hat{P}_\mu\Sigma^{\mu\nu} \tag{2.18a}$$

(which identifies $\Sigma_{\mu\nu}$ as the spin tensor since in the proper Lorentz frame, $\Sigma_{i0}=0$) and

$$d_\mu P^\mu=0=\hat{d}_\mu\hat{P}^\mu \tag{2.18b}$$

(which requires the relative position operator to be spacelike). From this information and Eq. (2.17) we have that $\Sigma_{\mu\nu}$ is the angular momentum with respect to the ordinary rest frame while $S_{\mu\nu}$ is the angular momentum with respect to the particle-position rest frame.

In addition to the $\hat{P}_\mu$ and $\hat{d}_\mu$, we also define the dimensionless particle-position operator:

$$\hat{Q}_\mu=Q_\mu M\quad(\neq MQ_\mu)\ . \tag{2.19}$$

Some consequences of Eq. (2.3a) and this definition are

$$[Q_\mu,M]=-i\hat{P}_\mu\ , \tag{2.20a}$$

$$[Q_\mu,M^{-1}]=i\hat{P}_\mu M^{-2}\ , \tag{2.20b}$$

$$[\hat{Q}_\mu,\hat{Q}_\nu]=i(\hat{Q}_\mu\hat{P}_\nu-\hat{Q}_\nu\hat{P}_\mu)\ , \tag{2.20c}$$

$$[\hat{Q}_\mu,\hat{P}_\nu]=-i(g_{\mu\nu}-\hat{P}_\mu\hat{P}_\nu)\ . \tag{2.20d}$$

Introducing the dimensionless center-of-mass operator

$$\hat{Y}_\mu=Y_\mu M\ , \tag{2.21}$$

and using Eq. (2.14) we have that the center of mass may be written in dimensionless form as

$$\hat{Y}_\mu=\hat{Q}_\mu+\hat{d}_\mu\ . \tag{2.22}$$

Inserting Eq. (2.17) into Eq. (2.2) and using Eqs. (2.19) and (2.21) we have

$$\begin{aligned}J_{\mu\nu}&=\Sigma_{\mu\nu}+\hat{Y}_\mu\hat{P}_\nu-\hat{Y}_\nu\hat{P}_\mu\\&=\Sigma_{\mu\nu}+Y_\mu P_\nu-Y_\nu P_\mu\ .\end{aligned} \tag{2.23}$$

III. QRR EQUATIONS OF MOTION

To obtain the time derivatives for the observables of the QRR, the quantum analog of constrained Hamiltonian mechanics will be used. Therefore the derivatives with respect to the evolution parameter τ are evaluated using $d\mathscr{O}/d\tau\equiv\dot{\mathscr{O}}=-i[\mathscr{O},\mathscr{H}]$ prior to imposing the constraint, Eq. (2.10a). Using Eq. (2.13) for $\mathscr{H}$ (i.e., $\mathscr{H}^{\text{Maj}}$), the following τ derivatives are obtained:

$$\dot{S}_{\mu\nu}=-\phi\lambda^2\{\hat{P}\cdot\Gamma,\hat{P}_\nu\Gamma_\mu-\hat{P}_\mu\Gamma_\nu\}\ , \tag{3.1a}$$

$$\dot{\hat{d}}_\mu=\dot{S}_{\mu\nu}\hat{P}^\nu=-\phi\lambda^2\{\hat{P}\cdot\Gamma,\Gamma_\mu-(\hat{P}\cdot\Gamma)\hat{P}_\mu\} \tag{3.1b}$$

(where $\dot{\hat{P}}_\mu=0$ has been used),

$$\dot{\Gamma}_\mu=\phi\lambda^2\{\hat{P}\cdot\Gamma,\hat{d}_\mu\}\ , \tag{3.1c}$$

$$\dot{Q}_\mu = -2\phi P_\mu + \phi\lambda^2\{\hat{P}\cdot\Gamma, \Gamma_\mu - (\hat{P}\cdot\Gamma)\hat{P}_\mu\}M^{-1} , \quad (3.1d)$$

and, using Eq. (2.14),

$$\dot{Y}_\mu = -2\phi P_\mu . \quad (3.1e)$$

Using the commutation relations of Eq. (2.5) and the definition for $\hat{d}_\mu$, Eq. (2.15b), we may write Eq. (3.1d) in the following alternate form:

$$\dot{Q}_\mu = -2\phi P_\mu + 2\phi\frac{\lambda^2}{M}(\hat{P}\cdot\Gamma)\Gamma_\mu - 2\phi\frac{\lambda^2}{M}(\hat{P}\cdot\Gamma)^2\hat{P}_\mu - i\phi\frac{\lambda^2}{M}\hat{d}_\mu . \quad (3.2)$$

