PHYSICAL REVIEW D VOLUME 32, NUMBER 6 15 SEPTEMBER 1985

Curvature tensor for Kaluza-Klein theories with homogeneous fibers

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(Received 25 January 1985; revised manuscript received 22 April 1985)

We give explicit formulas for all components of the Riemannian connection and curvature tensor for a class of metrics which describe low-energy deformations in Kaluza-Klein theories with homogeneous fibers.

I. INTRODUCTION

Einstein's theory of gravity is a theory in which the gravitational potential is interpreted as a manifestation of the (pseudo-) Riemannian structure of space-time. Kaluza-Klein theories[1–8] are theories in which the gravitational potential together with the gauge potentials of various interactions are interpreted as manifestations of (pseudo-) Riemannian structure of the Universe which is $4+K$ dimensional. In such theories, it is assumed that spontaneous compatification takes place around the Planck scale and a fiber structure with homogeneous fiber appears on the total manifold. A metric tensor can be thought of as describing deformations of the "ground state" which is given by a particular metric with "maximal symmetry" consistent with the assumed fiber structure. Without loss of much generality, the ground state is then taken to be $M^4\times G/H$ with Minkowskian metric on M^4 and a certain invariant metric on G/H. The low-energy excitations are then described by gravitational and gauge fields, where the gauge transformations are equivalent to certain fiber-preserving diffeomorphisms, and by the Brans-Dicke-Jordan-Thiry[9] scalar field, or the generalization of which, when the space of invariant metrics on G/H is more than one dimensional.

One example is given by the well-known metric in $4+1$ dimensions,

$$d\bar{s}^2=g_{\mu\nu}(x)dx^\mu\otimes dx^\nu +\phi(x)[dy-A_\mu(x)dx^\mu]\otimes[dy-A_\mu(x)dx^\mu] . \quad (1)$$

This metric is invariant under the transformation

$$y'=y+\Lambda(x),\quad A'_\mu=A_\mu+\partial_\mu\Lambda , \quad (2)$$

hence a gauge transformation is equivalent to an x-dependent y translation. One result of such a unification of gauge and coordinate transformations is that the gauge singularity in $A_\mu(x)$ may become a coordinate singularity. Thus we may have monopoles or instantons described by metric tensors possessing only coordinate singularities.[10] Looking for such solutions is one of our motivations for computing the curvature tensors in various cases.

How do we generalize Eq. (1) to the non-Abelian case so that the equivalence of a gauge transformation to a fiber-preserving diffeomorphism is maintained? Before answering this question, let us introduce some notation:

$$s^{-1}ds=\theta^a t_a,\quad ds\,s^{-1}=-\hat{\theta}^a t_a,\quad s\in G , \quad (3)$$

$$[t_a,t_b]=C^c{}_{ab}t_c,\quad \theta^a(Y_b)=\delta^a{}_b,\quad \hat{\theta}^a(\hat{Y}_b)=\delta^a{}_b , \quad (4)$$

where t_a are the generators of the Lie algebra of G, and Y_a and $\hat{Y}_a$ are the left- and right-invariant vector fields, respectively.

The group action on the fiber will be taken to be right multiplication so that the invariant metrics on the fiber are

$$\tilde{g}=\phi_{ab}(x)\hat{\theta}^a\otimes\hat{\theta}^b , \quad (5)$$

and the Killing vectors of the metrics are Y_a.

To generalize Eqs. (1) and (2) we have to add gauge potentials in a suitable way. The proper way can be found by considering certain fiber-preserving diffeomorphisms. There are two possibilities.

(i) The first possibility is

$$\begin{aligned} d\bar{s}^2&=g+\phi_{ab}(x)[\hat{\theta}^a(y)+A^a(x)]\otimes[\hat{\theta}^b(y)+A^b(x)]\\ &=g+\phi_{ab}(x)\,\mathrm{Tr}t^a[ds\,s^{-1}-A(x)]\\ &\qquad\otimes\mathrm{Tr}t^b[ds\,s^{-1}-A(x)] , \end{aligned} \quad (6)$$

where we have written

$$g=g_{\mu\nu}(x)dx^\mu\otimes dx^\nu,\quad \mathrm{Tr}t^a t_b=\delta^a{}_b . \quad (7)$$

