Field equations and spontaneous compactification in quasi-Riemannian theories of gravity

S. P. de Alwis* and S. Randjbar-Daemi
Institute for Theoretical Physics, University of Bern, Sidlerstrasse 5, CH-3012 Bern, Switzerland
(Received 7 February 1985)

We construct gravitational as well as fermionic action integrals for Weinberg's quasi-Riemannian theories of gravity in d dimensions. We investigate the possibilities of obtaining spontaneously compactifying solutions with $G_T = \mathrm{SO}(1,D-1)\times G'_T$ where $G'_T \subseteq \mathrm{SO}(d-D)$, $4 \le D \le d$. For $G'_T = \mathrm{SO}(d-D)$ we show that they are unlikely to have G-invariant solutions of the form $(\text{Minkowski})_4 \times G/H$, while we explicitly construct this type of solution for $G'_T = H$. We also present several fermionic Lagrangians and briefly discuss the chirality problem.

I. INTRODUCTION

The orthodox Kaluza idea of unification of all long-range forces in a single higher-dimensional gravitational field encounters at least two major obstacles. Theses are (i) the impossibility of obtaining four-dimensional chiral fermions from a higher-dimensional Einstein-Dirac action[1] and (ii) the absence of compactifying solutions, up to one loop, of the form $M_4 \times B_{d-4}$, where M_4 is the flat Minkowski space and B_d is a non-Ricci-flat compact $(d-4)$-dimensional manifold[2] (see, however, Ref. 3). In view of these problems if we want to obtain realistic four-dimensional physics, we are forced to depart from the orthodoxy.

At present three kinds of departures are known. The first, and the most extensively studied, is to introduce elementary gauge fields in the higher-dimensional action. These fields have the standard minimal coupling to gravity and fermions, and can solve both of the above-mentioned problems.[4]

The other two ways of departing from the pure gravity Kaluza theory are to abandon either[5] the compactness of the internal space B_{d-4} or[6] the Riemannian structure of the starting d-dimensional theory. Both of these ideas are in their primary stages and we believe they are worthwhile investigating. This paper is devoted to a study of the second possibility, i.e., Weinberg's quasi-Riemannian gravity.

In quasi-Riemannian theory the only elementary Bose fields are the ones which are given by the manifold structure, namely, the tangent vector fields to the manifold. In contrast to the ordinary Riemannian geometry where the tangent vectors form the fundamental representation of the (pseudo-)orthogonal group, Weinberg assumes that they transform according to some representation of a group G_T. It is then argued that G_T must have a product structure[7] of the form $\mathrm{SO}(1,D-1)\times G'_T$ where $4 \le D \le d$ and $G'_T \subseteq \mathrm{SO}(d-D)$. ($d$ is the total number of dimensions.)

In this paper we shall construct action integrals which are invariant under the tangent space group $G_T = \mathrm{SO}(1,D-1)\times G'_T$ and the general coordinate transformations in d dimensions. We shall then study the compactifying solutions to the equations derived from these actions. It will be shown that the generic G-invariant solutions are of the form $(\text{de Sitter})_4 \times G/H$, with a compact G/H. However for the special case of $G'_T = H$ we shall obtain solutions of the form $(\text{Minkowski})_4 \times G/H$, provided that $G'_T \subset \mathrm{SO}(d-D)$.

The plan of the paper is as follows. In Sec. II we construct the action for pure gravity with $G_T = \mathrm{SO}(1,D-1)\times \mathrm{SO}(d-D)$, where $4 \le D \le d$. The possibility of obtaining solutions with this tangent space group is studied in Sec. III. Section IV generalizes these results to a tangent space group of the form $G_T = \mathrm{SO}(1,D-1)\times G'_T$ where $G'_T \subset \mathrm{SO}(d-D)$. In Sec. V we construct fermionic Lagrangians and briefly discuss the problem of chirality. Section VI concludes the paper. The geometrical technicalities and more details on the construction of the Lagrangians and field equations are relegated to two Appendixes at the end of the paper.

