Massive fluid spheres in general relativity

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We present a class of nonsingular analytic solutions of the general-relativistic field equations in isotropic form for a static spherically symmetric material distribution. Within a sphere the outward variation of pressure, density, pressure-density ratio, and the adiabatic sound speed is monotonic decreasing. The solution has been used to construct causal models for neutron stars with a maximum mass $\approx 4M_{\odot}$ where we have assumed a surface density equivalent to the typical terrestrial nuclear density.

I. INTRODUCTION

A considerable number[1] of exact solutions of Einstein's field equations for the description of static fluid spheres have been obtained, most of them conveniently expressible in the standard form of the space-time metric. The applicability of an exact solution to astrophysical situations of relativistic nature depends upon the physical behavior of the corresponding equation of state, and also upon the stability of the model under small radial perturbations. It is well known that none of the known exact solutions has been found satisfactory in these respects. On the other hand, using fast computers stable relativistic models have been numerically computed satisfying physically sound equations of state.[2–4] Nevertheless, on account of the structural and operational simplicity of exact solutions it is desirable that we find new ones if they lead to an improvement in our understanding of the field equations. Since a solution having a simple form in one coordinate system may appear very complicated in another system we propose to explore yet unknown solutions that may not have required structural simplicity when expressed in the standard system of coordinates but have quite simple forms in some other system.

II. A SOLUTION IN THE ISOTROPIC FORM

We consider the line element in the isotropic form

$$ds^2=c^2e^{\nu(r)}dt^2-e^{\omega(r)}(dr^2+r^2d\Omega^2) \tag{2.1}$$

with

$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2 .$$

The sphere is of perfect fluid:

$$T^i_j=\mathrm{diag}(c^2\rho,-p,-p,-p) . \tag{2.2}$$

The field equations of general relativity reduce to the following:

$$\frac{8\pi G}{c^4}p=e^{-\omega}\left[\frac{(\omega')^2}{4}+\frac{\omega'}{r}+\frac{\omega'\nu'}{2}+\frac{\nu'}{r}\right] , \tag{2.3}$$

$$\frac{8\pi G}{c^4}p=e^{-\omega}\left[\frac{\omega''}{2}+\frac{\nu''}{2}+\frac{(\nu')^2}{4}+\frac{\omega'}{2r}+\frac{\nu'}{2r}\right] , \tag{2.4}$$

$$\frac{8\pi G}{c^2}\rho=-e^{-\omega}\left[\omega''+\frac{(\omega')^2}{4}+\frac{2\omega'}{r}\right] . \tag{2.5}$$

From (2.3) and (2.4) we get the following equation in ω and ν:

$$\nu''+\omega''+\frac{(\nu')^2}{2}-\frac{(\omega')^2}{2}-\nu'\omega'-\frac{1}{r}(\nu'+\omega')=0 . \tag{2.6}$$

A solution of (2.6) is

$$e^{\nu/2}=A\left[\frac{1-k\delta}{1+k\delta}\right],\quad e^{\omega/2}=\frac{(1+k\delta)^2}{1+r^2/a^2} \tag{2.7}$$

with

$$\delta(r)=(1+r^2/a^2)^{1/2}/(1+br^2/a^2)^{1/2} . \tag{2.8}$$

Here a, b, k, and A are constants. The expressions for pressure and density are as follows:

$$\frac{8\pi G}{c^4}p=\frac{4(bk^2\delta^6-1)}{a^2(1+k\delta)^5(1-k\delta)} , \tag{2.9}$$

$$\frac{8\pi G}{c^2}\rho=\frac{12(1+bk\delta^5)}{a^2(1+k\delta)^5} . \tag{2.10}$$

Clearly for $b=0$ the solution is Buchdahl's solution[5]—an analog of a classical polytrope of index 5. In fact, our solution consists of two different classes of solutions, according to whether $b>0$ or $b\leq 0$. The first case corresponds to finite boundary models whereas the latter case gives rise to a family of unbounded systems. Since we are interested in finite-sized models, we write

$$b=n^2 , \tag{2.11}$$

where n is a nonzero real number.