An explicit τ-dependent expression for the QRR particle position $Q_\mu(\tau)$ may be obtained by directly integrating Eq. (3.2) with respect to the evolution parameter τ:

$$Q_\mu(\tau) = -2\phi P_\mu\tau + 2\phi\lambda^2\frac{(\hat{P}\cdot\Gamma)}{M}\int\Gamma_\mu(\tau)d\tau - 2\phi\lambda^2\frac{(\hat{P}\cdot\Gamma)^2}{M}\hat{P}_\mu\tau - i\phi\frac{\lambda^2}{M}\int\hat{d}_\mu(\tau)d\tau + D_\mu \quad (3.3)$$

(where D_μ is a τ-independent constant of integration). We must now obtain explicit τ-dependent expressions for $\Gamma_\mu(\tau)$ and $\hat{d}_\mu(\tau)$. From Eq. (3.1c) it follows that

$$\dot{\Gamma}_\mu(\tau) = 2\phi\lambda^2(\hat{P}\cdot\Gamma)\hat{d}_\mu(\tau) + i\phi\lambda^2\Gamma_\mu(\tau) - i\phi\lambda^2(\hat{P}\cdot\Gamma)\hat{P}_\mu \quad (3.4)$$

(where the definition $\hat{d}_\mu = d_\mu M$ was used). We also obtain from Eq. (3.1b) that

$$\dot{\hat{d}}(\tau) = -2\phi\lambda^2(\hat{P}\cdot\Gamma)\Gamma_\mu(\tau) + 2\phi\lambda^2(\hat{P}\cdot\Gamma)^2\hat{P}_\mu + i\phi\lambda^2\hat{d}_\mu(\tau) . \quad (3.5)$$

We are now in a position to evaluate the second τ derivatives for $\Gamma_\mu(\tau)$ and $\hat{d}_\mu(\tau)$. From Eq. (3.4) we obtain

$$\ddot{\Gamma}_\mu(\tau) = 2\phi\lambda^2(\hat{P}\cdot\Gamma)\dot{\hat{d}}_\mu(\tau) + i\phi\lambda^2\dot{\Gamma}_\mu(\tau) , \quad (3.6)$$

where the fact that $(d/d\tau)(\hat{P}\cdot\Gamma)=0$ has been used. By substituting Eq. (3.5) into Eq. (3.6) and using Eq. (3.4), we obtain the following nonhomogeneous equation for the second τ derivative of the SO(3,2) vector operator $\Gamma_\mu(\tau)$:

$$\ddot{\Gamma}_\mu(\tau) - 2i\phi\lambda^2\dot{\Gamma}_\mu(\tau) + [4\phi^2\lambda^4(\hat{P}\cdot\Gamma)^2 - \phi^2\lambda^4]\Gamma_\mu(\tau) = [4\phi^2\lambda^4(\hat{P}\cdot\Gamma)^2 - \phi^2\lambda^4](\hat{P}\cdot\Gamma)\hat{P}_\mu . \quad (3.7)$$

From Eq. (3.5) we find that

$$\ddot{\hat{d}}_\mu(\tau) = -2\phi\lambda^2(\hat{P}\cdot\Gamma)\dot{\Gamma}_\mu(\tau) + i\phi\lambda^2\dot{\hat{d}}_\mu(\tau) , \quad (3.8)$$

which, after substitution of Eq. (3.4) and using Eq. (3.5), becomes

$$\ddot{\hat{d}}_\mu(\tau) - 2i\phi\lambda^2\dot{\hat{d}}_\mu(\tau) + [4\phi^2\lambda^4(\hat{P}\cdot\Gamma)^2 - \phi^2\lambda^4]\hat{d}_\mu(\tau) = 0 . \quad (3.9)$$

The solutions to the second-order differential equations for $\Gamma_\mu(\tau)$ and $\hat{d}_\mu(\tau)$ may be written as

$$\Gamma_\mu(\tau) = (\hat{P}\cdot\Gamma)\hat{P}_\mu - e^{i\phi\lambda^2[1+2(\hat{P}\cdot\Gamma)]\tau}\hat{A}_\mu - e^{-i\phi\lambda^2[-1+2(\hat{P}\cdot\Gamma)]\tau}\hat{B}_\mu , \quad (3.10)$$

$$\hat{d}_\mu(\tau) = i(-e^{i\phi\lambda^2[1+2(\hat{P}\cdot\Gamma)]\tau}\hat{A}_\mu + e^{-i\phi\lambda^2[-1+2(\hat{P}\cdot\Gamma)]\tau}\hat{B}_\mu) . \quad (3.11)$$

