Renormalization group and nonlocal terms in the curved-spacetime effective action: Weak-field results

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Renormalization-group methods are used to study interacting quantum field theory in curved spacetime. Details are presented explicitly showing how the short-distance behavior of the theory is linked to the existence of nonlocal terms in the effective action. We show how the usual value for the running electric charge in QED in flat spacetime may be obtained in this way by using the background-field method. A similar technique is then used to find nonlocal terms in the effective action arising from a real scalar field on a weakly curved gravitational background. It is shown how the results for the effective gravitational coupling constants obtained from analysis of these nonlocal terms agree with those found in our previous work.

I. INTRODUCTION

In recent papers[1,2] we have analyzed the renormalization-group behavior of interacting field theories in curved spacetime. We were particularly interested in the short-distance limit of the coupling constants which involve the curvature, and hence are not present in flat-spacetime quantum field theory. Examples of such coupling constants are provided by those which multiply curvature-squared terms in the generalized Einstein-Hilbert gravitational action, and those which link the Higgs scalar bosons to the curvature.

The method which we used was the curved-spacetime renormalization-group analysis. As originally introduced,[3] this method involved looking at what happened to the Green's functions under a rescaling $g_{\mu\nu}\rightarrow s^{-2}g_{\mu\nu}$ of the background metric. Increasing the dimensionless parameter s probes the short-distance behavior of the theory. In flat spacetime, this corresponds to the limit of high momentum; however, for a general curved spacetime, there is no natural global definition of momentum space (although a local momentum space may be introduced[4]). The curved-spacetime momentum-space method was later applied to the effective action,[5] and used extensively in Refs. 1 and 2. This approach is also adopted in the present paper.

Details of the calculations in Refs. 1 and 2 involved a computation of the pole part of the one-loop effective action using the background-field method[6] and dimensional regularization.[7] Once the pole terms were obtained, the renormalization was effected by adding the appropriate counterterms which were also expressed as pure pole terms (minimal subtraction). A standard renormalization-group analysis[8] was then applied to find out the behavior of the coupling constants at short distance. This enabled the short-distance limit of the effective action to be inferred.

The present paper is one of two which seeks to clarify the short-distance limit of the effective action as obtained in our earlier work. The origin of the short-distance behavior will be shown explicitly to lie in the fact that nonlocal terms are present in the effective action. (By nonlocal, we mean those which cannot be expressed as any finite polynomial with constant coefficients in the metric and its derivatives.) Because it is impossible to obtain an explicit expression for the effective action, approximation techniques must be resorted to. The one used in this paper involves looking at the weak-field expansion of the effective action, and is described in Sec. III below. The companion paper[9] uses the new form for the coincidence limit of the heat kernel found in Ref. 10 (see also Ref. 11) to calculate nonlocal terms in strong gravitational fields. Although the precise form of the nonlocal terms found are different with these differing methods, they lead to the same behavior for the effective action at short distance. A possible way of combining the two procedures is discussed in Sec. IV.

Related nonlocal terms in the effective action may be found in Refs. 12 and 13. The relationship between such terms, the renormalization group, and the effective coupling constants has not been discussed previously to our knowledge.

The outline of our paper is as follows. In Sec. II we discuss charge renormalization in QED from the background-field viewpoint adopted in this paper. We demonstrate how the existence of a nonlocal term in the effective action is responsible for the running behavior of the effective charge obtained using minimal subtraction. The third section contains our analysis of the nonlocal terms in the effective action which are responsible for the running behavior of the gravitational coupling constants. The details are presented for a real scalar field in a weak gravitational background. The final section gives a discussion of the results. A number of technical details may be found in the appendices.

II. CHARGE RENORMALIZATION IN QED AND NONLOCAL TERMS IN THE EFFECTIVE ACTION

In this section we wish to sketch the derivation of charge renormalization in QED in flat spacetime using the background-field method. The reason for including this material is to illuminate the subsequent sections which are concerned with renormalization on a curved background spacetime in a situation which is probably more familiar to the reader. There is a close parallel between the nonlocal terms in the effective action for QED which give rise to the running of the effective charge, and the nonlocal terms in the effective action for quantum field theory on a curved background spacetime which are responsible for the running of the effective gravitational coupling constants.

The bare Lagrangian for a Dirac spinor field $\psi(x)$ interacting with the electromagnetic field $A_\mu(x)$ will be taken as

$$\mathscr{L}(x) = -\tfrac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} + \bar{\psi}_B(\partial\!\!\!/ - im_B)\psi_B + ie_B\bar{\psi}_B A\!\!\!/_B\psi_B \ . \tag{2.1}$$

The subscript B denotes a bare quantity. The spinor fields are treated as anticommuting, and the choice of factors of i in (2.1) ensures that the classical action is real. We will adopt dimensional regularization[7] and work in n spacetime dimensions.

The background-field method[6] involves the replacement of all fields in the classical action by the sum of a background-field part and a quantum part. For the Lagrangian in Eq. (2.1) we may define renormalized background fields $\hat{A}_\mu(x)$, $\hat{\psi}(x)$ and a renormalized mass m, and charge e in terms of the bare ones by

$$\hat{A}_B^{\mu}(x) = \mu^{(n-4)/2} Z_A^{\ 1/2} \hat{A}^{\mu}(x) \ , \tag{2.2}$$

$$\hat{\psi}_B(x) = \mu^{(n-4)/2} Z_\psi^{\ 1/2} \hat{\psi}(x) \ , \tag{2.3}$$

$$m_B = Z_m m \ , \tag{2.4}$$

$$e = \mu^{-(n-4)/2} Z_e e \ . \tag{2.5}$$

The unit of mass μ has been introduced here so that the renormalized quantities occurring on the right-hand side (RHS) of Eqs. (2.2)—(2.5) have the same dimensions for all n as they do for $n=4$. (See Ref. 8.)

Because this section serves merely to illustrate our method in an easier and more familiar setting before proceeding to the gravitational case, we will specialize to the case where only the Dirac spinor field is quantized in the presence of a classical background electromagnetic field. (This is closely analogous to the quantization of matter fields on a classical gravitational background.) Owing to this simplifying assumption, there will be no renormalization of either the spinor field or its mass. Thus the renormalization constants Z_ψ and Z_m appearing in Eqs. (2.3) and (2.4) may be set equal to unity. The reason for this (using the language of Feynman diagrams) is that the contributions to Z_ψ and Z_m come from diagrams which contain internal photon lines which are absent if the gauge field is treated classically. Only a charge renormalization and a renormalization of the background gauge field need to be performed. The renormalization constants are restricted by the requirement of gauge invariance to satisfy

$$Z_e Z_A^{\ 1/2} = 1 \ . \tag{2.6}$$

This result follows most easily using the gauge-invariant background-field method.[14]

With the adoption of the background-field method we may write

$$I = \hat{I} + I_Q \ , \tag{2.7}$$

where

$$\hat{I} = \mu^{n-4} \int d^n x \left[-\tfrac{1}{4} Z_A \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \hat{\bar{\psi}}(\partial\!\!\!/ - im)\hat{\psi} + ie\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi} \right] \tag{2.8}$$

and

$$I_Q = \int d^n x \left[\bar{\psi}(\partial\!\!\!/ - im)\psi + ie\bar{\psi}\hat{A\!\!\!/}\psi \right] . \tag{2.9}$$

I_Q contains terms which are quadratic in the quantum fields. (Terms linear in the quantum fields may be ignored since they will not contribute to the effective action.[15]) The background fields will be taken to have no dependence on the $(n-4)$ extra coordinates. Thus in Eq. (2.8), the integral $\int d^n x$ will involve a factor of $(\text{length})^{n-4}$ or $(\text{mass})^{4-n}$ coming from the volume associated with the extra dimensions. We will choose μ to be associated with this volume so that (2.8) becomes

$$\hat{I} = \int d^4 x \left[-\tfrac{1}{4} Z_A \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} + \hat{\bar{\psi}}(\partial\!\!\!/ - im)\hat{\psi} + ie\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi} \right] . \tag{2.10}$$

The partition function Z is defined by

$$Z = \int d\mu[\bar{\psi},\psi]\exp(iI) \ , \tag{2.11}$$

where $d\mu[\bar{\psi},\psi]$ represents the functional measure for the quantum part of the fermion fields, and I is given in Eq. (2.7). The effective action may be obtained from Z by performing a functional Legendre transformation.[15] It proves convenient to regard the term in $\hat{A}$ which appears in I_Q as an interaction term I_{int},

$$I_{\text{int}} = ie \int d^n x \ \bar{\psi}(x)\hat{A\!\!\!/}(x)\psi(x) \ . \tag{2.12}$$