The solution is to be matched over the boundary with Schwarzschild's empty space-time,

$$ds^2=\left[1-\frac{2GM}{c^2R}\right]c^2dt^2-\left[1-\frac{2GM}{c^2R}\right]^{-1}dR^2-R^2d\Omega^2 , \tag{2.12}$$

where M is the mass of the ball as determined by an external observer and R is the radial coordinate in the exterior region. The usual boundary conditions are that the first and second fundamental forms be continuous across the boundary $r=r_b$ or equivalently $R=R_b$. Applying the boundary conditions we get the values of three constants, viz., A, k, and a in terms of the constant n and the Schwarzschild parameters M and R_b as follows:

$$A^2=\left[\frac{n+1}{n-1}\right]^2, \tag{2.13}$$

$$k=1/n\delta_b^3, \tag{2.14}$$

$$(r_b/a)^2=[1-(1-2u)^{1/2}]/n[1+(1-2u)^{1/2}], \tag{2.15}$$

$$R_b=(r_b/n)\{n+[1-(1-2u)^{1/2}]\times[1+(1-2u)^{1/2}]^{-1}\}^3(n+1)^{-2} \tag{2.16}$$

with

$$u\equiv GM/c^2R_b<\tfrac{1}{2} \tag{2.17}$$

and

$$n>1. \tag{2.18}$$

Since n is left arbitrary, we find in (2.7) a parametric class of exact solutions. We note that the case $0<n<1$ leads to the result similar to the case $n>1$.

We observe that it is not easy to transform (2.1) and (2.7) into the standard form. To transform the isotropic radial coordinate r into the standard radial coordinate R we require the transformation

$$\frac{r(1+k\delta)^2}{(1+r^2/a^2)}=R. \tag{2.19}$$

This implies that to find $r(R)$ one has to solve the following polynomial equation of eighth degree:

$$n^4R^2r^8-2R(k^2+n^2)a^2n^2r^7+[(n^2-k^2)^2a^2+2n^2(n^2+1)k^2]a^2r^6-2R(n^4+2n^2+k^2+2n^2k^2a^2)a^4r^5$$
$$+[2n^2a^2+2k^4a^2+R^2+n^4R^2+4n^2R^2-2k^2(n^2+1)a^2]a^4r^4-2R(1+k^2n^2+2n^2+2k^2)a^6r^3$$
$$+[a^2(1-k^2)^2+2(n^2+1)R^2]a^6r^2-2(1+k^2)Ra^8r+k^2a^8=0. \tag{2.20}$$

Apparently this equation has no simple roots.

III. PROPERTIES OF THE SOLUTION

In view of (2.9), (2.10), and (2.14) the expressions for pressure and density can be rewritten as follows:

$$\frac{8\pi G}{c^4}p=\frac{4[(\delta/\delta_b)^6-1]}{a^2(1+\delta/n\delta_b^3)^5(1-\delta/n\delta_b^3)}, \tag{3.1}$$

$$\frac{8\pi G}{c^2}\rho=\frac{12(1+n\delta^5/\delta_b^3)}{a^2(1+\delta/n\delta_b^3)^5}. \tag{3.2}$$

Clearly, p is non-negative and finite throughout the distribution if

$$n\delta_b^3>1, \tag{3.3}$$

which fixes an upper limit of u in terms of the parameter n,

$$0<u<2n^{1/3}(1+n^{2/3})(1+n^{1/3}+n^{2/3})^{-2}. \tag{3.4}$$

Clearly, the density is positive throughout the distribution and its outward variation up to the boundary is monotonically decreasing if

$$r_b<an^{-1/2}. \tag{3.5}$$

It is to be noted that as n approaches 1 (but not equal to 1 as it leads to a singularity in the metric) the upper limit of u approaches its maximum value $\frac{4}{9}$ and density approaches a uniform value throughout the sphere. Thus our model approaches Schwarzschild's uniform density model as $n\rightarrow 1$.

From (3.1) and (3.2) one obtains

$$p/c^2\rho=(n/3)(\delta^6-\delta_b^6)(n\delta_b^3-\delta)(\delta_b^3+n\delta^5)^{-1}, \tag{3.6}$$

so that

$$\left[\frac{p}{c^2\rho}\right]'=(n^2/3)\delta_b^3\delta'[\delta^{10}-(5/n^2)\delta^6+5\delta_b^6\delta^4-\delta_b^6/n^2]\times(n\delta_b^3-\delta)^{-1}(\delta_b^3+n\delta^5)^{-2}. \tag{3.7}$$

Thus extrema in p/ρ occur at the center and at radial points given by the roots of the quintic equation

$$f(\delta)\equiv\delta^{10}-(5/n^2)\delta^6+5\delta_b^6\delta^4-\delta_b^6/n^2=0. \tag{3.8}$$