Substituting these expressions into the QRR particle position, Eq. (3.3) gives

$$Q_\mu(\tau) = q_\mu - 2\phi P_\mu\tau + \frac{i}{M}e^{i\phi\lambda^2[1+2(\hat{P}\cdot\Gamma)]\tau}\hat{A}_\mu - \frac{i}{M}e^{-i\phi\lambda^2[-1+2(\hat{P}\cdot\Gamma)]\tau}\hat{B}_\mu , \quad (3.12)$$

where q_μ, $\hat{A}_\mu$, and $\hat{B}_\mu$ are τ-independent constants of integration. Using the fact that $\hat{d}_\mu = d_\mu M$, we may rewrite Eq. (3.12) in the following equivalent form:

$$Q_\mu(\tau) = q_\mu - 2\phi P_\mu\tau - d_\mu(\tau) . \quad (3.13)$$

IV. TIMELIKE CENTER-OF-MASS GAUGE

The form of Eq. (3.13) suggests that an agreement with the original definition of the QRR center of mass, Eq. (2.14), may be obtained by allowing

$$Y_\mu(\tau) = q_\mu - 2\phi P_\mu\tau \quad (4.1)$$

(where q_μ is a τ-independent constant of integration). The precise form for q_μ expressed in terms of the Poincaré generators is obtained by first examining the Finkelstein center operator, $b_\mu = \hat{b}_\mu M^{-1} = M^{-1}\hat{b}_\mu$, where

$$b_\mu = -\frac{1}{2M}\{J_{\rho\mu}, \hat{P}^\rho\} . \quad (4.2)$$

Using the form of $J_{\rho\mu}$ given by Eq. (2.2) along with the dimensionless quantities, $\hat{Q}_\mu$ and $\hat{P}_\mu$, we may write Eq. (4.2) as

$$\hat{b}_\mu = b_\mu M \equiv -\tfrac{1}{2}\{\hat{Q}_\rho(\tau)\hat{P}_\mu - \hat{Q}_\mu(\tau)\hat{P}_\rho + S_{\rho\mu}(\tau), \hat{P}^\rho\} \quad (4.3)$$

$$= -\tfrac{1}{2}\{\hat{Q}_\rho(\tau)\hat{P}_\mu - \hat{Q}_\mu(\tau)\hat{P}_\rho, \hat{P}^\rho\} + \hat{d}_\mu(\tau) , \quad (4.4)$$

where the definition, $\hat{d}_\mu = S_{\mu\rho}\hat{P}^\rho$, has been used. Using Eqs. (2.14) and (2.20), Eq. (4.4) may be written as

$$\hat{b}_\mu = \hat{Y}_\mu(\tau) - \hat{P}_\mu(\hat{P}\cdot\hat{Q}(\tau)) + i\tfrac{3}{2}\hat{P}_\mu , \quad (4.5)$$

or

$$\hat{Y}_\mu(\tau) = \hat{b}_\mu - i\tfrac{3}{2}\hat{P}_\mu + \hat{P}_\mu(\hat{P}\cdot\hat{Q}(\tau)) . \quad (4.6)$$

In dimensional form with $\hat{Y}_\mu = Y_\mu M$, $\hat{b}_\mu = b_\mu M$, and $\hat{Q}_\mu = Q_\mu M$:

$$Y_\mu(\tau) = b_\mu - i\frac{3}{2}\frac{\hat{P}_\mu}{M} + \hat{P}_\mu(\hat{P}\cdot Q(\tau)) . \quad (4.7)$$

Equations (4.1) and (4.7) imply that

$$q_\mu - 2\phi P_\mu \tau = b_\mu - i\frac{3}{2}\frac{\hat{P}_\mu}{M} + \hat{P}_\mu(\hat{P}\cdot Q(\tau)) . \tag{4.8}$$

This expression may be written as

$$\hat{P}_\mu(\hat{P}\cdot Q(\tau)) = -2\phi P_\mu \tau + i\frac{3}{2}\frac{\hat{P}_\mu}{M} + q_\mu - b_\mu$$
$$= \hat{P}_\mu(\hat{P}\cdot Y(\tau)) , \tag{4.9}$$

where we have used the definition of the center of mass, $Y_\mu = Q_\mu + d_\mu$, and the orthogonality relation, $\hat{P}\cdot d = 0$. We now evaluate Eq. (4.9), at $\tau = 0$:

$$i\frac{3}{2}\frac{\hat{P}_\mu}{M} + q_\mu - b_\mu = \hat{P}_\mu(\hat{P}\cdot Y(0)) . \tag{4.10}$$

From Finkelstein's center operator, Eq. (4.3), it may be shown that

$$P\cdot b = i\tfrac{3}{2} \quad \text{or} \quad \hat{P}_\mu(\hat{P}\cdot b) = i\frac{3}{2}\frac{\hat{P}_\mu}{M} . \tag{4.11}$$