Then the effective action is given by

$$\Gamma = \hat{I} + \Gamma^{(1)} \ , \tag{2.13}$$

where $\Gamma^{(1)}$ is the one-loop part defined by

$$\Gamma^{(1)} = -i\langle e^{iI_{\text{int}}} \rangle \ . \tag{2.14}$$

Here the angular brackets $\langle \cdots \rangle$ mean that only one-particle-irreducible graphs (which in the present case are simply connected graphs) are to be kept in the Wick reduction of the enclosed expression. The exponential may be expanded to give

$$\Gamma^{(1)}=-i\sum_{k=0}^{\infty}\frac{(-e)^k}{k!}\times\int d^nx_1\cdots d^nx_k\langle\bar{\psi}(x_1)\hat{A}(x_1)\psi(x_1)\cdots\times\bar{\psi}(x_k)\hat{A}(x_k)\psi(x_k)\rangle\ . \quad (2.15)$$

This expression may be evaluated by using the usual rules for integration over anticommuting variables.[16] The two-point function $\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle$, where α and β denote four-component spinor indices, is given in terms of the fermion propagator by

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(x')\rangle=iS_{\alpha\beta}(x,x')\ . \quad (2.16)$$

The fermion propagator satisfies (matrix notation)

$$(\partial\!\!\!/_x-im)S(x,x')=\delta(x,x')\ , \quad (2.17)$$

where $\delta(x,x')$ is the n-dimensional Dirac δ distribution. The momentum-space representation for the propagator is

$$S(x,x')=-i\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-x')}\frac{(p\!\!\!/+m)}{(p^2-m^2+i\epsilon)}\ , \quad (2.18)$$

where causal (or Feynman) boundary conditions have been adopted.

The $k=0$ term in Eq. (2.15) gives a vacuum contribution which may be ignored in flat Minkowski spacetime. The $k=1$ term involves

$$\langle\bar{\psi}(x)\hat{A}(x)\psi(x)\rangle=-i\,\mathrm{tr}[\hat{A}(x)S(x,x)] \quad (2.19)$$

if (2.16) is used. This may be seen to vanish using the momentum-space representation (2.18) and the fact that the γ matrices are traceless. In fact, all terms in Eq. (2.15) with k odd will vanish. This is a consequence of Furry's theorem.[17] Thus, the first nontrivial term in $\Gamma^{(1)}$ is

$$\Gamma^{(1)}=\frac{i}{2}e^2\int d^nx\,d^nx'\hat{A}^{\mu}(x)\Pi_{\mu\nu}(x,x')\hat{A}^{\nu}(x')+O(\hat{A}^4)\ , \quad (2.20)$$

where

$$\Pi_{\mu\nu}(x,x')=-\mathrm{tr}[\gamma_\mu S(x,x')\gamma_\nu S(x',x)]\ . \quad (2.21)$$

$\Pi_{\mu\nu}$ is the usual vacuum polarization tensor for QED. It is easily evaluated to be

$$\Pi_{\mu\nu}(x,x')=i(4\pi)^{-2}\Gamma(2-n/2)2^{1+n/2}(\partial_\mu\partial_\nu-\eta_{\mu\nu}\Box)\times\int_0^1 dt\,t(1-t)\left[\frac{m^2+t(1-t)\Box}{4\pi}\right]^{n/2-2}\delta(x,x')\ . \quad (2.22)$$

Because the background gauge field has no dependence on the extra spatial coordinates, it is easily shown that when the integration over the $(n-4)$ extra dimensions is performed in (2.20) using (2.22), a factor of μ^{4-n} arises from the volume of the extra dimensions. In addition, the components of momenta in the directions associated with the extra dimensions get set equal to zero. Expansion of (2.22) about the pole at $n=4$ then leads to

$$\Gamma^{(1)}=-\frac{e^2}{24\pi^2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]\times\int d^4x\,\hat{A}^{\mu}(x)(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)\hat{A}^{\nu}(x)-\frac{e^2}{4\pi^2}\int d^4x\,\hat{A}^{\mu}(x)(\eta_{\mu\nu}\Box-\partial_\mu\partial_\nu)F(\Box)\hat{A}^{\nu}(x)+O(\hat{A}^4)\ , \quad (2.23)$$

where

$$F(\Box)=\int_0^1 dt\,t(1-t)\ln[1+t(1-t)m^{-2}\Box]\ . \quad (2.24)$$

An effective Lagrangian may be defined by

$$\Gamma=\int d^4x\,\mathscr{L}_{\mathrm{eff}}(x)\ . \quad (2.25)$$

$\Gamma^{(1)}$ is easily seen to involve only the field strength $\hat{F}_{\mu\nu}$ after integration by parts. When $\Gamma^{(1)}$ is combined with $\hat{I}$ given in Eq. (10) to form the total effective action [see (2.13)], the effective Lagrangian is found to be

$$\mathscr{L}_{\mathrm{eff}}(x)=-\tfrac{1}{4}\hat{F}^{\mu\nu}\left[Z_A-\frac{e^2}{12\pi^2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]-\frac{e^2}{2\pi^2}F(\Box)\right]\hat{F}_{\mu\nu}+\hat{\bar{\psi}}(\partial\!\!\!/-im)\hat{\psi}+ie\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi}+\cdots\ . \quad (2.26)$$

This is observed to be manifestly gauge invariant. It is also seen to be a nonlocal expression due to the presence of the term which involves $F(\Box)$.

The usual measured value for the electron charge is determined in very-low-energy experiments for which the large-distance behavior is relevant. This leads us to define the electron charge in the limit where $|F(\Box)|\ll 1$. (The fact that this is the large-distance limit may be seen from scaling the background metric $n_{\mu\nu}\to s^{-2}\eta_{\mu\nu}$ and letting s tend to zero.) The field renormalization factor Z_A should then be fixed so that the coefficient of $\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}$ is $-\frac{1}{4}$:

$$Z_A=1+\frac{e^2}{12\pi^2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]\ . \quad (2.27)$$

If we call the value of the electron charge defined in this way e_L (L being mnemonic for low energy or large distance), then

$$\mathscr{L}_{\mathrm{eff}}(x)=-\tfrac{1}{4}\hat{F}^{\mu\nu}\left[1-\frac{e_L{}^2}{2\pi^2}F(\Box)\right]\hat{F}_{\mu\nu}+\hat{\bar{\psi}}(\partial\!\!\!/-im)\hat{\psi}+ie_L\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi}+\cdots \quad (2.28)$$

gives the renormalized effective Lagrangian.

If we now want to identify the electron charge measured at a different scale, then it is necessary to perform a further finite renormalization of the vector field. This is most easily formulated in momentum space, and is

$$\hat{A}^{\mu}(q)\to\left[1-\frac{e_L{}^2}{2\pi^2}F(-q^2)\right]^{-1/2}\hat{A}^{\mu}(q)\ . \tag{2.29}$$

This means that the interaction term [last term in (2.28)] involves a charge $e(q)$, where

$$e^2(q)=e_L{}^2\left[1-\frac{e_L{}^2}{2\pi^2}F(-q^2)\right]^{-1}\ . \tag{2.30}$$

For large values of $|q^2|\gg m^2$,

$$e^2(q)\simeq e_L{}^2\left[1-\frac{e_L{}^2}{12\pi^2}\ln(-q^2/m^2)\right]^{-1}\ . \tag{2.31}$$

This is the standard result for the running electric charge in QED.[18] It agrees with the result found using minimal subtraction. The leading term in the large-distance behavior of the effective Lagrangian may be obtained from Eq. (2.28) by expanding (2.25):

$$F(\Box)\simeq\tfrac{1}{30}m^{-2}\Box\ . \tag{2.32}$$

This gives

$$\mathscr{L}_{\text{eff}}(x)\simeq-\tfrac{1}{4}\hat{F}^{\mu\nu}\left[1-\frac{e_L{}^2}{60\pi^2m^2}\Box\right]\hat{F}_{\mu\nu}+\hat{\bar{\psi}}(\partial\!\!\!/-im)\hat{\psi}$$
$$+ie_L\hat{\bar{\psi}}\hat{A\!\!\!/}\hat{\psi}+\cdots\ . \tag{2.33}$$

The remainder of this paper will be concerned with obtaining analogous results for a scalar field on a gravitational background.