We obtain

$$\left[\frac{p}{c^2\rho}\right]''_0=-(2n/3)[\delta_b^3(n^2-1)(1+\delta_b^6)+4(n^2\delta_b^6-1)]\times(n\delta_b^3-1)^{-2}(n+\delta_b^3)^{-2} \tag{3.9}$$

implying that p/ρ is maximum at the center. Applying Sturm's theorem on $f(\delta)$ we find in view of (3.3) that (3.8) has no roots in the range $0<r\leq r_b$. It follows that p/ρ falls monotonically from its maximum central value to zero on the surface.

Again we have

$$\frac{1}{c^2}\frac{dp}{d\rho}=(2n/15)[2n\delta_b^6-2n\delta^6-3\delta_b^3\delta+3n^2\delta_b^3\delta^5]\times(n\delta_b^3-\delta)^{-2}(n^2\delta^4-1)^{-1}, \tag{3.10}$$

so that

$$\left[\frac{dp}{d\rho}\right]'=\frac{dp}{d\rho}\delta'\left[\frac{2}{(n\delta_b{}^3-\delta)}-\frac{4n^2\delta^3}{(n^2\delta^4-1)}+\frac{3(4n\delta^5-5n^2\delta_b{}^3\delta^4+\delta_b{}^3)}{(2n\delta^6-3n^2\delta_b{}^3\delta^5+3\delta\delta_b{}^3-2n\delta_b{}^6)}\right]. \tag{3.11}$$

Extrema of $dp/d\rho$ occur at the center and at radial positions which satisfy the following polynomial equation in δ:

$$-n^4\delta_b{}^3\delta^9+3n^5\delta_b{}^6\delta^8-8n\delta^6-8n^4\delta_b{}^9\delta^3+6n^2\delta_b{}^3\delta^5+3\delta_b{}^3\delta+6n^3\delta_b{}^6\delta^4-n\delta_b{}^6=0\,. \tag{3.12}$$

We obtain

$$\left[\frac{dp}{d\rho}\right]''_0=-2\left[\frac{dp}{d\rho}\right]_0(n\delta_b{}^3-1)^{-1}(n^2-1)^{-1}[2(n\delta_b{}^3-1)(n\delta_b{}^3+1)+(n^2-1)\delta_b{}^3]^{-1}$$
$$\times[(n\delta_b{}^3-1)(n^3\delta_b{}^6+n^4\delta_b{}^3+n^2\delta_b{}^3+6n)+8n\delta_b{}^3(1-n^3\delta_b{}^6)$$
$$+6n\delta_b{}^3(n^2\delta_b{}^3-1)+2(n-\delta_b{}^3)(n^4\delta_b{}^6-1)+n\delta_b{}^6(n^3\delta_b{}^3-1)+(n-1)^2\delta_b{}^3]\,. \tag{3.13}$$

The expression on the right-hand side of (3.13) is negative in view of (2.18), (3.3), and (3.5) showing thereby that $dp/d\rho$ is maximum at the center. Equation (3.12) is an equation of ninth degree in δ, and it has not been possible to show analytically the existence or nonexistence of its roots in $0<r\le r_b$; however, numerical study in different extreme cases suggests that $dp/d\rho$ will normally fall outward monotonically.

IV. PARAMETRIC BEHAVIOR OF THE STATIC BALL AND NEUTRON-STAR MODELS UNDER EXTREME CENTRAL CONDITIONS

In the last section we have shown analytically that the outward variation of p/ρ is monotonic decreasing. This indicates that if the equation of state is realistic in the central region it will be so in the rest of the fluid region. In view of this, since it has not been possible to find the equation of state in closed analytic form, it will be worth investigating the parametric behavior of the static ball when some limiting equation of state is assumed to hold at the center. Clearly, an assumption regarding the central equation of state fixes the parameter r_b.

The object of our present analysis is to determine the maximum value of the surface gravitational potential and hence the surface red-shift $Z\,[=(1-2u)^{-1/2}-1]$ obtainable from the solution in the parametric range $1<n<\infty$ under central conditions:

(i) $(dp/d\rho)_0=c^2$,

(ii) $(p/\rho)_0=c^2/3$,

(iii) $(p/\rho)_0=c^2$,

and when

(iv) $u=2n^{1/3}(1+n^{2/3})(1+n^{1/3}+n^{2/3})^{-2}$.