We may now substitute this expression into Eq. (4.10) to obtain

$$q_\mu - b_\mu + \hat{P}_\mu(\hat{P}\cdot b) = \hat{P}_\mu(\hat{P}\cdot Y(0)) . \tag{4.12}$$

Also, from Eq. (4.1) evaluated at $\tau = 0$ we have that

$$Y_\mu(0) = q_\mu \quad \text{or} \quad \hat{P}_\mu(\hat{P}\cdot Y(0)) = \hat{P}_\mu(\hat{P}\cdot q) , \tag{4.13}$$

which is consistent with Eq. (4.12) if and only if

$$q_\mu = b_\mu . \tag{4.14}$$

With this identification, Eq. (4.9) becomes

$$\hat{P}_\mu(\hat{P}\cdot Q(\tau)) = -2\phi P_\mu \tau + \hat{P}_\mu(\hat{P}\cdot b) , \tag{4.15}$$

where we have used Eq. (4.11). In the center-of-mass frame, where $\hat{P}_\mu = (1,0,0,0)$, Eq. (4.15) gives

$$Q_0(\tau) = b_0 + \tau \quad (\text{in center of mass}) , \tag{4.16}$$

if and only if

$$\phi \equiv -\frac{1}{2M} , \tag{4.17}$$

which agrees with Rohrlich's timelike center-of-mass gauge-fixing procedure for the string-time (evolution) parameter [see Ref. 7(c)]. The particular value for the previously undetermined velocity ϕ, given by Eq. (4.17), was also obtained in Ref. 8(a) where one imposed, $\dot{Y}\cdot\dot{Y} = 1$ and used Eq. (3.1e).

Therefore as a consequence of determining an explicit form for the constant of integration q_μ we have (i) obtained a timelike center-of-mass gauge which specifies τ as the proper time and (ii) obtained a value for the previously undetermined velocity parameter ϕ. In this particular gauge, we therefore obtain the following general expression for the QRR particle position [from Eq. (3.13) and using Eqs. (4.14) and (4.17)]:

$$Q_\mu(\tau) = b_\mu + \hat{P}_\mu \tau - d_\mu(\tau) \tag{4.18}$$

or

$$Q_\mu(\tau) = Y_\mu(\tau) - d_\mu(\tau) , \tag{4.19}$$

where

$$Y_\mu(\tau) = b_\mu + \hat{P}_\mu \tau . \tag{4.20}$$

We have now arrived at a form for the particle position, $Q_\mu(\tau)$, which agrees with the original assumption of Eq. (2.14) and which also coincides with Rohrlich's treatment of the quantum relativistic string [where the relative position operators for the two models are related by $d_\mu(\tau) \leftrightarrow -\xi^\mu(\tau)$; the particle position operators by $Q_\mu(\tau) \leftrightarrow x^\mu(\tau)$; and the center-of-mass operators by $Y_\mu(\tau) \leftrightarrow Q^\mu(\tau)$].

V. EXTERNAL/INTERNAL STRUCTURE

The similarity between the model of the QRR and Rohrlich's covariant noncanonical treatment of the relativistic string can be extended by comparing the algebraic structure of the two models. Substituting Eq. (4.6) into the expression for the total angular momentum, Eq. (2.23), gives

$$J_{\mu\nu} = \hat{b}_\mu \hat{P}_\nu - \hat{b}_\nu \hat{P}_\mu + \Sigma_{\mu\nu} \tag{5.1}$$

[where we have used Eqs. (2.20)]. Therefore, the total angular momentum may be expressed as

$$J_{\mu\nu} = b_\mu P_\nu - b_\nu P_\mu + \Sigma_{\mu\nu} \tag{5.2}$$

(where $\hat{b}_\mu = b_\mu M$ and $\hat{P}_\mu = P_\mu / M$ have been used). The center position (i.e., origin) of the system is characterized by b_μ [which is identical to the center of mass only when $\tau = 0$; see Eq. (4.20)], $\Sigma_{\mu\nu}$ is the spin tensor which satisfies $P_\mu \Sigma^{\mu\nu} = 0$ and the P_μ generate global translations. Using the definition of the Finkelstein center operator given in Eq. (4.2) and the commutation relations for the Poincaré group, Eqs. (2.1), we obtain the following set of algebraic relations:

$$\begin{aligned}
&[b_\mu, b_\nu] = iJ_{\mu\nu}M^{-2}, \quad [b_\mu, P_\nu] = -i(g_{\mu\nu} - \hat{P}_\mu \hat{P}_\nu) , \\
&[b_\mu, \Sigma_{\rho\sigma}] = i(\Sigma_{\mu\sigma}P_\rho - \Sigma_{\mu\rho}P_\sigma)M^{-2}, \quad [P_\mu, \Sigma_{\rho\sigma}] = 0 , \\
&[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = -i(g_{\mu\rho} - \hat{P}_\mu \hat{P}_\rho)\Sigma_{\nu\sigma} - i(g_{\nu\sigma} - \hat{P}_\nu \hat{P}_\sigma)\Sigma_{\mu\rho} \\
&\qquad + i(g_{\mu\sigma} - \hat{P}_\mu \hat{P}_\sigma)\Sigma_{\nu\rho} + i(g_{\nu\rho} - \hat{P}_\nu \hat{P}_\rho)\Sigma_{\mu\sigma} .
\end{aligned} \tag{5.3}$$