The total metric is invariant under the transformations

$$\begin{aligned} &s'=g(x)s,\quad A'=dg\,g^{-1}+gAg^{-1} ,\\ &\phi'_{ab}=\phi_{cd}\mathscr{D}^c{}_a(g^{-1})\mathscr{D}^d{}_b(g^{-1}) , \end{aligned} \quad (8)$$

where the matrix $\mathscr{D}$ of the adjoint representation is defined by

$$st_as^{-1}=t_b\mathscr{D}^b{}_a(s),\quad st^as^{-1}=\mathscr{D}^a{}_b(s^{-1})t^b . \quad (9)$$

It is easy to see from Eq. (6) that in this case the invariant fields Y_a are Killing vectors of the total metric. The Riemannian connection and curvature tensor of this case will be given in Sec. III.

(ii) The second possibility is

$$d\bar{s}^2 = g + \phi_{ab}(x)\,\mathrm{Tr}\,t^a(ds\,s^{-1} - sAs^{-1}) \otimes \mathrm{Tr}\,t^b(ds\,s^{-1} - sAs^{-1}) \; . \tag{10}$$

The metric is invariant under

$$s' = sg(x), \quad A' = g^{-1}dg + g^{-1}Ag, \quad \phi'_{ab} = \phi_{ab} \; . \tag{11}$$

Note that the scalars ϕ_{ab} are now singlets under group action, moreover Y_a are the Killing vectors of the metric only when restricted to the fiber.

Further generalization to the case when the fiber is a coset manifold G/H is possible. We shall consider the right cosets. Choose t_a so that $t_{\bar{a}}$ lies in the Lie algebra of H and t_i lies in the orthogonal complement with respect to Tr [see Eq. (7)]. Moreover, consider G as a fiber bundle over G/H and choose a local section σ: $G/H \to G$. Then we find the generalization

$$d\bar{s}^2 = g + \phi_{ij}(x)\,\mathrm{Tr}\,t^i(d\sigma\,\sigma^{-1} - \sigma A \sigma^{-1}) \otimes \mathrm{Tr}\,t^j(d\sigma\,\sigma^{-1} - \sigma A\sigma^{-1}) \; . \tag{12}$$

Under the group action, $\sigma' = h\sigma g$ for some $h \in H$ depending on $g \in G$ and the metric is invariant under

$$\sigma' = h\sigma g, \quad A' = g^{-1}dg + g^{-1}Ag, \quad \phi'_{ij} = \phi_{ij} \; . \tag{13}$$

It is clear that this generalizes case (ii) to the case when the fiber is a coset manifold. Case (i) cannot be generalized unless the normalizer N of H in G is nontrivial and one gets a gauge theory in N/H.[7] This case will not be considered here.

Note that in Eq. (13) we have assumed ϕ_{ij} to be AdH invariant. More generally, we have only to require

$$\phi'_{ij}(x) = \phi_{kl}(x)\mathscr{D}(h^{-1})^k{}_i \mathscr{D}(h^{-1})^l{}_j \; , \tag{14}$$

and assume G/H is reductive with $t_{\bar{a}}, t_i$ a proper choice of basis for the Lie algebra of G. In this case, $\phi_{ij}(x)$ are scalar fields which form nonlinear realizations of G on both indices. We shall not consider this case in the present work.

In the following section, we shall give explicit expressions of the Riemannian connection and curvature tensor corresponding to the metric (12) in terms of $g_{\mu\nu}(x)$, $\phi_{ij}(x)$, and $A_\mu(x)$, where $\phi_{ij}(x)$ satisfies (adjoint invariance)

$$\phi_{ij}(x)C^i{}_{\bar{a}k} + \phi_{ik}(x)C^i{}_{\bar{a}j} = 0 \; . \tag{15}$$