Figure 1 shows the variation of u and Z under these conditions and we find the following results:

Condition (i):

$\max u=0.3188,\ \ \max Z=0.6612\ \ \text{for } n=7.22$;

Condition (ii):

$\max u=0.2871,\ \ \max Z=0.5326\ \ \text{for } n=5.1727$;

Condition (iii):

$\max u=0.3757,\ \ \max Z=1.0055\ \ \text{for } n=2.1$;

Condition (iv):

$\max u\to0.4444,\ \ \max Z\to2\ \ \text{as } n\to1$.

We observe that the extreme model obtained under the condition (ii) satisfies the causality condition too (Table I). We note that the red-shift obtainable from this model is quite close to Bondi's limit[6] $Z<0.615$.

To illustrate the astrophysical application of the solution we compute the maximum mass of a neutron-star-like compact object. We assume $\rho_b=2\times10^{14}$ g cm^{-3} which fixes the value of u. We thus obtain a family of neutron-star models for values of n. Figure 2 shows the variation of $M/M_\odot$ with n under extreme conditions (i), (ii), (iii), and (iv). Mass maximization under these conditions results in the following models:

Condition (i):

$\max M=3.9231M_\odot,\ \ R_b=18.6944$ km ,

$\rho_0/\rho_b=2.2813$ for $n=4.5$;

Condition (ii):

$\max M=3.9926M_\odot,\ \ R_b=21.1215$ km ,

$\rho_0/\rho_b=1.0048$ for $n=1.1255$;

Condition (iii):

$\max M=6.2594M_\odot,\ \ R_b=24.5311$ km ,

$\rho_0/\rho_b=1.1565$ for $n=1.1255$;

Condition (iv):

$\max M=8.0684M_\odot,\ \ R_b=11.8606$ km ,

$\rho_0/\rho_b=1.0077$ for $n=1.1255$.