In the remaining part of the Introduction we recollect various definitions in quasi-Riemannian geometry which we use in the subsequent sections.

Let M_d denote a d-dimensional manifold and z^M, $M = 1, \ldots, d$ a set of coordinate functions. Applying any local $\mathrm{GL}(d,R)$ matrix $e^A_M(z)$ we can transform the natural basis dz^M of the cotangent space to a (nonholonomic) basis $e^A = e^A_M(z)dz^M$. The functions $e^A_M(z)$ are the vielbeins. In the standard pseudo-Riemannian geometry it is assumed that the index A transforms according to the fundamental representation of $\mathrm{SO}(1,d-1)$. In quasi-Riemannian geometry it will be postulated that A transforms according to some representation of a smaller tangent space group $G_T \equiv \mathrm{SO}(1,D-1)\times G'_T$, where $4 \le D \le d$ and $G'_T \subseteq \mathrm{SO}(d-D)$.

In order to generate Yang-Mills fields as fluctuations of the e^A_M fields it is imperative to have a nonvanishing torsion.[8,9] Here we shall introduce torsion by adopting a proposal of Weinberg.[9]

Let Ω denote a connection form for the gauge group $\mathrm{SO}(1,d-1)$. As a Lie-algebra-valued one-form it has a unique decomposition, viz.,

$$\Omega = \omega + \bar{\omega} , \tag{1}$$

where ω is a one-form with values in the algebra of G_T, while $\bar{\omega}$ lies in the complementary coset subspace

$SO(1,d-1)/G_T$.

Under an element $g \in G_T$ we have

$$\Omega \to g^{-1}(\Omega + d)g \ , \tag{2a}$$

$$\omega \to g^{-1}(\omega + d)g \ , \tag{2b}$$

$$\bar{\omega} \to g^{-1}\bar{\omega} g \ , \tag{2c}$$

i.e., ω transforms as a connection while $\bar{\omega}$ transforms covariantly. We shall demand invariance under local G_T transformations. Therefore all derivatives should be G_T covariant. This will be ensured by using ω as a G_T connection.

Since our philosophy is to regard e_M^A as the only fundamental Bose field, we must express ω and $\bar{\omega}$ in terms of e. To achieve this we demand Ω, as an $SO(1,d-1)$ linear connection, to be torsion free, i.e.,

$$de + \Omega \wedge e = 0 \ , \tag{3}$$

where e stands for the column matrix of vielbein one-forms $e^A = e_M^A dz^M$. Equation (3) can be solved in the usual way[10] and gives

$$\Omega_{AB} = \Omega_{ABC} e^C \ ,$$

where

$$\Omega_{ABC} = e_A^M e_B^N e_{C[M,N]} - e_B^M e_C^N e_{A[M,N]} - e_C^M e_A^N e_{B[M,N]} \ ,$$
$$e_{C[M,N]} = \partial_N e_{CM} - \partial_M e_{CN} \ . \tag{3a}$$

Substituting this solution in Eq. (1) we get a unique solution for ω and $\bar{\omega}$ as functions of e_M^A.

As we already remarked, in the absence of G_T torsion it is impossible to generate Yang-Mills fields from compactification of a higher-dimensional quasi-Riemannian manifold. The G_T connection ω has in fact a nonvanishing torsion. To see this it is sufficient to substitute from (1) in (3) to obtain

$$de + \omega \wedge e = -\bar{\omega} \wedge e \ . \tag{4}$$

This indicates that $-\bar{\omega} \wedge e$ is the torsion two-form of the connection ω.

The action integral in the next section will contain $\bar{\omega}$ alongside the curvature two-form R defined by

$$R = d\omega + \omega \wedge \omega \ . \tag{5}$$

In the construction of the action we shall never use the coordinate basis in the manifold. Therefore every equation we write will be automatically invariant under general coordinate transformations.