II. CONNECTION AND CURVATURE TENSOR

Our notation on coset manifold will follow the Appendix of Ref. 11. In particular, we shall write

$$\sigma^*(ds\,s^{-1}) = (-t_i + H_i)\hat{e}_i \; ,$$

i.e.,

$$\sigma^*(\hat{\theta}^i) = \hat{e}^i, \quad \sigma^*(\hat{\theta}^{\bar{a}}) = -H_i{}^{\bar{a}}\hat{e}^i \; , \tag{16}$$

and write $\hat{Y}_i$ as the dual vector field of $\hat{e}^i$ on G/H. We shall compute connection and curvature in the basis $(e_\mu, \hat{Y}_i)$ where

$$e_\mu \equiv \partial_\mu + A_\mu{}^a(x)Y_a \; . \tag{17}$$

The dual forms are then (dx^μ, Θ^i) where

$$\Theta^i \equiv \hat{e}^i + A_\mu{}^a(x)\mathscr{D}^i{}_a(y)dx^\mu = \hat{e}^i + \mathscr{D}^i{}_a A^a \tag{18}$$

and $\mathscr{D}^i{}_a(y)$ is shorthand for $\mathscr{D}^i{}_a(\sigma(y))$. In this basis, the metric tensor (12) is simply

$$d\bar{s}^2 \equiv \bar{g} = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu + \phi_{ij}(x)\Theta^i \otimes \Theta^j \; . \tag{19}$$

Writing $e_A = (e_\mu, \hat{Y}_i)$ we have

$$[e_A, e_B] = \mathscr{F}^i{}_{AB}\hat{Y}_i \; , \tag{20}$$

$$\mathscr{F}^i{}_{\mu\nu} \equiv -F^a{}_{\mu\nu}\mathscr{D}^i{}_a \; ,$$

$$F^a{}_{\mu\nu} = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + C^a{}_{bc}A_\mu{}^b A_\nu{}^c \; , \tag{21}$$

$$\mathscr{F}^j{}_{\mu i} \equiv (h_\mu)^j{}_i \equiv A_\mu{}^a \omega_a{}^{\bar{a}} C^j{}_{\bar{a}i}, \quad \mathscr{F}^k{}_{ij} = C^k{}_{ij} + f^k{}_{ij} \; ,$$

where

$$\omega_a(\sigma) = H_b(\sigma)\mathscr{D}^b{}_a(\sigma), \quad f^k{}_{ij} \equiv -H_i{}^{\bar{a}}C^k{}_{\bar{a}j} + H_j{}^{\bar{a}}C^k{}_{\bar{a}i} \; . \tag{22}$$

The Riemannian connection satisfies

$$\bar{\Gamma}^C{}_{AB} - \bar{\Gamma}^C{}_{BA} = [e_A, e_B]^C \; , \tag{23}$$

and is given by

$$\begin{aligned}
&\bar{\Gamma}^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\mu\nu} = \tfrac{1}{2}g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \; , \\
&\bar{\Gamma}^i{}_{\mu\nu} = \tfrac{1}{2}\mathscr{F}^i{}_{\mu\nu}, \quad \bar{\Gamma}^\alpha{}_{\mu i} = \bar{\Gamma}^\alpha{}_{i\mu} = \tfrac{1}{2}g^{\alpha\nu}\mathscr{F}^j{}_{\nu\mu}\phi_{ji} \; , \\
&\bar{\Gamma}^k{}_{\mu i} = \tfrac{1}{2}(\phi^{-1}\partial_\mu\phi)^k{}_i + (h_\mu)^k{}_i \; , \\
&\bar{\Gamma}^k{}_{i\mu} = \tfrac{1}{2}(\phi^{-1}\partial_\mu\phi)^k{}_i, \quad \bar{\Gamma}^\mu{}_{ij} = -\tfrac{1}{2}\partial^\mu\phi_{ij} \; , \\
&\bar{\Gamma}^k{}_{ij} = \tilde{\Gamma}^k{}_{ij} = \tfrac{1}{2}\phi^{kl}(C_{lij} + C_{jli} + C_{ilj}) - H_i{}^{\bar{a}}C^k{}_{\bar{a}j} \\
&\qquad \equiv \bar{\Gamma}'^k{}_{ij} - H_i{}^{\bar{a}}C^k{}_{\bar{a}j} \; ,
\end{aligned} \tag{24}$$