Taking note of Rohrlich's choice of metric, with $P^2 = -M^2$, we see that this set of relations agrees with the covariant noncanonical set of "exterior" algebra for the Rohrlich string [see Ref. 7(c)]. Also in agreement with Rohrlich's results are the commutation relations of the relative (internal) variables with the system's center-of-mass momentum:

$$[P_\mu, d_\nu] = 0, \quad [P_\mu, \dot{d}_\nu] = 0 , \tag{5.4}$$

and the commutation relations of the relative variables with the system's center operator:

$$[b_\mu,d_\nu]=id_\mu P_\nu M^{-2},\ \ [b_\mu,\hat{d}_\nu]=i\hat{d}_\mu P_\nu M^{-2}\ . \qquad (5.5)$$

Equations (5.4) display the fact that the relative variables are translationally invariant while Eqs. (5.5) show the nontrivial mixing between the relative (internal) position and the system's (external) center operator.

The "internal" structure for the system may be expressed by computing the commutation relations of the relative position operators, d_μ and $\hat{d}_\mu$:

$$\begin{aligned}
&[d_\mu,d_\nu]=-i\Sigma_{\mu\nu}M^{-2}\ ,\\
&[d_\mu,\hat{d}_\nu]=-i\lambda^2(g_{\mu\nu}-2\hat{P}_\mu\hat{P}_\nu)(\hat{P}\cdot\Gamma)^2M^{-2}+i\lambda^2\Gamma_\mu\Gamma_\nu M^{-2}-\frac{\lambda^2}{2}\Sigma_{\mu\nu}M^{-2}-i\lambda^2[(\hat{P}\cdot\Gamma)\Gamma_\nu\hat{P}_\mu+\Gamma_\mu\hat{P}_\nu(\hat{P}\cdot\Gamma)]M^{-2}\ ,\\
&[\hat{d}_\mu,\hat{d}_\nu]=-i\lambda^4\Sigma_{\mu\nu}[\tfrac{1}{4}+2(\hat{P}\cdot\Gamma)^2]M^{-2}\ ,
\end{aligned} \qquad (5.6)$$

which indicate a high degree of noncommutativity and display a drastic departure from Rohrlich's set of noncanonical commutation relations for his internal variables where

$$\begin{aligned}
&[\xi^\mu,\xi^\nu]=0\ ,\\
&[\xi^\mu,\Pi^\nu]=i(g^{\mu\nu}-P^\mu P^\nu/P^2)\ ,\\
&[\Pi^\mu,\Pi^\nu]=0\ .
\end{aligned} \qquad (5.7)$$

The particular form of the commutation relations of Eq. (5.6) can be shown to be a direct consequence of both the rotatorlike character of the QRR Hamiltonian [i.e., bilinear in $(\hat{P}\cdot\Gamma)$] and the rich structure imposed upon the model by way of its relativistic symmetry [in particular, the choice of the anti–de Sitter $SO(3,2)_{\Gamma_\mu,S_{\mu\nu}}$ as the spectrum generating group which is responsible for the nontrivial internal dynamics and the mass-spin spectrum].

VI. FUNDAMENTAL MODE OPERATORS

It has been shown that an explicit τ-dependent expression for the QRR particle position may be written as

$$Q_\mu(\tau)=b_\mu+\hat{P}_\mu\tau-d_\mu(\tau)\ , \qquad (6.1)$$

where the center-of-mass operator is

$$Y_\mu(\tau)=b_\mu+\hat{P}_\mu\tau\ , \qquad (6.2)$$

and the relative position operator is

$$d_\mu(\tau)=-\frac{i}{M}\exp\left[-i\frac{\lambda^2}{2M}[1+2(\hat{P}\cdot\Gamma)]\tau\right]\hat{A}_\mu+\frac{i}{M}\exp\left[i\frac{\lambda^2}{2M}[-1+2(\hat{P}\cdot\Gamma)]\tau\right]\hat{B}_\mu\ . \qquad (6.3)$$