II. THE ACTION FOR $G_T = SO(1,D-1) \times SO(d-D)$

To clarify the principles of constructing invariant actions, in this section we shall assume that $G_T = SO(D-1) \times SO(d-D)$. Within the context of quasi-Riemannian gravity this is the most restrictive G_T. Nevertheless we shall show that the most general G_T-invariant action which does not involve more than two derivatives contains nine independent parameters. To see this let the index A be decomposed into $\alpha = 1, \ldots, D$ and $a = D+1, \ldots, d$, such that e^α and e^a transform, respectively, as $SO(1,D-1)$ and $SO(d-D)$ vectors. Then the Lorentz connection $\Omega = \frac{1}{2} Q_{AB} \Omega_{[AB]C} e^c$, can be decomposed as follows:

$$\frac{1}{2} Q_{AB} \Omega_{[AB]C} e^c = \frac{1}{2} Q_{\alpha\beta} (\Omega_{[\alpha\beta]\gamma} e^\gamma + \Omega_{[\alpha\beta]a} e^a)$$
$$+ \frac{1}{2} Q_{[ab]} (\Omega_{[ab]\gamma} e^\gamma + \Omega_{[ab]c} e^c)$$
$$+ \frac{1}{2} Q_{[a\beta]} (\Omega_{[a\beta]\gamma} e^\gamma + \Omega_{[a\beta]b} e^b) \ , \tag{6}$$

where Q_{AB}, $Q_{\alpha\beta}$, and Q_{ab} are the generators of $SO(1,d-1)$, $SO(1,D-1)$, and $SO(d-D)$, respectively. Upon comparison with Eq. (1) we conclude that

$$\omega = \frac{1}{2} Q_{[\alpha\beta]} (\Omega_{\alpha\beta\gamma} e^\gamma + \Omega_{\alpha\beta a} e^a)$$
$$+ \frac{1}{2} Q_{[ab]} (\Omega_{[ab]\gamma} e^\gamma + \Omega_{[ab]c} e^c) \ , \tag{7a}$$

$$\bar{\omega} = Q_{a\beta} (-\Omega_{\beta a b} e^b + \Omega_{a\beta\gamma} e^\gamma) \ . \tag{7b}$$

As mentioned in the previous section Eq. (3) has as the solution Eq. (3a) giving Ω as functions of e^A. Hence ω and $\bar{\omega}$ will be functions of e^A and their first derivatives.

To construct the action we shall construct all G_T invariants which contain ω, $\bar{\omega}$, and at most their first derivatives. First consider ω. Since it transforms as a G_T connection we can form its curvature tensor. It is given by Eq. (5). Relative to our adopted basis it may be expanded as follows:

$$R = \frac{1}{4} (R^{\alpha\beta}{}_{\gamma\delta} Q_{[\alpha\beta]} + R^{ab}{}_{\gamma\delta} Q_{[ab]}) e^\gamma \wedge e^\delta$$
$$+ \frac{1}{2} (R^{\alpha\beta}{}_{c\delta} Q_{[\alpha\beta]} + R^{ab}{}_{c\delta} Q_{[ab]}) e^c \wedge e^\delta$$
$$+ \frac{1}{4} (R^{\alpha\beta}{}_{cd} Q_{[\alpha\beta]} + R^{ab}{}_{cd} Q_{[ab]}) e^c \wedge e^d \ . \tag{8}$$

Subject to our restriction of not having more than two derivatives, the required G_T-invariant tensors should be formed out of various components of R. Recalling the antisymmetry of $R_{AB}{}^{CD}$ with respect to $A \leftrightarrow B$ and $C \leftrightarrow D$, the only possible G_T invariants are

$$R^{\alpha\beta}{}_{\alpha\beta}, \ R^{ab}{}_{ab} \ . \tag{9}$$

Next consider $\bar{\omega}$. Since $\bar{\omega}$ contains the first derivatives of e, we should employ only bilinears in $\bar{\omega}$. It is fairly easy to see from Eq. (7b) and Appendix A that the only permissible G_T invariants are