III. RENORMALIZATION OF THE GRAVITATIONAL COUPLING CONSTANTS

Consider a real scalar field with a quartic self-interaction on a gravitational background. The action will be taken to be

$$I=\int d^nx(-g)^{1/2}\left[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi_B\partial_\nu\phi_B-\tfrac{1}{2}m_B{}^2\phi_B{}^2-\tfrac{1}{2}\xi_BR\phi_B{}^2-\frac{\lambda_B}{4!}\phi_B{}^4\right], \tag{3.1}$$

where ϕ_B is the bare field, m_B is the bare mass, and λ_B is the bare quartic self-coupling. The constant ξ_B is a bare quantity which describes a nonminimal coupling to the scalar curvature.

Again the theory will be quantized using the background-field method. Because we are on a curved background spacetime, it is necessary to add in additional gravitational terms to renormalize the theory. The required terms take the form

$$I_G=\int d^nx(-g)^{1/2}(\Lambda_B+\kappa_BR+\alpha_{1B}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+\alpha_{2B}R^{\mu\nu}R_{\mu\nu}+\alpha_{3B}R^2)\ . \tag{3.2}$$

Renormalized quantities are defined by

$$\hat{\phi}_B(x)=\mu^{(n-4)/2}Z_\phi{}^{1/2}\hat{\phi}(x)\ , \tag{3.3}$$
$$m_B{}^2=Z_mm^2\ , \tag{3.4}$$
$$\lambda_B=\mu^{4-n}Z_\lambda\lambda\ , \tag{3.5}$$
$$\xi_B=\xi+\delta\xi\ , \tag{3.6}$$
$$\Lambda_B=\mu^{n-4}(\Lambda+\delta\Lambda)\ , \tag{3.7}$$
$$\kappa_B=\mu^{n-4}(\kappa+\delta\kappa)\ , \tag{3.8}$$
$$\alpha_{iB}=\mu^{n-4}(\alpha_i+\delta\alpha_i)\ \ (i=1,2,3)\ . \tag{3.9}$$

Here $\hat{\phi}(x)$ denotes the background scalar field.

Make the background field split, and as before take the background fields $\hat{\phi}$, $g_{\mu\nu}$ to be independent of the extra dimensions. Because we are only working to one-loop order, we have

$$I=\hat{I}+I_Q\ , \tag{3.10}$$

where

$$\hat{I}=\int d^4x(-g)^{1/2}\left[\tfrac{1}{2}Z_\phi g^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}-\tfrac{1}{2}Z_\phi Z_mm^2\hat{\phi}^2-\tfrac{1}{2}Z_\phi(\xi+\delta\xi)R\hat{\phi}^2-Z_\lambda Z_\phi{}^2\frac{\lambda}{4!}\hat{\phi}^4\right]$$
$$+\int d^4x(-g)^{1/2}[\Lambda+\delta\Lambda+(\kappa+\delta\kappa)R+(\alpha_1+\delta\alpha_1)R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}+(\alpha_2+\delta\alpha_2)R^{\mu\nu}R_{\mu\nu}+(\alpha_3+\delta\alpha_3)R^2] \tag{3.11}$$

and

$$I_Q=\tfrac{1}{2}\int d^nx(-g)^{1/2}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\phi^2-\xi R\phi^2-\frac{\lambda}{2}\hat{\phi}^2\phi^2\right]\ . \tag{3.12}$$

I_Q is the part of the action which gives rise to one-loop effects. As in Sec. II, the linear term has been neglected since it will not give any contribution to the effective action. Also, because we are only working to one-loop order, the counterterms and renormalization constants may be dropped in I_Q.

As explained in the Introduction, we want to examine the nonlocal terms in the effective action which give rise to the running behavior in the gravitational coupling constants. We will do this by making the following weak-field expansion of the metric:

$$g_{\mu\nu}(x)=\eta_{\mu\nu}+h_{\mu\nu}(x) \tag{3.13}$$

and treating $h_{\mu\nu}(x)$ as an external field. This means that flat-spacetime Feynman rules can be used. As we have already mentioned, an alternate method which does not use this expansion is given in Ref. 9. This leads to different nonlocal terms in the effective action.

As with the QED calculation in a background electromagnetic field, terms in $h_{\mu\nu}$ will be treated as interaction terms. We will write

$$I_Q=I_Q^{(0)}+I_{\rm int}\ , \tag{3.14}$$

where

$$I_Q^{(0)}=\tfrac{1}{2}\int d^nx(\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-m^2\phi^2) \tag{3.15}$$

is the free part of the action which defines the scalar field propagator. The interaction part is obtained using results given in Appendix A. We may write

$$I_{\rm int}=I_{\rm int}^{(1)}+I_{\rm int}^{(2)}+\cdots\ , \tag{3.16}$$

where the superscript denotes the power of $h_{\mu\nu}$ which occurs. It is easily shown that

$$I_{\rm int}^{(1)}=-\tfrac{1}{2}\int d^nx\left[(h^{\mu\nu}-\tfrac{1}{2}h\eta^{\mu\nu})\partial_\mu\phi\partial_\nu\phi+\tfrac{1}{2}hm^2\phi^2+\frac{\lambda}{4}h\hat{\phi}^2\phi^2+\tfrac{1}{2}\xi(\Box h)\phi^2\right] \tag{3.17}$$

and

$$I_{\rm int}^{(2)}=\tfrac{1}{2}\int d^nx\left[[h^{\mu\lambda}h_\lambda{}^\nu-\tfrac{1}{2}hh^{\mu\nu}-\tfrac{1}{4}(h^{\lambda\sigma}h_{\lambda\sigma}-\tfrac{1}{2}h^2)\eta^{\mu\nu}]\partial_\mu\phi\partial_\nu\phi-(\tfrac{1}{8}h^2-\tfrac{1}{4}h^{\mu\nu}h_{\mu\nu})\left[m^2+\frac{\lambda}{2}\hat{\phi}^2\right]\phi^2\right.$$
$$\left.+\xi(h^{\mu\nu}\Box h_{\mu\nu}+\tfrac{3}{4}h^{\mu\nu,\lambda}h_{\mu\nu,\lambda}-\tfrac{1}{2}h^{\mu\nu,\lambda}h_{\mu\lambda,\nu}-\tfrac{1}{4}h\Box h)\phi^2\right]\ . \tag{3.18}$$

The one-loop contribution to the effective action, expressed in powers of $h_{\mu\nu}$, may be obtained from

$$\Gamma=-i\langle e^{iI_{\rm int}}\rangle\ . \tag{3.19}$$

(The term independent of h is ignored here.) The Wick reduction is obtained using

$$\langle\phi(x)\phi(x')\rangle=i\Delta(x-x')\ , \tag{3.20}$$

where

$$(\Box_x+m^2-i\epsilon)\Delta(x-x')=-\delta(x-x') \tag{3.21}$$

defines the flat-spacetime propagator.

A. ξ renormalization

The ξ renormalization is fixed by terms in the one-loop effective action of the form $R\hat{\phi}^2$. In the weak-field expansion this means that the lowest-order nonvanishing term is of order $h\hat{\phi}^2$. It is easy to see that the relevant part of the one-loop effective action is

$$\Gamma_\xi^{(1)}=-\frac{\lambda}{8}\int d^nx\,h(x)\hat{\phi}^2(x)\langle\phi^2(x)\rangle$$
$$+\frac{i}{8}\lambda\int d^nx\,d^nx'\hat{\phi}^2(x')\{[h^{\mu\nu}(x)-\tfrac{1}{2}\eta^{\mu\nu}h(x)]\langle\phi^2(x')\partial_\mu\phi(x)\partial_\nu\phi(x)\rangle$$
$$+\tfrac{1}{2}m^2h(x)\langle\phi^2(x)\phi^2(x')\rangle+\tfrac{1}{2}\xi[\Box h(x)]\langle\phi^2(x)\phi^2(x')\rangle\}\ . \tag{3.22}$$