where

$$\phi^{ij}\phi_{jk} = \delta^i{}_k, \quad C_{ijk} = \phi_{il}C^l{}_{jk} \; .$$

The curvature tensor is given by

$$\bar{R}^D{}_{CAB} = e_A(\bar{\Gamma}^D{}_{BC}) - e_B(\bar{\Gamma}^D{}_{AC}) + \bar{\Gamma}^E{}_{BC}\bar{\Gamma}^D{}_{AE} - \bar{\Gamma}^E{}_{AC}\bar{\Gamma}^D{}_{BE} - \mathscr{F}^i{}_{AB}\bar{\Gamma}^D{}_{iC} \; , \tag{25}$$

and explicitly,

$$\bar{R}_{\lambda\kappa\mu\nu} = R_{\lambda\kappa\mu\nu} + \phi_{ij}(-\tfrac{1}{4}\mathscr{F}^i{}_{\lambda\mu}\mathscr{F}^j{}_{\kappa\nu} + \tfrac{1}{4}\mathscr{F}^i{}_{\lambda\nu}\mathscr{F}^j{}_{\kappa\mu} - \tfrac{1}{2}\mathscr{F}^i{}_{\lambda\kappa}\mathscr{F}^j{}_{\mu\nu}) \; , \tag{26}$$

$$\bar{R}_{i\lambda\mu\nu} = \tfrac{1}{2}\phi_{ij}(\nabla'_\mu\mathscr{F}^j{}_{\nu\lambda} - \nabla'_\nu\mathscr{F}^j{}_{\mu\lambda}) + \tfrac{1}{4}\partial_\mu\phi_{ij}\mathscr{F}^j{}_{\nu\lambda} - \tfrac{1}{4}\partial_\nu\phi_{ij}\mathscr{F}^j{}_{\mu\lambda} - \tfrac{1}{2}\partial_\lambda\phi_{ij}\mathscr{F}^j{}_{\mu\nu} \; , \tag{27}$$

$$\nabla'_\mu\mathscr{F}_{\nu\lambda} \equiv D_\mu\mathscr{F}_{\nu\lambda} - \Gamma^\alpha{}_{\mu\nu}\mathscr{F}_{\alpha\lambda} - \Gamma^\alpha{}_{\mu\lambda}\mathscr{F}_{\nu\alpha}, \quad D_\mu = \partial_\mu + [A_\mu, \cdot\,] \; , \tag{28}$$

$$\bar{R}_{i\mu j\nu} = \tfrac{1}{2}C_{i\bar{a}j}F^a{}_{\mu\nu}\mathscr{D}^{\bar{a}}{}_a + \tfrac{1}{2}\bar{\Gamma}'_{ikj}\mathscr{F}^k{}_{\nu\mu} + \tfrac{1}{4}g^{\alpha\beta}\phi_{jk}\phi_{il}\mathscr{F}^k{}_{\alpha\mu}\mathscr{F}^l{}_{\beta\nu} - \tfrac{1}{2}\nabla_\nu(\partial_\mu\phi_{ij}) + \tfrac{1}{4}(\partial_\nu\phi\phi^{-1}\partial_\mu\phi)_{ij} \; , \tag{29}$$

$$\bar R_{ij\mu\nu}=C_{i\bar a j}F^a{}_{\mu\nu}\mathscr{D}^{\bar a}{}_a+\bar\Gamma'{}_{ikj}\mathscr{F}^k{}_{\nu\mu} + \tfrac14 g^{\alpha\beta}\phi_{jk}\phi_{il}(\mathscr{F}^k{}_{\alpha\mu}\mathscr{F}^l{}_{\beta\nu}-\mathscr{F}^k{}_{\alpha\nu}\mathscr{F}^l{}_{\beta\mu}) + \tfrac14(\partial_\nu\phi\phi^{-1}\partial_\mu\phi-\partial_\mu\phi\phi^{-1}\partial_\nu\phi)_{ij}\ , \quad (30)$$

$$\bar R_{\mu ijk}=\tfrac12(\partial_\mu\phi_{kl}\bar\Gamma'^l{}_{ji}-\partial_\mu\phi_{jl}\bar\Gamma'^l{}_{ki}+\partial_\mu\phi_{il}C^l{}_{jk}) + \tfrac14(\phi_{kl}\partial^\nu\phi_{ij}-\phi_{jl}\partial^\nu\phi_{ik})\mathscr{F}^l{}_{\mu\nu}\ , \quad (31)$$

$$\bar R_{ijkl}=\tilde R_{ijkl}+\tfrac14(\partial^\mu\phi_{kj}\partial_\mu\phi_{il}-\partial^\mu\phi_{lj}\partial_\mu\phi_{ik})\ . \quad (32)$$

$\tilde R_{ijkl}$ is the curvature tensor of the internal space and will be given below in terms of ϕ.