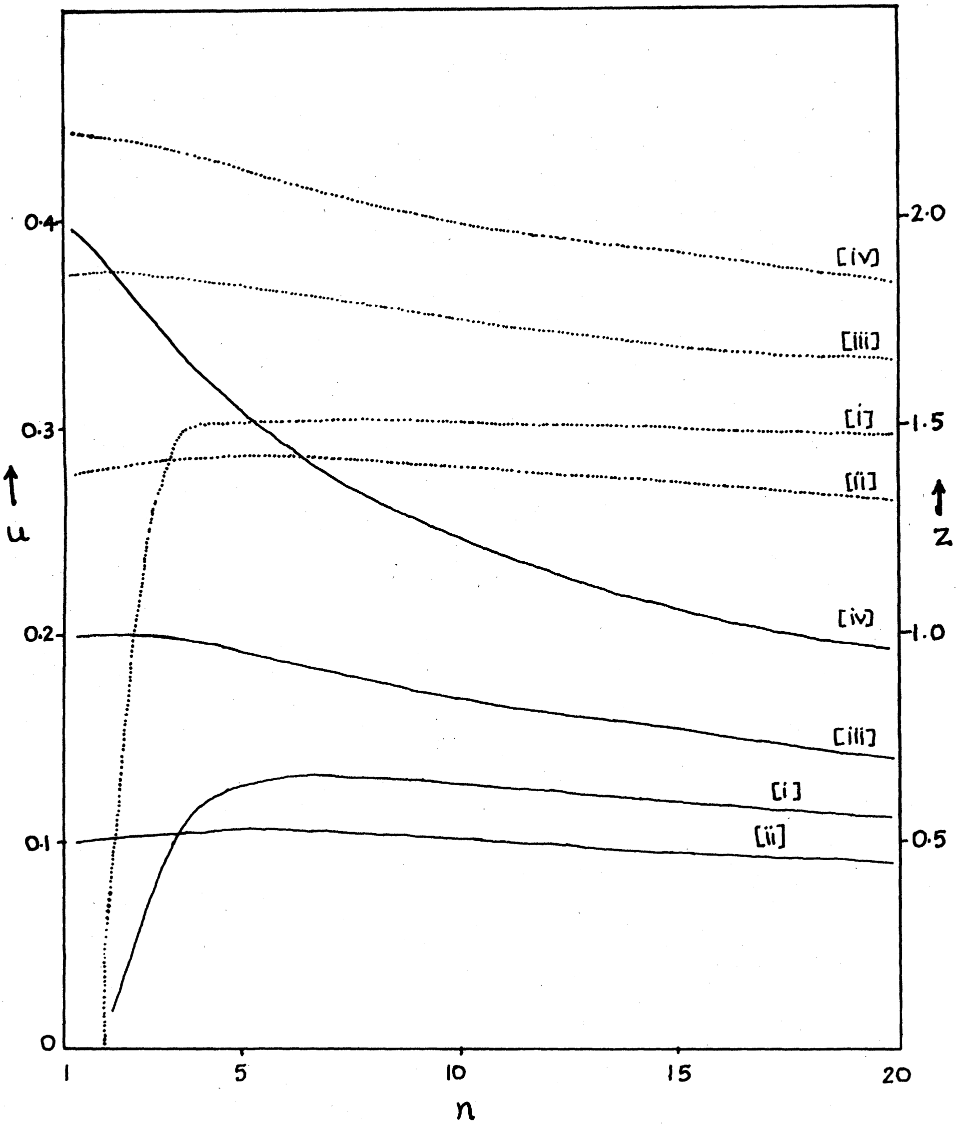

FIG. 1. Variation of u (dotted lines) and Z (solid lines) with n under the extreme conditions (i) $(dp/d\rho)_0=c^2$, (ii) $(p/\rho)_0=c^2/3$, (iii) $(p/\rho)_0=c^2$, (iv) $u=2n^{1/3}(1+n^{2/3})(1+n^{1/3}+n^{2/3})^{-2}$.

TABLE I. Variation of $dp/d\rho$ with r in (I) the extreme model satisfying the conditions $p/\rho \le c^2/3$ and $dp/d\rho \le c^2$, (II) the neutron-star model with $(dp/d\rho)_0=c^2$, and (III) the neutron-star model with $(dp/d\rho)_0=c^2$ and $(p/\rho)_0=c^2/3$.

	$dp/c^2d\rho$		
$(r/r_b)^2$	(I) $n=5.1727$, $u=0.2871$	(II) $n=4.5$, $u=0.308\,48$	(III) $n=3.415$, $u=0.2854$
0	0.7045	1.0000	1.0000
0.1	0.6251	0.9215	0.8671
0.2	0.5676	0.8610	0.7795
0.3	0.5244	0.8138	0.7134
0.4	0.4848	0.7767	0.6641
0.5	0.4667	0.7474	0.6269
0.6	0.4474	0.7246	0.5988
0.7	0.4326	0.7070	0.5802
0.8	0.4216	0.6939	0.5701
0.9	0.4135	0.6846	0.5515
1.0	0.3852	0.6786	0.5448