This result follows directly from expanding the exponential in (3.19) and keeping the relevant terms. Using Wick's theorem and (3.20) gives

$$\langle\phi^2(x)\phi^2(x')\rangle=-2\Delta^2(x-x')\ , \tag{3.23}$$

$$\langle\phi^2(x')\partial_\mu\phi(x)\partial_\nu\phi(x)\rangle=-2\partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x')\ . \tag{3.24}$$

Note that the prescription for T^* products has been employed here. Then

$$\Gamma_\xi^{(1)}=-\frac{i}{8}\lambda\int d^nx\,h(x)\Delta(0)\hat{\phi}^2(x)$$
$$-\frac{i}{4}\lambda\int d^nx\,d^nx'\hat{\phi}^2(x')\{[h^{\mu\nu}(x)-\tfrac{1}{2}h(x)\eta^{\mu\nu}]\partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x')+\tfrac{1}{2}m^2h(x)\Delta^2(x-x')$$
$$+\tfrac{1}{2}\xi[\Box h(x)]\Delta^2(x-x')\}\ . \tag{3.25}$$

The products of Green's functions which occur in (3.25) may be evaluated by introducing the Fourier transform

$$\Delta(x-x')=\int \frac{d^nk}{(2\pi)^n}e^{ik\cdot(x-x')}(k^2-m^2+i\epsilon)^{-1} . \tag{3.26}$$

A number of relevant integrals are listed in Appendix B. Everything may be expanded about $n=4$ to identify the pole term and finite term. It is necessary to separate the finite term into two pieces. One involves only non-negative powers of masses and is local. The second piece is a nonlocal expression which, if it is expanded in powers of m^2, begins at order m^{-2}. This procedure leads to

$$\Delta^2(x-x')=-i(4\pi)^{-2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]\delta(x-x')+F_1(x-x') , \tag{3.27}$$

$$m^2\Delta^2(x-x')=-im^2(4\pi)^{-2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]\delta(x-x')-\frac{i}{6}(4\pi)^{-2}\Box\delta(x-x')+F_2(x-x') , \tag{3.28}$$

$$\begin{aligned}\partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x')=&-\frac{i}{2}m^2(4\pi)^{-2}\eta_{\mu\nu}[2(n-4)^{-1}+\gamma-1+\ln(m^2/4\pi\mu^2)]\delta(x-x')\\&-\frac{i}{12}(4\pi)^{-2}\eta_{\mu\nu}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]\Box\delta(x-x')\\&-\frac{i}{6}(4\pi)^{-2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)]\partial_\mu\partial_\nu\delta(x-x')+F_{3\mu\nu}(x-x') ,\end{aligned} \tag{3.29}$$

where

$$F_1(x-x')=-i(4\pi)^{-2}\int_0^1 dt\,\ln[1+t(1-t)m^{-2}\Box]\delta(x-x') , \tag{3.30}$$

$$F_2(x-x')=-i(4\pi)^{-2}\int_0^1 dt\{m^2\ln[1+t(1-t)m^{-2}\Box]-\tfrac{1}{6}\Box\}\delta(x-x') , \tag{3.31}$$

$$\begin{aligned}F_{3\mu\nu}(x-x')=&-\frac{i}{2}(4\pi)^{-2}\eta_{\mu\nu}\int_0^1 dt\,t(1-t)\ln[1+t(1-t)m^{-2}\Box]\Box\delta(x-x')\\&-i(4\pi)^{-2}\int_0^1 dt\,t(1-t)\ln[1+t(1-t)m^{-2}\Box]\partial_\mu\partial_\nu\delta(x-x')+\tfrac{1}{2}\eta_{\mu\nu}F_2(x-x') .\end{aligned} \tag{3.32}$$

These results may be substituted into (3.25) and the result simplified using the gauge condition (A5). Note also that

$$\eta_{\mu\nu}(h^{\mu\nu}-\tfrac{1}{2}h\eta^{\mu\nu})=[-1-\tfrac{1}{2}(n-4)]h \tag{3.33}$$

must be used. After a straightforward calculation it is found that

$$\begin{aligned}\Gamma_\xi^{(1)}=&-\tfrac{1}{8}\lambda m^2(4\pi)^{-2}[2(n-4)^{-1}+\gamma-1+\ln(m^2/4\pi\mu^2)]\int d^4x\,h(x)\hat{\phi}^2(x)\\&-\tfrac{1}{8}\lambda(4\pi)^{-2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)](\xi-\tfrac{1}{6})\int d^4x\,\hat{\phi}^2(x)\Box h(x)-\tfrac{1}{8}\lambda(4\pi)^{-2}\int d^4x\,\hat{\phi}^2(x)G(\Box)\Box h(x) ,\end{aligned} \tag{3.34}$$

where

$$G(\Box)=\int_0^1 dt[\xi-t(1-t)]\ln[1+t(1-t)m^{-2}\Box] . \tag{3.35}$$

The complete terms of order $h\hat{\phi}^2$ in the one-loop effective action also involve contributions coming from $\hat{I}$ as given in Eq. (3.11). Noting that no field renormalization is required at one-loop order (i.e., $Z_\phi=1$), the needed terms are found to be

$$\begin{aligned}\hat{I}_\xi=-\tfrac{1}{2}\int d^4x\{&(h^{\mu\nu}-\tfrac{1}{2}h\eta^{\mu\nu})\partial_\mu\hat{\phi}\partial_\nu\hat{\phi}\\&+\tfrac{1}{2}m^2(1+\delta Z_m)h(x)\hat{\phi}^2(x)\\&+\tfrac{1}{2}(\xi+\delta\xi)[\Box h(x)]\hat{\phi}^2(x)\} ,\end{aligned} \tag{3.36}$$

where we have written $Z_m=1+\delta Z_m$. In order to identify the physical values of the coupling constants we must describe how to fix the counterterms. Just as in the case of charge renormalization in QED, we will identify the physical coupling constants to be those measured at large distances (or low momenta). With this choice, the counterterms δZ_m and $\delta\xi$ will cancel off the complete dependence on $h(x)\hat{\phi}^2(x)$ and $[\Box h(x)]\hat{\phi}^2(x)$, respectively, in $\hat{I}_\xi+\Gamma_\xi^{(1)}$:

$$\delta Z_m=-\frac{\lambda}{32\pi^2}[2(n-4)^{-1}+\gamma-1+\ln(m^2/4\pi\mu^2)] , \tag{3.37}$$

$$\delta\xi=-\frac{\lambda}{32\pi^2}(\xi-\tfrac{1}{6})[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)] . \tag{3.38}$$

[Note that the nonlocal term $G(\Box)$ will vanish in the large-distance limit. The large-distance limit is obtained by the metric rescaling $\eta_{\mu\nu}\to s^{-2}\eta_{\mu\nu}$ and letting $s\to 0$.]

Adopting this definition for the coupling constants, we have

$$\Gamma_\xi = \int d^4x \left[-\tfrac{1}{2}(h^{\mu\nu} - \tfrac{1}{2}h\eta^{\mu\nu})\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - \tfrac{1}{4}m^2 h\hat{\phi}^2 - \tfrac{1}{4}\hat{\phi}^2 \left[\xi + \frac{\lambda}{32\pi^2}G(\Box) \right] (\Box h) \right] \quad (3.39)$$

as the complete part of the renormalized one-loop effective action involving terms of order $h\hat{\phi}^2$. By letting $\eta_{\mu\nu} \to s^{-2}\eta_{\mu\nu}$ in Eq. (3.35) and taking $s \to 0$, it is easily seen that

$$G(\Box) \simeq 2(\xi - \tfrac{1}{6})\ln s \ . \quad (3.40)$$

The effective ξ parameter may be identified from Eq. (3.39) to be

$$\xi(s) = \xi + \frac{\lambda}{16\pi^2}(\xi - \tfrac{1}{6})\ln s \quad (3.41)$$

if it is noted that to order h, $R = \frac{1}{2}\Box h$. This is identical to that obtained from the use of minimal subtraction and standard renormalization-group methods. It clearly demonstrates that the origin lies in the nonlocal term involving $G(\Box)$ in the effective action.

B. Calculation of $O(h^2)$ terms in the gravitational action

In order to examine the terms in the effective action which are purely gravitational it is necessary to work to order h^2. [This is obvious from the results given in Eqs. (A10)–(A13).] We may set the background scalar field to zero for the gravitational terms since they are "vacuum" contributions. The order h^2 contribution to the gravitational part of the one-loop effective action is

$$\Gamma_G^{(2)} = \langle I_{\rm int}^{(2)} \rangle + \frac{i}{2}\langle (I_{\rm int}^{(1)})^2 \rangle \ , \quad (3.42)$$

where $\hat{\phi} = 0$ is taken in Eqs. (3.17) and (3.18).