We now obtain explicit forms for the integration constants $\hat{A}_\mu$ and $\hat{B}_\mu$ in terms of the QRR variables. From Eq. (6.1) evaluated at $\tau=0$, we have that

$$\begin{aligned}
Q_\mu(0)&=b_\mu+\frac{i}{M}\hat{A}_\mu-\frac{i}{M}\hat{B}_\mu\\
&=Y_\mu(0)+\frac{i}{M}\hat{A}_\mu-\frac{i}{M}\hat{B}_\mu\ ,
\end{aligned} \qquad (6.4)$$

where we have used Eqs. (6.2) and (6.3). Also, from the definition of the center of mass, Eq. (2.14), evaluated at $\tau=0$ we obtain

$$Q_\mu(0)=Y_\mu(0)-d_\mu(0)\ . \qquad (6.5)$$

Equating Eq. (6.4) with Eq. (6.5) yields

$$\frac{\hat{B}_\mu}{M}=\frac{\hat{A}_\mu}{M}-i\hat{d}_\mu\ . \qquad (6.6)$$

From Eqs. (6.1) and (6.3) we now calculate the first τ derivative of $Q_\mu(\tau)$ evaluated at $\tau=0$:

$$\dot{Q}_\mu(0)=\hat{P}_\mu+\frac{\lambda^2}{2M^2}[1+2(\hat{P}\cdot\Gamma)]\hat{A}_\mu+\frac{\lambda^2}{2M^2}[-1+2(\hat{P}\cdot\Gamma)]\hat{B}_\mu\ . \qquad (6.7)$$

This form of $\dot{Q}_\mu(0)$ may be equated with the expression for $\dot{Q}_\mu(0)$ obtained from Eq. (3.2) where

$$\dot{Q}_\mu(0)=\hat{P}_\mu+\frac{\lambda^2}{M^2}\left[(\hat{P}\cdot\Gamma)^2\hat{P}_\mu-(\hat{P}\cdot\Gamma)\Gamma_\mu(0)+i\frac{\hat{d}_\mu(0)}{2}\right]\ , \qquad (6.8)$$

to give the following equality:

$$\hat{P}_\mu+\frac{\lambda^2}{2M^2}[1+2(\hat{P}\cdot\Gamma)]\hat{A}_\mu+\frac{\lambda^2}{2M^2}[-1+2(\hat{P}\cdot\Gamma)]\hat{B}_\mu=\hat{P}_\mu+\frac{\lambda^2}{M^2}\left[(\hat{P}\cdot\Gamma)^2\hat{P}_\mu-(\hat{P}\cdot\Gamma)\Gamma_\mu(0)+i\frac{\hat{d}_\mu(0)}{2}\right]\ . \qquad (6.9)$$

We now substitute Eq. (6.6) into this expression to obtain

$$\hat{A}_\mu=\frac{(\hat{P}\cdot\Gamma)\hat{P}_\mu}{2}-\frac{\Gamma_\mu(0)}{2}+i\frac{\hat{d}_\mu(0)}{2}\ . \qquad (6.10)$$

This form for $\hat{A}_\mu$ is now substituted back into Eq. (6.6) to yield

$$\hat{B}_\mu=\frac{(\hat{P}\cdot\Gamma)\hat{P}_\mu}{2}-\frac{\Gamma_\mu(0)}{2}-i\frac{\hat{d}_\mu(0)}{2}=\hat{A}_\mu^\dagger\ . \qquad (6.11)$$

From the orthogonality relation $\hat{P}\cdot\hat{d}=0$, and the explicit τ-dependent form of $d_\mu(\tau)$ given by Eq. (6.3), we anticipate the fact that

$$\hat{P}\cdot\hat{A}=\hat{P}\cdot\hat{A}^\dagger=0 \qquad (6.12)$$

(i.e., the mode operators are spacelike four-vectors), which is seen to hold by analyzing Eqs. (6.10) and (6.11) using $\hat{P}\cdot\dot{\hat{d}}=0$. Therefore for the model of the QRR we have, as a consequence of the definition of the relative position variables, an orthogonality relation which directly leads to an operator identity, Eq. (6.12), ensuring the elimination of ghost states. This method of ghost elimination is identical to that of the Rohrlich string where the application of the timelike gauge in the center-of-mass frame demands $P\cdot\xi=0$ which implies $P\cdot a_n=0$, $n>0$, guaranteeing a positive-definite Hilbert space.