$$\bar{\omega}_{a[\beta\gamma]} \bar{\omega}^{a[\beta\gamma]}, \ \bar{\omega}_{a\{\beta\gamma\}} \bar{\omega}^{a\{\beta\gamma\}}, \ \bar{\omega}_{a\beta}{}^{\beta} \bar{\omega}^{a}{}_{\gamma}{}^{\gamma} \ ,$$
$$\bar{\omega}_{\alpha[bc]} \bar{\omega}^{\alpha[bc]}, \ \bar{\omega}_{\alpha\{bc\}} \bar{\omega}^{\alpha\{bc\}}, \ \bar{\omega}_{\alpha b}{}^{b} \bar{\omega}^{\alpha}{}_{c}{}^{c} \ , \tag{10}$$

where $\bar{\omega}_{\beta[ab]} \equiv \frac{1}{2}(\bar{\omega}_{\beta ab} - \bar{\omega}_{\beta ba})$ and $\bar{\omega}_{\beta\{ab\}} \equiv \frac{1}{2}(\bar{\omega}_{\beta ab} + \bar{\omega}_{\beta ba}) - (1/D)\eta_{ab}\bar{\omega}_{\beta c}{}^{c}$. Similar definitions hold for $\bar{\omega}_{a[\beta\gamma]}$ and $\bar{\omega}_{a\{\beta\gamma\}}$. In Eq. (10) the raising and lowering of the indices have been performed with $\eta_{\alpha\beta}$ and δ_{ab}. Now from (9) and (10) we can construct the invariant action

$$S[e]=\int \det e\, d^dz(c_1R^{\alpha\beta}{}_{\alpha\beta}+c_2R^{ab}{}_{ab}+c_3\bar\omega_{a[\beta\gamma]}\bar\omega^{a[\beta\gamma]}+c_4\bar\omega_{a\{\beta\gamma\}}\bar\omega^{a\{\beta\gamma\}}+c_5\bar\omega_{\alpha\beta}{}^{\beta}\bar\omega^{a}{}_{\gamma}{}^{\gamma}$$
$$+c_6\bar\omega_{\alpha[bc]}\bar\omega^{\alpha[bc]}+c_7\bar\omega_{\alpha\{bc\}}\bar\omega^{\alpha\{bc\}}+c_8\bar\omega_{\alpha b}{}^{b}\bar\omega^{\alpha}{}_{c}{}^{c}+c_9)\ , \tag{11}$$

where $c_1,\ldots,c_9$ are constants.

In principle one can vary S with respect to e and obtain the equations of motion. In practice, however, it turns out to be simpler to regard ω, $\bar\omega$, and e as independent variables and impose the Eq. (4) as a constraint by introducing a Lagrange multiplier ϕ_A (see Appendix B). Thus instead of (11) we consider the following:

$$\bar S[e,\omega,\bar\omega,\phi]\equiv S[e,\omega,\bar\omega]+\int \phi_A\wedge{}^*(de^A+\omega^A{}_B\wedge e^B+\bar\omega^A{}_B\wedge e^B)\ , \tag{12}$$

where ϕ is a column vector of two-forms and the * operation in Eq. (12) has been defined in Appendix A. Clearly the variation of S with respect to ϕ yields the desired constraint, namely, Eq. (4).

We conclude that the most general $G_T=\mathrm{SO}(1,D-1)\times\mathrm{SO}(d-D)$ invariant action depends on nine arbitrary constants. c_9 is a d-dimensional cosmological constant.

III. VACUUM SOLUTIONS FOR $G_T=\mathrm{SO}(1,D-1)\times\mathrm{SO}(d-D)$

In this section we shall look for compactifying solutions to the field equations derived from Eq. (11). These solutions will be required to have a product structure $M_4\times B$ where M_4 is a four-dimensional maximally symmetric space-time and B is a compact homogeneous space.