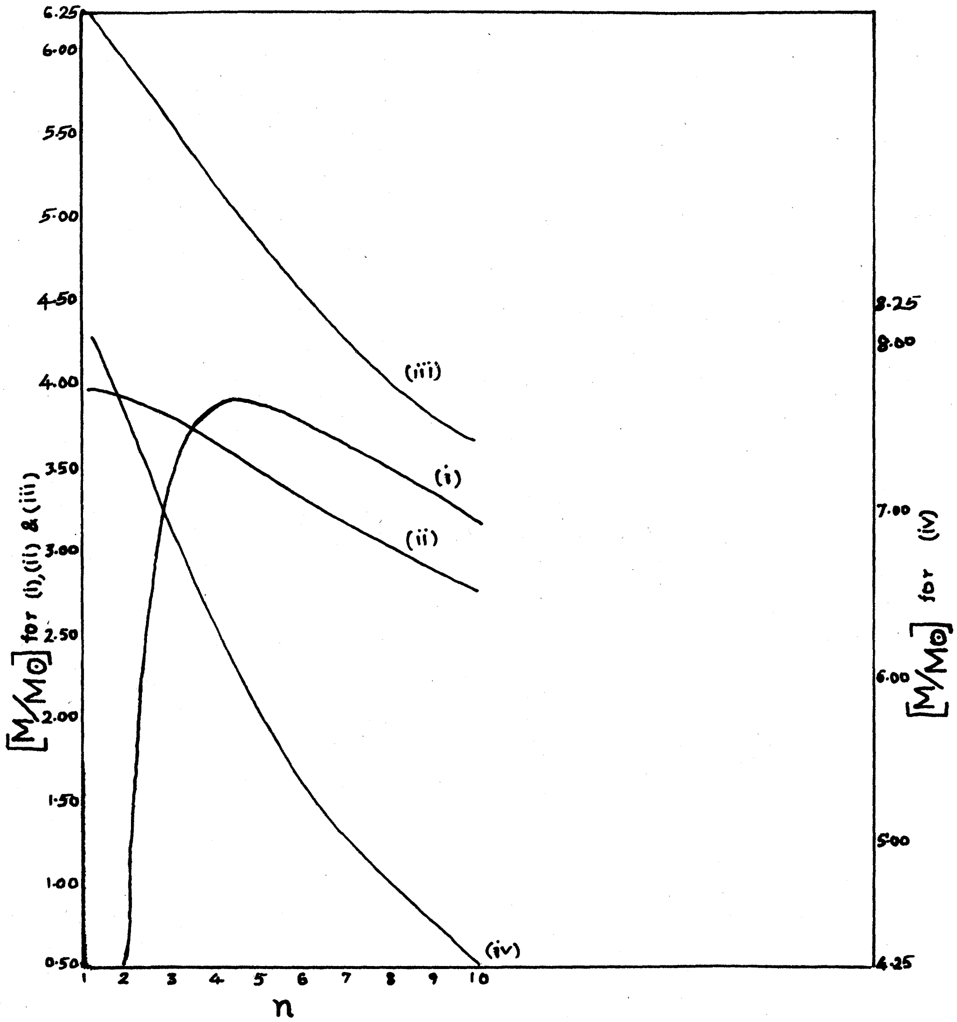

FIG. 2. Variation of $M/M_\odot$ with n under the extreme conditions (i) $(dp/d\rho)_0=c^2$, (ii) $(p/\rho)_0=c^2/3$, (iii) $(p/\rho)_0=c^2$, (iv) $u=2n^{1/3}(1+n^{2/3})(1+n^{1/3}+n^{2/3})^{-2}$.

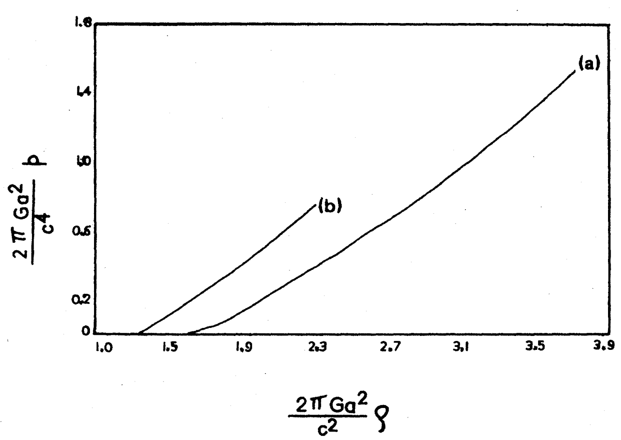

FIG. 3. p vs ρc^2 for (a) $n=4.5$ and $(dp/d\rho)_0=c^2$, (b) $n=3.415$, $(dp/d\rho)_0=c^2$, and $(p/\rho)_0=c^2/3$.

For the neutron-star model satisfying each of the conditions (i) and (ii) the parameters are as follows:

$$\max M=3.731M_\odot,\quad R_b=19.214\text{ km},$$

$$\rho_0/\rho_b=1.6933 \quad \text{for } n=3.415 .$$

Figure 3 shows the equation of state for two neutron-star models, viz., for $n=4.5$ and $n=3.415$. Table I shows that in these two models the outward variation of sound speed is monotonically decreasing throughout.

As such all the above-described calculations are based on a particular surface density, viz., $\rho_b=2\times10^{14}$ g cm^{-3}. A decrease in ρ_b is found to correspond to an increase in $M/M_\odot$ and also in R_b which shows that neutron-star models bigger in mass and size are obtainable from our solution. This is possible if one assumes a significant crust of subnuclear density surrounding the nuclear-density core.

[1]D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Deutscher, Berlin, 1980).
[2]J. R. Oppenheimer and G. M. Volkoff, Phys. Rev. **55**, 374 (1939).
[3]R. F. Tooper, Astrophys. J. **140**, 434 (1964).
[4]B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitational Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965).
[5]H. A. Buchdahl, Astrophys. J. **140**, 1512 (1964).
[6]H. Bondi, Proc. R. Soc. London **A281**, 39 (1964).