The Ricci tensor is given by

$$\bar R_{\mu\nu}=R_{\mu\nu}-\tfrac12 g^{\alpha\beta}\phi_{ij}\mathscr{F}^i{}_{\mu\alpha}\mathscr{F}^j{}_{\nu\beta}-\tfrac12\nabla_\nu\operatorname{Tr}(\phi^{-1}\partial_\mu\phi) - \tfrac14\operatorname{Tr}(\phi^{-1}\partial_\mu\phi\phi^{-1}\partial_\nu\phi)\ , \quad (33)$$

$$\bar R_{ij}=\tilde R_{ij}+\tfrac14 g^{\mu\nu}g^{\alpha\beta}\phi_{ik}\phi_{jl}\mathscr{F}^k{}_{\mu\alpha}\mathscr{F}^l{}_{\nu\beta}-\tfrac12\nabla_\mu(\partial^\mu\phi_{ij}) + \tfrac12(\partial^\mu\phi\phi^{-1}\partial_\mu\phi)_{ij}-\tfrac14\partial^\mu\phi_{ij}\operatorname{Tr}(\phi^{-1}\partial_\mu\phi)\ , \quad (34)$$

$$\bar R_{i\mu}=-\tfrac12 g^{\nu\lambda}\nabla'_\nu(\phi_{ij}\mathscr{F}^j{}_{\mu\lambda})-\tfrac14\operatorname{Tr}(\phi^{-1}\partial^\nu\phi)\phi_{ij}\mathscr{F}^i{}_{\mu\nu} + \tfrac12 C^k{}_{ij}(\phi^{-1}\partial_\mu\phi)^j{}_k+\tfrac12 C^j{}_{jk}(\phi^{-1}\partial_\mu\phi)^k{}_i\ , \quad (35)$$

and the scalar curvature is given by

$$\bar R=R+\tilde R-\tfrac14 g^{\mu\nu}g^{\alpha\beta}\phi_{ij}\mathscr{F}^i{}_{\mu\alpha}\mathscr{F}^j{}_{\nu\beta}-\nabla_\mu(\operatorname{Tr}\phi^{-1}\partial^\mu\phi) - \tfrac14\operatorname{Tr}(\phi^{-1}\partial_\mu\phi\phi^{-1}\partial^\mu\phi)-\tfrac14(\operatorname{Tr}\phi^{-1}\partial_\mu\phi)(\operatorname{Tr}\phi^{-1}\partial^\mu\phi)\ . \quad (36)$$

The curvature tensor of the internal space with the metric $\phi_{ij}\hat e^i\otimes\hat e^j$ is given by

$$\tilde R_{ijkl}=\tfrac14\phi^{mn}[(C_{ilm}+C_{lim})(C_{kjn}+C_{jkn})-(C_{ikm}+C_{kim})(C_{ljn}+C_{jln})] - \tfrac14\phi_{mn}(C^m{}_{ik}C^n{}_{jl}+C^m{}_{il}C^n{}_{kj}+2C^m{}_{ij}C^n{}_{kl})+\tfrac14[(C_{ijm}-C_{jim})C^m{}_{kl}+(C_{klm}-C_{lkm})C^m{}_{ij}]+\tfrac12(C_{ij\bar a}C^{\bar a}{}_{kl}+C_{kl\bar a}C^{\bar a}{}_{ij})\ , \quad (37)$$

and the Ricci tensor is given by

$$\tilde R_{ij}=-\tfrac12(C^k{}_{\bar a i}C^{\bar a}{}_{kj}+C^k{}_{\bar a j}C^{\bar a}{}_{ki}+C^k{}_{li}C^l{}_{kj}) + \tfrac14\phi^{kl}\phi^{mn}C_{ikm}C_{jln}-\tfrac12\phi_{kl}\phi^{mn}C^k{}_{mi}C^l{}_{nj} - \tfrac12\phi^{mn}C^k{}_{km}(C_{ijn}+C_{jin})\ . \quad (38)$$