The result for $\langle I_{\rm int}^{(2)} \rangle$ is obtained easily, involving only $\langle \phi^2(x) \rangle$ and $\langle \partial_\mu\phi(x)\partial_\nu\phi(x) \rangle$. From (3.20),

$$\langle \phi^2(x) \rangle = i\Delta(0) \ . \quad (3.43)$$

A simple calculation (see Appendix B) leads to

$$\langle \partial_\mu\phi(x)\partial_\nu\phi(x) \rangle = i\frac{m^2}{n}\Delta(0)\eta_{\mu\nu} \ . \quad (3.44)$$

After some integrations by parts and use of the gauge condition (A6), it is observed that

$$\langle I_{\rm int}^{(2)} \rangle = \frac{i}{2}\Delta(0) \times \int d^n x \left[\frac{m^2}{n}(h^{\mu\nu}h_{\mu\nu} - \tfrac{1}{2}h^2) + \tfrac{1}{4}\xi(h^{\mu\nu}\Box h_{\mu\nu} - \tfrac{1}{2}h\Box h) \right] . \quad (3.45)$$

This contains only local terms.

The evaluation of $(I_{\rm int}^{(1)})^2$ is a good deal more tedious. In addition to the results in Eqs. (3.23) and (3.24), we require

$$\langle \partial_\mu\phi(x)\partial_\nu\phi(x)\partial'_\rho\phi(x')\partial'_\sigma\phi(x') \rangle = -\partial_\mu\partial'_\rho\Delta(x-x')\partial_\nu\partial'_\sigma\Delta(x-x') - \partial_\mu\partial'_\sigma\Delta(x-x')\partial_\nu\partial'_\rho\Delta(x-x') \ . \quad (3.46)$$

It is found that

$$\begin{aligned} \langle (I_{\rm int}^{(1)})^2 \rangle = -\tfrac{1}{2}\int d^n x\, d^n x' \{ & [h^{\mu\nu}(x) - \tfrac{1}{2}\eta^{\mu\nu}h(x)][h^{\rho\sigma}(x') - \tfrac{1}{2}\eta^{\rho\sigma}h(x')]\partial_\mu\partial'_\rho\Delta(x-x')\partial_\nu\partial'_\sigma\Delta(x-x') \\ & + m^2 h(x')[h^{\mu\nu}(x) - \tfrac{1}{2}\eta^{\mu\nu}h(x)]\partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x') \\ & + \tfrac{1}{4}m^4 h(x)h(x')\Delta^2(x-x') + \xi[\Box' h(x')][h^{\mu\nu}(x) - \tfrac{1}{2}\eta^{\mu\nu}h(x)] \\ & \times \partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x') + \tfrac{1}{2}\xi m^2 h(x')[\Box h(x)]\Delta^2(x-x') \\ & + \tfrac{1}{4}\xi^2[\Box h(x)][\Box' h(x')]\Delta^2(x-x') \} \ . \end{aligned} \quad (3.47)$$

Expressions for the required products of Green's functions may be found in Appendix B.

As before, we must separate the terms in the effective action into parts which contain no inverse powers of mass when expanded, and those which begin at order m^{-2}. For simplicity we refer to these parts as local and nonlocal, respectively. It is found that

$$\begin{aligned} [\partial_\mu\partial'_\rho\Delta(x-x')\partial_\nu\partial'_\sigma\Delta(x-x')]_L = & -\frac{i}{8(4\pi)^2}\{ m^4[2(n-4)^{-1} + L - \tfrac{3}{2}] + \tfrac{1}{3}m^2[2(n-4)^{-1} + L - 1]\Box' + \tfrac{1}{30}[2(n-4)^{-1} + L]\Box'^2 \} \\ & \times \delta(x-x')(\eta_{\mu\nu}\eta_{\rho\sigma} + \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}) - \frac{i}{2(4\pi)^2}\{ \tfrac{1}{6}m^2[2(n-4)^{-1} + L - 1] + \tfrac{1}{30}[2(n-4)^{-1} + L]\Box' \} \\ & \times (\eta_{\mu\nu}\partial'_\rho\partial'_\sigma + \eta_{\mu\sigma}\partial'_\nu\partial'_\rho + \eta_{\nu\rho}\partial'_\mu\partial'_\sigma + \eta_{\rho\sigma}\partial'_\mu\partial'_\nu)\delta(x-x') \\ & + \frac{i}{2(4\pi)^2}\{ \tfrac{1}{3}m^2[2(n-4)^{-1} + L - 1] + \tfrac{1}{20}[2(n-4)^{-1} + L]\Box' \}(\eta_{\nu\sigma}\partial'_\mu\partial'_\rho + \eta_{\mu\rho}\partial'_\nu\partial'_\sigma)\delta(x-x') \\ & - \frac{i}{30(4\pi)^2}[2(n-4)^{-1} + L]\partial'_\mu\partial'_\nu\partial'_\rho\partial'_\sigma\delta(x-x') \ , \end{aligned} \quad (3.48)$$

where

$$L=\gamma+\ln(m^2/4\pi\mu^2)\ . \tag{3.49}$$

The subscript L on the LHS of Eq. (3.48) refers to the local contribution. The part which leads to nonlocal (NL) terms in the effective action is

$$\begin{aligned}[\partial_\mu\partial'_\rho\Delta(x-x')\partial_\nu\partial'_\sigma\Delta(x-x')]_{\rm NL}=&-\frac{i}{8(4\pi)^2}(\eta_{\mu\nu}\eta_{\rho\sigma}+\eta_{\mu\rho}\eta_{\nu\sigma}+\eta_{\mu\sigma}\eta_{\nu\rho})G_4(\Box')\delta(x-x')\\&-\frac{i}{2(4\pi)^2}[\eta_{\rho\sigma}\partial'_\mu\partial'_\nu+\eta_{\nu\rho}\partial'_\mu\partial'_\sigma+\eta_{\mu\sigma}\partial'_\nu\partial'_\rho+\eta_{\mu\nu}\partial'_\rho\partial'_\sigma]G_5(\Box')\delta(x-x')\\&+\frac{i}{2(4\pi)^2}(\eta_{\nu\sigma}\partial'_\mu\partial'_\rho+\eta_{\mu\rho}\partial'_\nu\partial'_\sigma)G_6(\Box')\delta(x-x')-\frac{i}{(4\pi)^2}G_7(\Box')\partial'_\mu\partial'_\nu\partial'_\rho\partial'_\sigma\delta(x-x')\ ,\end{aligned} \tag{3.50}$$

where

$$G_4(\Box)=\int_0^1 dt\{[m^2+t(1-t)\Box]^2\ln[1+t(1-t)m^{-2}\Box]-t(1-t)m^2\Box-\tfrac{3}{2}t^2(1-t)^2\Box^2\}\ , \tag{3.51}$$

$$G_5(\Box)=\int_0^1 dt\,t(1-t)\{[m^2+t(1-t)\Box]\ln[1+t(1-t)m^{-2}\Box]-t(1-t)\Box\}\ , \tag{3.52}$$

$$G_6(\Box)=\int_0^1 dt\,t^2\{[m^2+t(1-t)\Box]\ln[1+t(1-t)m^{-2}\Box]-t(1-t)\Box\}\ , \tag{3.53}$$

$$G_7(\Box)=\int_0^1 dt\,t^2(1-t)^2\ln[1+t(1-t)m^{-2}\Box]\ . \tag{3.54}$$

The other expressions needed are

$$m^4\Delta^2(x-x')=-\frac{im^4}{(4\pi)^2}[2(n-4)^{-1}+L]\delta(x-x')-\frac{im^2}{6(4\pi)^2}\Box'\delta(x-x')+\frac{i}{60(4\pi)^2}\Box'^2\delta(x-x')+G_8(\Box')\delta(x-x')\ , \tag{3.55}$$

where

$$G_8(\Box)=-\frac{im^4}{(4\pi)^2}\int_0^1 dt\{\ln[1+t(1-t)m^{-2}\Box]-\tfrac{1}{6}m^{-2}\Box+\tfrac{1}{60}m^{-4}\Box^2\} \tag{3.56}$$