First we notice that for a manifold like $M_4\times B$ the tangent space group G_T given above will break into $\mathrm{SO}(1,3)\times\mathrm{SO}(D-4)\times\mathrm{SO}(d-D)$. Let the index set $\{\alpha\}$ be broken up accordingly as $\{\dot\alpha\}\cup\{\bar\alpha\}$, $\dot\alpha=(1,\ldots,4)$, $\bar\alpha=(5,\ldots,D-4)$. Let us also write $\bar A,\bar B=5,6,\ldots,d-4$, i.e., $\{\bar A\}=\{\bar\alpha\}\cup\{a\}$. Hence (7b) decomposes into

$$\bar\omega=Q_{a\dot\beta}(\Omega_{a\dot\beta\bar C}e^{\bar C}+\Omega_{a\dot\beta\dot\gamma}e^{\dot\gamma})+Q_{a\bar\beta}(\Omega_{a\bar\beta\bar C}e^{\bar C}+\Omega_{a\bar\beta\dot\gamma}e^{\dot\gamma})\ . \tag{13}$$

For $\Omega_{\bar A\dot\beta\dot\gamma}$ we use the explicit expression, Eq. (3a), giving us

$$\bar\omega_{\bar A\dot\beta\dot\gamma}=\Omega_{\bar A\dot\beta\dot\gamma}=e^m_{\bar A}e^\nu_{\dot\beta}e_{\dot\gamma[m,\nu]}-e^\mu_{\dot\beta}e^\nu_{\dot\gamma}e_{\bar A[\mu,\nu]}-e^\mu_{\dot\gamma}e^n_{\bar A}e_{\dot\beta[\mu,n]}\ .$$

This vanishes by the product structure of the manifolds $M_4\times B$. Note that in the above equation $\mu,\nu=1,\ldots,4$ and $m,n=5,\ldots,d$ are space-time indices. Then the maximal symmetry of M_4 implies that (the form index is suppressed)

$$\bar\omega_{\bar A\dot\beta}=0\ . \tag{14}$$

First let us ask whether B can be a homogeneous (G/H) space. In every homogeneous space G/H there exist a set of Maurer-Cartan forms $e^{\hat A}$, $\hat A=1,\ldots,\dim G$, satisfying[11]

$$de^{\hat A}+\tfrac{1}{2}c_{\hat B\,\hat C}{}^{\hat A}e^{\hat B}\wedge e^{\hat C}=0\ , \tag{15}$$

where $c_{\hat B\,\hat C}{}^{\hat A}$, $\hat A=1,\ldots,\dim G$ are the structure constants of G. Denoting the indices in the algebra of H by i and the rest by $\bar A$ we get from (15)

$$de^{\bar A}+c_{i\bar C}{}^{\bar A}e^i\wedge e^{\bar C}+\tfrac{1}{2}c_{\bar B\,\bar C}{}^{\bar A}e^{\bar B}\wedge e^{\bar C}=0\ . \tag{16}$$

Now we consider the $\bar A$ component of Eq. (4) and subtract it from Eq. (16) to get [we set $\omega^{\bar A}{}_{\dot\alpha}=0=\bar\omega^{\dot\alpha}{}_{\bar A}$, see Eq. (14)]

$$(\omega^{\bar A}{}_{\bar B}-c_{i\bar B}{}^{\bar A}e^i)\wedge e^{\bar B}+(\bar\omega^{\bar A}{}_{\bar B}-\tfrac{1}{2}c_{\bar C\,\bar B}{}^{\bar A}e^{\bar C})\wedge e^{\bar B}=0\ .$$

From this we conclude that

$$\omega^{\bar A}{}_{\bar B}-c_{i\bar B}{}^{\bar A}e^i+\bar\omega^{\bar A}{}_{\bar B}-\tfrac{1}{2}c_{\bar C\,\bar B}{}^{\bar A}e^{\bar C}=0\ . \tag{17a}$$

Thus, we have the following solution for ω and $\bar\omega$ (note that e^i is an H connection):

$$\omega^{\bar A}{}_{\bar B}=c_{i\bar B}{}^{\bar A}e^i+\tfrac{1}{2}(1-\lambda)c_{\bar C\,\bar B}{}^{\bar A}e^{\bar C}\ , \tag{17b}$$

$$\bar\omega^{\bar A}{}_{\bar B}=\frac{\lambda}{2}c_{\bar C\,\bar B}{}^{\bar A}e^{\bar C}, \tag{17c}$$

where λ is an arbitrary parameter.