The scalar curvature is given by

$$\tilde R=-\phi^{ij}(C^k{}_{\bar a i}C^{\bar a}{}_{kj}+\tfrac12 C^k{}_{li}C^l{}_{kj}) - \tfrac14\phi_{ij}\phi^{kl}\phi^{mn}C^i{}_{km}C^j{}_{ln} - \phi^{mn}C^i{}_{im}C^j{}_{jn}\ . \quad (39)$$

Note that we have not made assumptions about the nature of G or G/H. Besides the definition of the structure constants given in (4), the only assumption is the AdH invariance of ϕ as expressed in Eq. (15).

III. CONNECTION AND CURVATURE TENSOR FOR CASE (i)

We shall compute the Riemannian connection and curvature tensor in the frame

$$e_A=(e_\mu,\hat Y_a),\quad e_\mu=\partial_\mu+A_\mu{}^a\hat Y_a\ . \quad (40)$$

The dual forms are

$$(dx^\mu,\Theta^a),\quad \Theta^a=\hat\theta^a-A_\mu{}^a dx^\mu\ , \quad (41)$$

and the metric can be written as

$$\bar g=g_{\mu\nu}(x)dx^\mu\otimes dx^\nu+\phi_{ab}(x)\Theta^a\otimes\Theta^b\ . \quad (42)$$

Define

$$[e_A,e_B]=\mathscr{F}^a{}_{AB}\hat Y_a\ , \quad (43)$$

then

$$\mathscr{F}^a{}_{\mu\nu}=F^a{}_{\mu\nu}=\partial_\mu A_\nu{}^a-\partial_\nu A_\mu{}^a+C^a{}_{bc}A_\mu{}^b A_\nu{}^c\ , \quad (44)$$

$$\mathscr{F}^b{}_{\mu a}=-\mathscr{F}^b{}_{a\mu}=C^b{}_{ca}A_\mu{}^c\ . \quad (45)$$

Let us also define the covariant derivative ∇_μ to be the "total" covariant derivative with respect to the four-dimensional Riemannian connection and the gauge connection of the adjoint representation. For example, we have

$$\nabla_\mu F^a{}_{\nu\lambda}=\partial_\mu F^a{}_{\nu\lambda}-\Gamma^\alpha{}_{\mu\nu}F^a{}_{\alpha\lambda}-\Gamma^\alpha{}_{\mu\lambda}F^a{}_{\nu\alpha}+\mathscr{F}^a{}_{\mu c}F^c{}_{\nu\lambda}\ , \quad (46)$$

$$\nabla_\mu\phi_{ab}=\partial_\mu\phi_{ab}-\phi_{bc}\mathscr{F}^c{}_{\mu a}-\phi_{ac}\mathscr{F}^c{}_{\mu b}\ , \quad (47)$$

$$\nabla_\mu\phi^b{}_a=\partial_\mu\phi^b{}_a+\mathscr{F}^b{}_{\mu c}\phi^c{}_a-\phi^b{}_c\mathscr{F}^c{}_{\mu a}\ . \quad (48)$$

The connection is given by

$$\bar\Gamma^\alpha{}_{\mu\nu}=\Gamma^\alpha{}_{\mu\nu}=\tfrac12 g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu}+\partial_\nu g_{\lambda\mu}-\partial_\lambda g_{\mu\nu})\ , \quad (49)$$

$$\bar\Gamma^a{}_{\mu\nu}=\tfrac12 F^a{}_{\mu\nu},\quad \bar\Gamma^\mu{}_{a\nu}=\bar\Gamma^\mu{}_{\nu a}=\tfrac12 g^{\mu\alpha}F^b{}_{\alpha\nu}\phi_{ab}\ , \quad (50)$$

$$\bar\Gamma^\mu{}_{ab}=-\tfrac12\nabla^\mu\phi_{ab},\quad \bar\Gamma^a{}_{\mu b}=\mathscr{F}^a{}_{\mu b}+\tfrac12(\phi^{-1}\nabla_\mu\phi)^a{}_b\ , \quad (51)$$

$$\bar\Gamma^a{}_{b\mu}=\tfrac12(\phi^{-1}\nabla_\mu\phi)^a{}_b\ ,\qquad \bar\Gamma^c{}_{ab}=\tfrac12\phi^{cd}(\phi_{be}C^e{}_{da}+\phi_{de}C^e{}_{ab}+\phi_{ae}C^e{}_{db})\ . \quad (52)$$