and

$$\begin{aligned}m^2\partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x')=&-\frac{im^4}{32\pi^2}[2(n-4)^{-1}+L-1]\eta_{\mu\nu}\delta(x-x')\\&-\frac{im^2}{12(4\pi)^2}[2(n-4)^{-1}+L]\eta_{\mu\nu}\Box\delta(x-x')-\frac{im^2}{6(4\pi)^2}[2(n-4)^{-1}+L]\partial_\mu\partial_\nu\delta(x-x')\\&-\frac{i}{30(4\pi)^2}\Box\partial_\mu\partial_\nu\delta(x-x')-\frac{i}{120(4\pi)^2}\eta_{\mu\nu}\Box^2\delta(x-x')+G_{9\mu\nu}(\Box)\delta(x-x')\ ,\end{aligned} \tag{3.57}$$

where

$$\begin{aligned}G_{9\mu\nu}(\Box)=&-\frac{im^2}{32\pi^2}\eta_{\mu\nu}\int_0^1 dt\,t(1-t)\{\ln[1+t(1-t)m^{-2}\Box]-t(1-t)m^{-2}\Box\}\Box\\&-\frac{im^2}{(4\pi)^2}\int_0^1 dt\,t(1-t)\{\ln[1+t(1-t)m^{-2}\Box]-t(1-t)m^{-2}\Box\}\partial_\mu\partial_\nu\\&-\frac{im^4}{32\pi^2}\eta_{\mu\nu}\int_0^1 dt\{\ln[1+t(1-t)m^{-2}\Box]-\tfrac{1}{6}m^{-2}\Box+\tfrac{1}{60}m^{-4}\Box^2\}\ .\end{aligned} \tag{3.58}$$

It now follows from these results that the contribution to the local part (i.e., terms not involving the G_i) of $(i/2)\langle(I_{\rm int}^{(1)})^2\rangle$ is

$$\begin{aligned}\frac{i}{2}\langle(I_{\rm int}^{(1)})^2\rangle_L=(64\pi^2)^{-1}\int d^nx\{&-\tfrac{1}{4}m^4[2(n-4)^{-1}+L-\tfrac{3}{2}](h^{\mu\nu}h_{\mu\nu}-\tfrac{1}{2}h^2)\\&-\tfrac{1}{12}m^2[2(n-4)^{-1}+L-1](h^{\mu\nu}\Box h_{\mu\nu}-\tfrac{1}{2}h\Box h)\\&-\tfrac{1}{120}[2(n-4)^{-1}+L]h^{\mu\nu}\Box^2h_{\mu\nu}-\tfrac{1}{4}(\xi^2-\tfrac{1}{3}\xi+\tfrac{1}{60})[2(n-4)^{-1}+L]h\Box^2h\}\ .\end{aligned} \tag{3.59}$$

Using the results of (3.59) and (3.48) in (3.45) shows that the local contribution to the one-loop gravitational effective action is

$$(\Gamma_G^{(2)})_L = \int d^4x \left[\frac{m^4}{16(4\pi)^2}[2(n-4)^{-1}+L-\tfrac{3}{2}](h^{\mu\nu}h_{\mu\nu}-\tfrac{1}{2}h^2)+\frac{m^2}{8(4\pi)^2}(\xi-\tfrac{1}{6})[2(n-4)^{-1}+L-1](h^{\mu\nu}\Box h_{\mu\nu}-\tfrac{1}{2}h\Box h) - \frac{1}{480(4\pi)^2}[2(n-4)^{-1}+L]h^{\mu\nu}\Box^2 h_{\mu\nu} - \frac{1}{16(4\pi)^2}(\xi^2-\tfrac{1}{3}\xi+\tfrac{1}{60})[2(n-4)^{-1}+L]h\Box^2 h \right] . \quad (3.60)$$

The nonlocal part of $\Gamma_G^{(2)}$ is also obtained easily from the above results. After performing a number of integrations by parts, and using the gauge condition (A5), it is found that

$$(\Gamma_G^{(2)})_{\rm NL} = (64\pi^2)^{-1}\int d^4x \{ \tfrac{1}{4}m^2 h f_2(\Box)h + \tfrac{1}{8}m^4 h f_1(\Box)h + \tfrac{1}{2}\xi h f_2(\Box)\Box^2 h - \tfrac{1}{4}\xi^2 h f_1(\Box)\Box^2 h - \tfrac{1}{8}h f_3(\Box)\Box^2 h + \tfrac{1}{80}(h^{\mu\nu}\Box^2 h_{\mu\nu} - \tfrac{1}{2}h\Box^2 h) + \tfrac{1}{24}m^2(h^{\mu\nu}\Box h_{\mu\nu}-\tfrac{1}{2}h\Box h) - \tfrac{1}{4}m^4 h^{\mu\nu}f_1(\Box)h_{\mu\nu} - \tfrac{1}{2}m^2 h^{\mu\nu}f_2(\Box)\Box h_{\mu\nu} - \tfrac{1}{4}h^{\mu\nu}f_3(\Box)\Box^2 h_{\mu\nu} \} , \quad (3.61)$$

where

$$f_1(\Box) = \int_0^1 dt \ln[1+t(1-t)m^{-2}\Box] , \quad (3.62)$$

$$f_2(\Box) = \int_0^1 dt\, t(1-t)\ln[1+t(1-t)m^{-2}\Box] , \quad (3.63)$$

$$f_3(\Box) = \int_0^1 dt\, t^2(1-t)^2\ln[1+t(1-t)m^{-2}\Box] . \quad (3.64)$$

As before, we will define the "physical" values of the gravitational coupling constants to be those measured at large distance (or low momentum). $(\Gamma_G^{(2)})_{\rm NL}$ has been constructed explicitly so that it vanishes in the large-distance limit. Thus, the gravitational counterterms must cancel off all of the contribution from $(\Gamma_G^{(2)})_L$. Using Eq. (3.11) and the weak-field expansions in Appendix A leads to

$$\hat{I}_G^{(2)} = \int d^4x [(\Lambda+\delta\Lambda)(1+\tfrac{1}{2}h+\tfrac{1}{8}h^2-\tfrac{1}{4}h^{\mu\nu}h_{\mu\nu}) + (\kappa+\delta\kappa)(-\tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu}+\tfrac{1}{8}h\Box h) + (\alpha_1+\delta\alpha_1+\tfrac{1}{4}\alpha_2+\tfrac{1}{4}\delta\alpha_2)h^{\mu\nu}\Box^2 h_{\mu\nu} + \tfrac{1}{4}(\alpha_3-\alpha_1+\delta\alpha_3-\delta\alpha_1)h\Box^2 h] . \quad (3.65)$$

This means that we can identify $\delta\Lambda$ to cancel terms in $(\Gamma_G^{(2)})_L$ which involve no derivatives of h, $\delta\kappa$ to cancel terms which involve one derivative, etc.:

$$\delta\Lambda = \frac{m^4}{64\pi^2}[2(n-4)^{-1}+\gamma-\tfrac{3}{2}+\ln(m^2/4\pi\mu^2)] , \quad (3.66)$$

$$\delta\kappa = \frac{m^2}{32\pi^2}(\xi-\tfrac{1}{6})[2(n-4)^{-1}+\gamma-1+\ln(m^2/4\pi\mu^2)] , \quad (3.67)$$

$$\delta\alpha_1+\tfrac{1}{4}\delta\alpha_2 = \frac{1}{480(4\pi)^2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)] , \quad (3.68)$$

$$\delta\alpha_3-\delta\alpha_1 = \frac{1}{64\pi^2}(\xi^2-\tfrac{1}{3}\xi+\tfrac{1}{60})[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)] . \quad (3.69)$$

We can disentangle $\delta\alpha_1$ from $\delta\alpha_3$ in (3.69) by use of the argument given in Appendix C showing that $\delta\alpha_1$ must be independent of ξ, and that $\delta\alpha_3=0$ when $\xi=\frac{1}{6}$. This leads to

$$\delta\alpha_1 = \frac{1}{360(4\pi)^2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)] , \quad (3.70)$$

$$\delta\alpha_3 = \frac{1}{64\pi^2}(\xi-\tfrac{1}{6})^2[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)] . \quad (3.71)$$

Knowledge of $\delta\alpha_1$ enables $\delta\alpha_2$ to be obtained from (3.68):

$$\delta\alpha_2 = -\frac{1}{360(4\pi)^2}[2(n-4)^{-1}+\gamma+\ln(m^2/4\pi\mu^2)] . \quad (3.72)$$

These results allow the renormalized one-loop effective gravitational action to be obtained easily.