Using the explicit forms of the QRR mode operators $\hat{A}_\mu$ and $\hat{A}^\dagger_\mu$ we may now write a detailed solution to the particle position equation of motion which takes on the following form:

$$Q_\mu(\tau)=b_\mu+\hat{P}_\mu\tau+\frac{i}{M}\exp\left[-i\frac{\lambda^2}{2M}[1+2(\hat{P}\cdot\Gamma)]\tau\right]\hat{A}_\mu -\frac{i}{M}\exp\left[i\frac{\lambda^2}{2M}[-1+2(\hat{P}\cdot\Gamma)]\tau\right]\hat{A}^\dagger_\mu\ , \quad (6.13)$$

where the center of mass $Y_\mu(\tau)=b_\mu+\hat{P}_\mu\tau$ and b_μ is the Finkelstein center operator. This expression has the same form as the general solution to the particle position equation of motion for the quantum relativistic string where

$$x^\mu(\sigma,\tau)=Q^\mu+\frac{P^\mu\tau}{\pi}+\frac{i}{\sqrt{\pi}}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\alpha_n^\mu}{n}e^{-in\tau}\cos n\sigma \quad (6.14)$$

[where π is a constant of dimension (mass)^2], when the string's excitations are restricted to the lowest mode (where $n=\pm1$) and one considers the dynamics at one end point [i.e., $\sigma=0$ where $x^\mu(\sigma,\tau)$ characterizes individual positions along the surface with $\tau_1\le\tau\le\tau_2$ and $0\le\sigma\le L$, where L is the string's length]. This similarity becomes transparent when Eq. (6.13) is written in the following alternate form:

$$Q_\mu(\tau)=b_\mu+\hat{P}_\mu\tau+\frac{i}{M}\exp\left[-i\frac{\lambda^2}{2M}\tau\right]\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{\exp\left[-in\frac{\lambda^2}{M}(\hat{P}\cdot\Gamma)\tau\right]}{n}\hat{A}_{\mu n} \quad (6.15)$$

(where for the QRR we must restrict the modes to $n=\pm1$ and $\hat{A}^\dagger_{\mu 1}\equiv\hat{A}_{\mu-1}$).

The QRR mode operators may be expressed in terms of the relative position variables by using Eq. (3.5) where

$$\dot{\hat{d}}_\mu(\tau)=\frac{\lambda^2}{M}(\hat{P}\cdot\Gamma)\Gamma_\mu(\tau)-\frac{\lambda^2}{M}(\hat{P}\cdot\Gamma)^2\hat{P}_\mu -i\frac{\lambda^2}{2M}\hat{d}_\mu(\tau)\ . \quad (6.16)$$

From this expression, evaluated at $\tau=0$, we obtain

$$\Gamma_\mu(0)=\left[\frac{\lambda^2}{M}(\hat{P}\cdot\Gamma)\right]^{-1}\dot{\hat{d}}_\mu(0)+(\hat{P}\cdot\Gamma)\hat{P}_\mu +\frac{i}{2}(\hat{P}\cdot\Gamma)^{-1}\hat{d}_\mu(0)\ . \quad (6.17)$$

We now insert this expression into Eq. (6.10) to obtain

$$\hat{A}_\mu=-\frac{M}{2\lambda^2(\hat{P}\cdot\Gamma)}\{\dot{\hat{d}}_\mu(0)-i\lambda^2[(\hat{P}\cdot\Gamma)-\tfrac{1}{2}]d_\mu(0)\} \quad (6.18a)$$

(where we have used the fact that $\hat{d}_\mu=d_\mu M$). By inserting Eq. (6.17) into Eq. (6.11) we find

$$\hat{A}^\dagger_\mu=-\frac{M}{2\lambda^2(\hat{P}\cdot\Gamma)}\{\dot{\hat{d}}_\mu(0)+i\lambda^2[(\hat{P}\cdot\Gamma)+\tfrac{1}{2}]d_\mu(0)\}\ . \quad (6.18b)$$

Equations (6.18a) and (6.18b) display forms for the non-Hermitian QRR mode operators which are reminiscent of those for the creation and annihilation operators of the harmonic oscillator (or the lowest mode of the relativistic string) where

$$a^{\mu\dagger}=\frac{1}{\sqrt{2\kappa}}(\Pi^\mu+i\kappa\xi^\mu) \quad (6.19a)$$

and

$$a^\mu=\frac{1}{\sqrt{2\kappa}}(\Pi^\mu-i\kappa\xi^\mu)\ , \quad (6.19b)$$

which (for the string model under consideration) fulfill the following covariant noncanonical commutation relations:

$$[a_m^\mu,a_n^\nu]=0=[a_m^{\mu\dagger},a_n^{\nu\dagger}]\ ,$$
$$[a_m^\mu,a_n^{\nu\dagger}]=\left[g^{\mu\nu}-\frac{P^\mu P^\nu}{P^2}\right]\delta_{m,n}\quad (m,n>0)\ . \quad (6.20)$$

But the QRR mode operators satisfy the following set of commutation relations:[16]

$$[\hat{A}_\mu,\hat{A}_\nu]=[\hat{A}^\dagger_\mu,\hat{A}^\dagger_\nu]=0\ ,$$
$$[\hat{A}_\mu,\hat{A}^\dagger_\nu]=-\tfrac{1}{2}(g_{\mu\nu}-\hat{P}_\mu\hat{P}_\nu)(\hat{P}\cdot\Gamma)-\frac{i}{2}\Sigma_{\mu\nu}\ , \quad (6.21)$$

which does not agree with the covariant noncanonical algebra of Eq. (6.20). Therefore, the $\hat{A}_\mu$ and $\hat{A}^\dagger_\mu$ are not the "usual" creation and annihilation operators of the harmonic-oscillator type. However, they do play a similar role in that they serve as the fundamental ladder operators acting on spin states of the QRR.