We have already concluded that the only nonvanishing components of $\bar\omega$ are $\bar\omega_{\bar\alpha a}$, i.e., the components $\bar\omega^a{}_b$, $\bar\omega^{\bar\alpha}{}_{\bar\beta}$ should vanish. As a result we must have [from Eq. (17c)] either $\lambda=0$ or

$$c_{\bar\beta\,\bar\gamma}{}^{\bar\alpha}=c_{\bar\beta C}{}^{\bar\alpha}=c_{b\bar\gamma}{}^{a}=c_{bc}{}^{a}=0\ .$$

Then

$$\bar\omega^{\bar\alpha}{}_b=-\tfrac{1}{2}\lambda(c_{b\bar\gamma}{}^{\bar\alpha}e^{\bar\gamma}+c_{bc}{}^{\bar\alpha}e^c)=-\tfrac{1}{2}\lambda c_{bc}{}^{\bar\alpha}e^c\ , \tag{18a}$$

$$\bar\omega^b{}_{\bar\alpha}=-\tfrac{1}{2}\lambda(c_{\bar\alpha\,\bar\gamma}{}^{b}e^{\bar\gamma}+c_{\bar\alpha c}{}^{b}e^c)=-\tfrac{1}{2}\lambda c_{\bar\alpha\,\bar\gamma}{}^{b}e^{\bar\gamma}\ . \tag{18b}$$

But $\omega_{\bar\alpha b}+\omega_{b\bar\alpha}=0$, so that from the linear independence of e^c and $e^{\bar\gamma}$, it follows that $c_{\bar\alpha\,\bar\gamma}{}^{b}=c_{bc}{}^{\bar\alpha}=0$. So either $\lambda=0$ or all the relevant structure constants vanish. Hence $\bar\omega^{\bar A}{}_{\bar B}=0$. The outcome of these geometrical considerations is (see Appendix B) that the only terms contributing to the field equations are the first two and the last terms in Eq. (11) and they yield the following:

$$R_{\alpha\beta}-\tfrac{1}{2}\eta_{\alpha\beta}(R_D+sR_{d-D}+\Lambda)=0\ , \tag{19a}$$

$$R_{ab}-\tfrac{1}{2}\delta_{ab}(R_D+sR_{d-D}+\Lambda)=0\ , \tag{19b}$$

$$R_{\beta c}=R_{c\beta}=0\ , \tag{19c}$$

where $s\equiv c_2/c_1$, $\Lambda\equiv c_9/c_1$, and

$$R_{\alpha\beta}\equiv R^{\gamma}{}_{\alpha\gamma\beta},\quad R_{ab}\equiv R^{c}{}_{acb}\ ,$$

$$R_D\equiv R^{\alpha}{}_{\alpha},\quad R_{d-D}\equiv R^{a}{}_{a}\ .$$

We observe that Eq. (19a) does not admit a Ricci-flat solution for M_4 unless at the same time the internal space is also Ricci flat. The generic G-invariant solution is thus (de Sitter)$_4 \times G/H$.

The argument above for the vanishing of $\bar{\omega}$ depends on the standard construction of G-invariant Maurer-Cartan forms on G/H. Perhaps this argument can be circumvented if we drop the requirement of maximal symmetry, i.e., if we look for solutions which are invariant only under a subgroup of G.