The curvature tensor is given by

$$\bar R_{\lambda\kappa\mu\nu}=R_{\lambda\kappa\mu\nu}+\phi_{ab}(\tfrac14 F^a{}_{\nu\kappa}F^b{}_{\lambda\mu} - \tfrac14 F^a{}_{\mu\kappa}F^b{}_{\lambda\nu}-\tfrac12 F^a{}_{\mu\nu}F^b{}_{\lambda\kappa})\ , \quad (53)$$

$$\bar R_{a\lambda\mu\nu}=\tfrac12\phi_{ab}(\nabla_\mu F^b{}_{\nu\lambda}-\nabla_\nu F^b{}_{\mu\lambda})+\tfrac14\nabla_\mu\phi_{ab}F^b{}_{\nu\lambda} - \tfrac14\nabla_\nu\phi_{ab}F^b{}_{\mu\lambda}-\tfrac12\nabla_\lambda\phi_{ab}F^b{}_{\mu\nu}\ , \quad (54)$$

$$\bar R_{a\mu b\nu}=\tfrac14 g^{\alpha\beta}\phi_{ac}\phi_{bd}F^c{}_{\alpha\nu}F^d{}_{\beta\mu}+\tfrac12\phi_{ad}\bar\Gamma^d{}_{bc}F^c{}_{\nu\mu} - \tfrac12\nabla_\nu(\nabla_\mu\phi_{ab})+\tfrac14(\nabla_\nu\phi\phi^{-1}\nabla_\mu\phi)_{ab}\ , \quad (55)$$

$$\bar{R}_{\mu\nu ab}=\tfrac{1}{4}g^{\alpha\beta}\phi_{ad}\phi_{bc}(F^c{}_{\alpha\mu}F^d{}_{\beta\nu}-F^c{}_{\alpha\nu}F^d{}_{\beta\mu}) -\tfrac{1}{2}C^c{}_{ab}\phi_{cd}F^d{}_{\mu\nu}+\tfrac{1}{4}(\nabla_\nu\phi\phi^{-1}\nabla_\mu\phi -\nabla_\mu\phi\phi^{-1}\nabla_\nu\phi)_{ab}\ , \tag{56}$$

$$\bar{R}_{\mu abc}=\tfrac{1}{2}(\nabla_\mu\phi_{cd}\bar{\Gamma}^d{}_{ba}-\nabla_\mu\phi_{bd}\bar{\Gamma}^d{}_{ca}+\nabla_\mu\phi_{ad}C^d{}_{bc}) +\tfrac{1}{4}(\phi_{cd}\nabla^\nu\phi_{ba}-\phi_{bd}\nabla^\nu\phi_{ca})F^d{}_{\mu\nu}\ , \tag{57}$$

$$\bar{R}_{abcd}=\tilde{R}_{abcd}+\tfrac{1}{4}(\nabla^\mu\phi_{cb}\nabla_\mu\phi_{ad}-\nabla^\kappa\phi_{db}\nabla_\mu\phi_{ac})\ . \tag{58}$$

$\tilde{R}_{abcd}$ is the Riemannian curvature tensor of the internal space as given in Eq. (37) with obvious notation change.