We may clarify this fact by evaluating the commutation relations of the QRR mode operators with the $(\hat{P}\cdot\Gamma)$:

$$[(\hat{P}\cdot\Gamma),\hat{A}_\mu]=-\hat{A}_\mu,\quad [(\hat{P}\cdot\Gamma),\hat{A}_\mu^\dagger]=\hat{A}_\mu^\dagger\,, \tag{6.22a}$$

and using Eq. (6.21):

$$[\hat{A}_\mu,\hat{A}^{\mu\dagger}]=-\tfrac{3}{2}(\hat{P}\cdot\Gamma)\,. \tag{6.22b}$$

Since the $(\hat{P}\cdot\Gamma)$ have the spectrum n $(=s+\frac{1}{2}$ for the special case of the Majorana representation), it is evident from Eq. (6.22a) that the basic action of the QRR mode operators is to raise and lower spin for this particular representation.

VII. CONCLUDING REMARKS

It has been shown that the models of the QRR and the covariant noncanonical treatment of the Rohrlich string utilize identical techniques in their basic construction which result in many similarities between the two (despite the fact that they ultimately account for the hadron spectrum in two distinct manners).

On one hand, we have the quantum relativistic string which explains the spectrum in terms of arbitrary-dimensional harmonic-oscillator states [where, for the special case of the lowest mode of excitation, the spin spectrum is a result of a three-dimensional oscillator at rest, see Ref. 7(c)]. Its relativistic Hamiltonian is obtained by quantizing the orthogonal gauge constraint which leads to (when restricted to the lowest mode) a linear mass-spin relationship of the form [see Refs. 7(a)—7(c)]

$$m^2={m_0}^2+\frac{1}{\alpha'}s \tag{7.1}$$

(i.e., the usual Regge trajectories). The arbitrary constant ${m_0}^2$ is a consequence of the normal-ordering procedure and determines the energy of the unexcited string which is identified with the vacuum. Since the string is a physical system (with P_μ timelike), $m^2\geq 0$ which requires ${m_0}^2\geq 0$.

On the other hand, the QRR explains the hadron spectrum in terms of a set of infinite "towers" which consist of an arbitrary number of rotational states [where the spin spectrum comes from the relativistic spectrum generating group, SO(3,2)]. The Hamiltonian for the QRR is obtained by imposing a constraint on the second-order Casimir operator of a de Sitter SO(4,1) which leads to a rotatorlike mass-spin spectrum [see Refs. 8(a)—8(c)]:

$$m^2=\lambda^2(\alpha^2-\tfrac{9}{4})+\lambda^2 s(s+1)\,. \tag{7.2}$$

The constant $\lambda^2(\alpha^2-\frac{9}{4})$ determines the energy of the unexcited system which may be identified with the vacuum. The requirement that we are dealing with a physical system implies the restriction that $m^2\geq 0$ which in turn demands $\lambda^2(\alpha^2-\frac{9}{4})\geq 0$.

Regardless of the difference in how they account for the hadron spectrum, we obtained the following basic elements which both models have in common.

(a) a unique specification of the system's evolution parameter τ through the use of a timelike gauge in the center-of-mass frame;

(b) an operator identity restricting the relative position variables to spacelike oscillations only which directly leads to the elimination of ghost states;

(c) identical "external" structures as obtained from the covariant noncanonical algebraic relations involving the center operators, center-of-mass momenta and spin tensors;

(d) identical commutation relations for the relative position variables with the system's center operators and also with the center-of-mass momenta (the relative variables are translationally invariant);

(e) both models may be expressed in terms of fundamental modes where for the quantum relativistic string, $a_\mu^\dagger$ and a_μ are the usual creation and annihilation operators of the harmonic oscillator type and for the QRR, the $A_\mu^\dagger$ and A_μ are ladder operators acting on the spin; and

(f) the elimination of tachyons is ensured: by the appearance of an arbitrary constant ${m_0}^2$ (due to the normal-ordering procedure) for the Rohrlich string and by restricting ourselves to the principal series representations of SO(4,1) which demands, $\alpha^2\geq\frac{9}{4}$, for the QRR.

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