Alternatively we may assume that B is not homogeneous. It should of course possess some isometries for the Kaluza idea to work. In this case the components $\bar{\omega}_{\bar{a}b\bar{C}}$ given by

$$\bar{\omega}_{\bar{a}b\bar{C}} = e^m_{\bar{a}} e^n_b e_{\bar{C}[mn]} - e^m_b e^n_{\bar{C}} e_{\bar{a}[mn]} - e^m_{\bar{C}} e^n_{\bar{a}} e_{b[mn]}$$

are nonzero, and can act as a source term for compactification. Now the ansatz $e^{\dot{\alpha}}_{\mu} = \delta^{\dot{\alpha}}_{\mu}$, $\omega^{\dot{\alpha}}_{\dot{\beta}} = 0$ (Minkowski four-space) will be compatible with $R_{\bar{A}\,\bar{B}} = \theta_{\bar{A}\,\bar{B}}$ where $\theta_{\bar{A}\,\bar{B}}$ is a (nonvanishing) function of $e^{\bar{A}}_M$. In other words we may have solutions of the form (Minkowski)$_4 \times B$ where B is a nonhomogeneous space.

IV. GENERALIZATION TO $G_T = \mathrm{SO}(1, D-1) \times G'_T$

To construct an action for the case of $G'_T \subset \mathrm{SO}(d-D)$ it is sufficient to embed G'_T in $\mathrm{SO}(d-D)$ and decompose each term of Eq. (11) in accordance with $\mathrm{SO}(d-D) \rightarrow G'_T$. Hence apart from the first and the last terms in Eq. (11), every single term will give rise to several new G_T invariants. The number of arbitrary parameters will increase correspondingly.

Additional sources of new G_T invariants are the components $\bar{\omega}^a{}_{bc}$ and $\bar{\omega}^a{}_{b\gamma}$ which vanish whenever $G'_T = \mathrm{SO}(d-D)$. These components of $\bar{\omega}$ give rise to qualitatively new contributions to the action. In fact they make it possible to find compactifying solutions of the form $M_4 \times G/H$. The general Lagrangian for $G_T = \mathrm{SO}(1, D-1) \times G'_T$ is thus a rather unwieldy object with many terms. Clearly we need some additional symmetry principle—for instance, supersymmetry to reduce the number of free parameters.

In this section we restrict ourselves to the simplest modification of Eq. (11) so that the field equations admit a compactifying solution of the form (Minkowski)$_4$ $\times G/H$. To this end we supplement Eq. (11) by the following:

$$\int \det e \, d^d z \, (c_{10} \bar{\omega}_{ab\gamma} \bar{\omega}^{ab\gamma} + c_{11} \bar{\omega}_{abc} \bar{\omega}^{abc}) \, . \tag{20}$$

Here the latin indices are raised and lowered by δ_{ab}, as this tensor is necessarily an invariant tensor of any $G'_T \subset \mathrm{SO}(d-D)$. To exhibit the solution let us further simplify the problem by setting $D=4$ and consider the following ansatz (note that now α, β range over the first four dimensions):

$$\omega_{\alpha\beta} = 0 \, , \quad \bar{\omega}_{ab} = -\bar{\omega}_{ba} = 0 \, . \tag{21}$$

The components $\bar{\omega}_{\alpha\beta}$, ω_{ab}, and $\bar{\omega}_{ab\gamma}$ vanish by virtue of the product structure $M_4 \times G/H$.

To write down the nonvanishing components of ω and $\bar{\omega}$ we return to the Maurer-Cartan equation (15) and divide the set of one-forms $\{e^{\hat{A}}\}$ into two nonoverlapping subsets $\{e^a\}$, $a = 1, \ldots, \dim G/H$ and $\{e^i\}$, $i = 1, \ldots, \dim H$. We recall that under an H operation the one-forms e^a transform covariantly while e^i transforms as an H connection.[11] Therefore the subset e^a may be identified as a basis of the one-forms in G/H.