The short-distance limit of the gravitational effective action (to order h^2) is seen to be

$$\Gamma_G^{(2)} = (32\pi^2)^{-1}\int d^4x \{ -\tfrac{1}{120}\ln s\, h^{\mu\nu}\Box^2 h_{\mu\nu} + [\tfrac{1}{360}-\tfrac{1}{4}(\xi-\tfrac{1}{6})^2]\ln s\, h\Box^2 h \} . \quad (3.73)$$

This follows upon performing a rescaling $\eta_{\mu\nu}\to s^{-2}\eta_{\mu\nu}$ and looking at the behavior for large s. By using the order-h^2 expansions in Eqs. (A15)–(A17), and the knowledge that R can only occur in the combination $(\xi-\frac{1}{6})R$ at one-loop order (see Appendix C), it follows that for large s

$$\alpha_1(s) \simeq -(2880\pi^2)^{-1}\ln s , \quad (3.74)$$

$$\alpha_2(s) \simeq (2880\pi^2)^{-1}\ln s , \quad (3.75)$$

$$\alpha_3(s) \simeq -(32\pi^2)^{-1}(\xi-\tfrac{1}{6})^2\ln s . \quad (3.76)$$

This scaling behavior is identical to that found in Ref. 2 using minimal subtraction and the 't Hooft[8] renormalization-group analysis. It also agrees with the method of Ref. 9 which involved looking at curvature-dependent nonlocal terms in the effective action. These results demonstrate explicitly how the behavior of the effective gravitational action at short distances is linked to the presence of nonlocal terms.

IV. DISCUSSION AND CONCLUSIONS

The preceding section has shown how the existence of nonlocal terms in the one-loop effective action leads to a short-distance limit which agrees with that found in our previous work. The specific model studied was a real scalar field, and the background spacetime was taken to be weakly curved with $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$. By working to appropriate order in $h_{\mu\nu}$ the result for arbitrary curvature was inferred.

The method used has both advantages and disadvantages. One advantage is that since we were perturbing about a flat background, the usual flat-spacetime Feynman propagator could be used. This simplifies the analysis considerably. In addition, the short-distance behavior could be linked to the large-momentum behavior in the usual way.

A disadvantage is that it is not completely obvious whether by working to higher order in $h_{\mu\nu}$ the same results would be obtained. This would require more detailed calculations. Also, because we worked only to order h^2 when computing the gravitational part of the effective action, any dependence on curvature invariants whose order is higher than two would not show up. However, because our results are consistent both with our earlier ones, and also with those using the method of Ref. 9, we regard both of these possibilities as extremely unlikely.

It is also possible to obtain the effective cosmological and gravitational constants in the short-distance limit from the results presented in Sec. III. The cosmological constant is found to be in complete agreement with that found in Refs. 2 and 9. (See Ref. 9 for a discussion of the significance of the result.) For the gravitational constant, the method appears to give the result found in Refs. 2 and 9 for a minimally coupled scalar field only. It is not clear to us why this is the case.

Finally, we wish to comment on a possible way in which the results of this paper could be combined with those of Ref. 9. Rather than perturbing about a flat-spacetime background, it should be possible to take a general curved background. In place of the flat-spacetime propagators used here, it would be possible to use the analysis described in Ref. 10 (and used in Ref. 9) to obtain the propagator, and then follow the perturbative type of analysis contained in the present paper. This would presumably lead to both types of nonlocal terms being present in the effective action. A conjecture is that terms of the form $\ln(m^{-2}\Box)$ found in the present paper would be replaced by $\ln\{m^{-2}[\Box+(\xi-\frac{1}{6})R]\}$.

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APPENDIX A

We summarize a number of useful expansions needed in the main part of the paper which follow from assuming

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\ . \tag{A1}$$

The inverse metric is given by

$$g^{\mu\nu}=\eta^{\mu\nu}-h^{\mu\nu}+h^{\mu\lambda}h_{\lambda}{}^{\nu}+\cdots \tag{A2}$$

to order h^2. The convention is that indices are raised and lowered using the Minkowski metric. By expanding the determinant of (A1) it may be shown that

$$(-g)^{1/2}=1+\tfrac{1}{2}h+\tfrac{1}{8}h^2-\tfrac{1}{4}h^{\mu\nu}h_{\mu\nu}+\cdots\ , \tag{A3}$$

where $h=h^{\mu}{}_{\mu}$. The Christoffel symbols are given by

$$\Gamma^{\lambda}_{\mu\nu}=\tfrac{1}{2}(h^{\lambda}{}_{\mu,\nu}+h^{\lambda}{}_{\nu,\mu}-h_{\mu\nu}{}^{,\lambda}) -\tfrac{1}{2}h^{\lambda\rho}(h_{\mu\rho,\nu}+h_{\rho\nu,\mu}-h_{\mu\nu,\rho})+\cdots\ . \tag{A4}$$

In order to shorten some of the expressions, it is convenient to adopt a gauge condition on $h_{\mu\nu}$. We will choose the background-field gauge[19]

$$(h^{\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}h)_{,\nu}=0 \tag{A5}$$

from which it follows that

$$h^{\mu\nu}{}_{,\mu\nu}=\tfrac{1}{2}\Box h\ . \tag{A6}$$

With this choice of gauge it may be shown that

$$R=\tfrac{1}{2}\Box h-h^{\mu\nu}\Box h_{\mu\nu}-\tfrac{3}{4}h^{\mu\nu,\lambda}h_{\mu\nu,\lambda}+\tfrac{1}{2}h^{\mu\nu,\lambda}h_{\mu\lambda,\nu}+\cdots\ , \tag{A7}$$

$$R_{\mu\nu}=\tfrac{1}{2}\Box h_{\mu\nu}+\cdots\ , \tag{A8}$$

$$R_{\lambda\mu\sigma\nu}=\tfrac{1}{2}(h_{\lambda\sigma,\mu\nu}+h_{\mu\nu,\lambda\sigma}-h_{\mu\sigma,\lambda\nu}-h_{\lambda\nu,\mu\sigma})+\cdots\ . \tag{A9}$$

(We use the curvature conventions of our previous paper.[2])

Using (A3), (A6), and (A7), performing integration by parts, and discarding total divergences leads to

$$\int d^4x(-g)^{1/2}R=\int d^4x(-\tfrac{1}{4}h^{\mu\nu}\Box h_{\mu\nu}+\tfrac{1}{8}h\Box h)\ , \tag{A10}$$

where terms of higher order in h have been dropped. Similarly, it is found that

$$\int d^4x(-g)^{1/2}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}=\int d^4x(h^{\mu\nu}\Box^2h_{\mu\nu}-\tfrac{1}{4}h\Box^2h)\ , \tag{A11}$$

$$\int d^4x(-g)^{1/2}R^{\mu\nu}R_{\mu\nu}=\int d^4x(\tfrac{1}{4}h^{\mu\nu}\Box^2h_{\mu\nu})\ , \tag{A12}$$

$$\int d^4x(-g)^{1/2}R^2=\int d^4x(\tfrac{1}{4}h\Box^2h)\ . \tag{A13}$$

It follows from these results that we must work to order h^2 in order to calculate the purely gravitational terms in the one-loop effective action. It may be noted from Eqs. (A11)–(A13) that, to order h^2,

$$\int d^4x(-g)^{1/2}(R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}-4R^{\mu\nu}R_{\mu\nu}+R^2)=0\ . \tag{A14}$$

This is a direct consequence of the Gauss-Bonnet theorem. By discarding total divergences we are restricted to trivial spacetime topologies.

Further useful identities (to order h^2) are

$$\int d^4x(-g)^{1/2}RF(\Box)R=\tfrac{1}{4}\int d^4x\,hF(\Box)\Box^2h\ , \tag{A15}$$

$$\int d^4x(-g)^{1/2}R^{\mu\nu}F(\Box)R_{\mu\nu}=\tfrac{1}{4}\int d^4x\,h^{\mu\nu}F(\Box)\Box^2h_{\mu\nu}\ , \tag{A16}$$

$$\int d^4x(-g)^{1/2}R^{\mu\nu\rho\sigma}F(\Box)R_{\mu\nu\rho\sigma} = \int d^4x[h^{\mu\nu}F(\Box)\Box^2 h_{\mu\nu}-\tfrac{1}{4}hF(\Box)\Box^2 h]\,, \quad (A17)$$

where $F(\Box)$ is any function.