The Ricci tensor is given by

$$\bar{R}_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g^{\alpha\beta}\phi_{ab}F^a{}_{\mu\alpha}F^b{}_{\nu\beta}-\tfrac{1}{2}\nabla_\nu\mathrm{Tr}(\phi^{-1}\nabla_\mu\phi) +\tfrac{1}{2}C^a{}_{ac}F^c{}_{\nu\mu}-\tfrac{1}{4}\mathrm{Tr}(\phi^{-1}\nabla_\nu\phi\phi^{-1}\nabla_\mu\phi)\ , \tag{59}$$

$$\bar{R}_{ab}=\tilde{R}_{ab}+\tfrac{1}{4}g^{\mu\nu}g^{\alpha\beta}\phi_{ac}\phi_{bd}F^c{}_{\mu\alpha}F^d{}_{\nu\beta}-\tfrac{1}{2}\nabla_\mu(\nabla^\mu\phi_{ab}) +\tfrac{1}{2}(\nabla_\mu\phi\phi^{-1}\nabla^\mu\phi)_{ab}-\tfrac{1}{4}\nabla^\mu\phi_{ab}\,\mathrm{Tr}(\phi^{-1}\nabla_\mu\phi)\ , \tag{60}$$

$$\bar{R}_{\mu a}=-\tfrac{1}{2}g^{\nu\lambda}\nabla_\nu(\phi_{ab}F^b{}_{\mu\lambda})-\tfrac{1}{4}\phi_{ab}F^b{}_{\mu\nu}\mathrm{Tr}(\phi^{-1}\nabla^\nu\phi) +\tfrac{1}{2}C^c{}_{ab}(\phi^{-1}\nabla_\mu\phi)^b{}_c+\tfrac{1}{2}C^b{}_{bc}(\phi^{-1}\nabla_\mu\phi)^c{}_a\ . \tag{61}$$

Finally, the scalar curvature is given by

$$\bar{R}=R+\tilde{R}-\tfrac{1}{4}g^{\mu\nu}g^{\alpha\beta}\phi_{ab}F^a{}_{\mu\alpha}F^b{}_{\nu\beta}-\nabla_\mu(\mathrm{Tr}\phi^{-1}\nabla^\mu\phi) -\tfrac{1}{4}\mathrm{Tr}(\phi^{-1}\nabla_\mu\phi\phi^{-1}\nabla^\mu\phi) -\tfrac{1}{4}\mathrm{Tr}(\phi^{-1}\nabla_\mu\phi)\mathrm{Tr}(\phi^{-1}\nabla^\mu\phi)\ . \tag{62}$$

This agrees with Ref. 7 except for a factor of 2 in the last two terms (however, see Ref. 12). Again, $\tilde{R}$ was given in Eq. (39).

IV. DISCUSSIONS

In this paper, we have presented the expressions for the components of the Riemannian connection and curvature tensor in chosen coordinate systems for a class of metrics usually occurring in Kaluza-Klein theories. The metrics are often expressed in terms of the Killing vectors of the total metric or the submetric obtained by restriction to the fiber. Here we rewrite the metrics in terms of the invariant forms of the structure group G and show that one of the main properties of such metrics, namely, the equivalence of the gauge transformations to fiber-preserving diffeomorphisms, is obvious in this formalism.

In the existing literature, usually only the Ricci tensor or part of the components of the curvature tensor related to the Ricci tensor are given. However, in some applications the complete curvature tensor may be needed. For example, it is known that in Kaluza-Klein theories with no matter fields the spontaneous compactification cannot occur. It may occur, however, if one considers theories with $R\cdot R$ terms.[13] In such theories one will need all the components of the curvature tensor. Another example is when one wants to check whether a singularity in the solution of the field equations is a physical singularity. This usually requires computation of $R^A{}_{BCD}R_A{}^{BCD}$.

Although we have assumed the space-time to be four dimensional, the formulas can certainly be used in any space-time dimensions. One may find that the expressions we gave are useful even outside the context of the Kaluza-Klein theories. For example, we found them useful when computing the Riemannian connection and curvature tensor for four-dimensional gravitational theory with axial symmetry.

Finally, we comment on a few references which contain similar computations. In Ref. 5, the connection, the Ricci tensor, and the curvature scalar were given in a different basis. The curvature tensor for the internal space was not given. It appears that it may be possible to simplify their expressions. We find it difficult to compare with our result without doing some computations. In Ref. 6, a similar result was given in the same basis as in Ref. 5, but different scalar fields were used. Reference 7 considered the case when the normalizer of H is nontrivial. The (Ricci) scalar curvature was given. In particular, the Ricci scalar for the internal manifold as given in Eq. (39) agrees with their result.

ACKNOWLEDGMENT

This work is supported in part by a grant from National Science Council, Taipei, Taiwan, Republic of China.

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