APPENDIX B

We list a number of integrals here which are needed in the main part of the text. They are all evaluated using the standard techniques of dimensional regularization[7] after combining denominators by introducing a parametric form.[20]

From Eq. (3.26),

$$\Delta(0)=-\frac{im^2}{(4\pi)^2}\Gamma\left[1-\frac{n}{2}\right]\left[\frac{m^2}{4\pi}\right]^{n/2-2}. \quad (B1)$$

We also have

$$\begin{Bmatrix} I(p) \\ I_\mu(p) \\ I_{\mu\nu}(p)\end{Bmatrix} = \int \frac{d^nk}{(2\pi)^n}\begin{Bmatrix} 1 \\ k_\mu \\ k_\mu k_\nu\end{Bmatrix}(k^2-m^2+i\epsilon)^{-1}[(k+p)^2-m^2+i\epsilon]^{-1} \quad \begin{matrix}(B2a)\\(B2b)\\(B2c)\end{matrix}$$

$$= \int_0^1 dt\begin{Bmatrix} J_2(t) \\ -tp_\mu J_2(t) \\ t^2p_\mu p_\nu J_2(t)+\tfrac{1}{2}\eta_{\mu\nu}J_1(t)\end{Bmatrix}, \quad \begin{matrix}(B3a)\\(B3b)\\(B3c)\end{matrix}$$

where

$$J_\alpha(t)=i(-1)^\alpha(4\pi)^{-n/2}\frac{\Gamma(\alpha-n/2)}{\Gamma(\alpha)}[m^2-t(1-t)p^2]^{n/2-\alpha}. \quad (B4)$$

The products of Green's functions encountered in Sec. III A may all be expressed in terms of the given integrals:

$$\Delta^2(x-x')=\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-x')}I(p)\,, \quad (B5)$$

$$\partial_\mu\Delta(x-x')\partial_\nu\Delta(x-x')=\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-x')}[I_{\mu\nu}(p)+p_\mu I_\nu(p)]\,. \quad (B6)$$

A further result needed in Sec. III B involves the integrals

$$\begin{Bmatrix} I_{\mu\nu\lambda}(p) \\ I_{\mu\nu\lambda\sigma}(p)\end{Bmatrix} = \int \frac{d^nk}{(2\pi)^n}\begin{Bmatrix} k_\mu k_\nu k_\lambda \\ k_\mu k_\nu k_\lambda k_\sigma\end{Bmatrix}(k^2-m^2+i\epsilon)^{-1}[(k+p)^2-m^2+i\epsilon]^{-1}\,. \quad \begin{matrix}(B7a)\\(B7b)\end{matrix}$$

They may be shown to give the results

$$I_{\mu\nu\lambda}(p)=-\int_0^1 dt[\tfrac{1}{2}t(p_\lambda\eta_{\mu\nu}+p_\mu\eta_{\nu\lambda}+p_\nu\eta_{\lambda\mu})J_1(t)+t^3p_\mu p_\nu p_\lambda J_2(t)]\,, \quad (B8)$$

$$I_{\mu\nu\lambda\sigma}(p)=\int_0^1 dt\left[\frac{1}{2n}(\eta_{\mu\nu}\eta_{\sigma\lambda}+\eta_{\mu\sigma}\eta_{\nu\lambda}+\eta_{\mu\lambda}\eta_{\nu\sigma})[m^2-t(1-t)p^2]J_1(t)\right.$$
$$\left.+\tfrac{1}{2}t^2(p_\mu p_\nu\eta_{\lambda\sigma}+p_\nu p_\lambda\eta_{\mu\sigma}+p_\mu p_\lambda\eta_{\nu\sigma}+p_\lambda p_\sigma\eta_{\mu\nu}+p_\sigma p_\nu\eta_{\mu\lambda}+p_\mu p_\sigma\eta_{\nu\lambda})J_1(t)+t^4p_\mu p_\nu p_\lambda p_\sigma J_2(t)\right]. \quad (B9)$$

The following product of Green's functions is then found,

$$\partial_\mu\partial'_\rho\Delta(x-x')\partial_\nu\partial'_\sigma\Delta(x-x')=\int\frac{d^np}{(2\pi)^n}e^{ip\cdot(x-x')}[I_{\mu\nu\rho\sigma}(p)+p_\nu I_{\mu\rho\sigma}(p)+p_\sigma I_{\mu\rho\nu}(p)+p_\nu p_\sigma I_{\mu\rho}(p)]\,. \quad (B10)$$

APPENDIX C

In this appendix we wish to make some general comments concerning the gravitational counterterms in the effective action and to justify the claim made in Sec. III B that R can only occur in the one-loop effective action in the combination $(\xi-\frac{1}{6})R$.

First of all, the gravitational part of the effective action can be expressed in terms of the propagator $\Delta(x,x')$ which satisfies

$$(\Box+m^2+\xi R)\Delta(x,x')=-\delta(x,x') \ . \qquad \text{(C1)}$$

The only other way in which ξ can enter the effective action is through the $\delta\xi R\phi^2$ counterterm. Thus ξ always occurs in the effective action in the combination ξR. In particular, there can be no dependence on ξ in the counterterms which multiply $R^{\mu\nu\rho\epsilon}R_{\mu\nu\rho\epsilon}$ or $R^{\mu\nu}R_{\mu\nu}$. This result is true to all orders in perturbation theory.

Because $\delta\alpha_1$ and $\delta\alpha_2$ are dimensionless, and $\delta\Lambda$ has dimensions of (mass)^4, we may write

$$\delta\alpha_i=F_{\alpha_i}(\lambda,m,\mu) \ \ (i=1,2) \ , \qquad \text{(C2)}$$

$$\delta\Lambda=m^4F_\Lambda(\lambda,m,\mu) \qquad \text{(C3)}$$

for some functions F_{α_i}, F_Λ (which will contain pole terms as well as finite terms.) The dependence on m in F_{α_i} and F_Λ can only be through dimensionless quantities. This means that the dependence on m and μ can only be through $\ln(m^2/\mu^2)$. (Logarithms arise because μ only occurs as μ^{n-4} which gives rise to logarithms when expressions are expanded about $n=4$.)

$\delta\kappa$ can depend at most linearly on ξ since it multiplies a term which is linear in R. We may write

$$\delta\kappa=m^2[\xi F_\kappa^{(1)}(\lambda,m,\mu)+F_\kappa^{(0)}(\lambda,m,\mu)] \qquad \text{(C4)}$$

for functions $F_\kappa^{(1)}$ and $F_\kappa^{(0)}$. Again the only dependence on m and μ can be through $\ln(m^2/\mu^2)$.

Finally, $\delta\alpha_3$ multiplies a term in R^2 and therefore can have a quadratic dependence on ξ. We may write

$$\delta\alpha_3=\xi^2F_{\alpha_3}^{(2)}(\lambda,m,\mu)+\xi F_{\alpha_3}^{(1)}(\lambda,m,\mu)+F_{\alpha_3}^{(0)}(\lambda,m,\mu) \qquad \text{(C5)}$$

for functions $F_{\alpha_3}^{(i)}$ $(i=1,2,3)$.

These results give the general forms that the counterterms must take. The pole terms which contribute to $\delta\alpha_3$ and $\delta\kappa$ involve ξ only in the combination of $(\xi-\frac{1}{6})$ at one-loop order. Both counterterms vanish if $\xi=\frac{1}{6}$. (See Ref. 2, for example.) However, the resummation of the Schwinger-Dewitt proper-time series used in Refs. 9—11 shows that this is also true if finite renormalizations are made. The conclusion is that at one-loop order, $\delta\alpha_1$, $\delta\alpha_2$, and $\delta\Lambda$ are independent of ξ, $\delta\kappa$ involves ξ only as a multiplicative factor of $(\xi-\frac{1}{6})$, and $\delta\alpha_3$ involves ξ only as a multiplicative factor of $(\xi-\frac{1}{6})^2$. We emphasize that these conclusions concerning $\delta\alpha_3$ and $\delta\kappa$ would no longer be valid at higher-loop order. (It is not even true for the pole part of the counterterms. See Ref. 21, for example.)

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