In general H must be a subgroup of G'_T, where now G'_T is the tangent space group of G/H. Therefore to exhibit the existence of a compactifying solution it is sufficient to consider the case for which $G'_T = H$. Then the most natural G-invariant ansatz for the nonvanishing components of ω and $\bar{\omega}$ is

$$\omega^a{}_b = -c^a{}_{bi} e^i \, , \tag{22a}$$

$$\bar{\omega}^a{}_b = -\tfrac{1}{2} c^a{}_{bc} e^c \, , \tag{22b}$$

where e^i and e^A are the Maurer-Cartan forms. Our ansatz should be compatible with Eq. (4). This compatibility is immediately demonstrated with the help of the Maurer-Cartan equation (15). Substituting (22a) into Eq. (5) gives us

$$R_{abcd} = C_{iab} C_{cd}{}^i \, .$$

Now we can substitute our ansatz in the field equations derived from Eqs. (11) and (20). The result is (for details, see Appendix B)

$$R_{\alpha\beta} = 0 \, , \tag{23a}$$

$$R_{a\alpha} = 0 \, , \tag{23b}$$

$$R_{ab} = -\theta_{ab} \tag{23c}$$

with $c_1 = c_2$ and $c_9 = 0$. In the above

$$\theta_{ab} = (c/4) C_{cda} C^{cd}{}_b = (c/4)[2C_2(H) + 1]\delta_{ab}$$

with $c = c_{11}/c_1$, $C_2(H)$ being the quadratic Casimir of H in the representation (assumed for simplicity to be irreducible) of the vielbein. We have also normalized $C_{\hat{B}\,\hat{C}}{}^{\hat{A}}$ such that $C_{\hat{C}\,\hat{A}}{}^{\hat{D}} C_{\hat{D}\,\hat{B}}{}^{\hat{C}} = -\delta_{\hat{A}\,\hat{B}}$. Noting that

$$R_{ab} = C_{ac}{}^i C_{ib}{}^c = -C_2(H)\delta_{ab} \, ,$$

we see that the length scale of the G/H manifold (relative to the length scale of d-dimensional gravity) is determined by the field equations to be

$$-c = 4C_2(H)/[2C_2(H) + 1] \, . \tag{24}$$

The conclusion is that an appropriate choice of only three out of the eleven parameters of the action leads to a solution of the desired form (Minkowski)$_4 \times G/H$. It should be emphasized though, that the G/H manifold is necessarily nonsymmetric ($C_{abc} \neq 0$). Examples of such spaces (with the vielbein in a single irreducible representation of H) are SU(6)/SU(3) with the embedding of SU(3) in SU(6) such that **6** of SU(6) branches into **6** of SU(3) with the vielbein e^a in the **27** of SU(3) or SU(10)/SU(5) with e^a in the **75** of SU(5), etc.

V. SPINOR LAGRANGIANS

One of the original motivations for introducing quasi-Riemannian gravity was the possibility of obtaining four-dimensional chiral fermions from a higher-dimensional Einstein-Dirac theory.[6] It is therefore appropriate to construct a suitable action integral for the Fermi fields.

The case of $G_T = \mathrm{SO}(1,D-1)\times\mathrm{SO}(d-D)$ is simple and will be considered first. We denote by γ^α and γ^a the γ matrices of $\mathrm{SO}(1,D-1)$ and $\mathrm{SO}(d-D)$, respectively, and define Γ^M by

$$\Gamma^M = (\gamma^\alpha \otimes 1)e_\alpha^M + (1\otimes\gamma^a)e_a^M \,. \tag{25}$$

Then the following Lagrangian satisfies all of our invariance requirements:

$$\mathscr{L} = \det e\, \bar\psi\, i\Gamma^M D_M \psi + \mathrm{H.c.}\,, \tag{26}$$

where ψ transforms as a spinor of $\mathrm{SO}(1,D-1)$ and $\mathrm{SO}(d-D)$, and D_M is defined by

$$D_M = \partial_M + \tfrac{1}{4}\omega_{[\alpha\beta]M}\gamma^\alpha\gamma^\beta + \tfrac{1}{4}\omega_{[ab]M}\gamma^a\gamma^b \,. \tag{27}$$

It should be mentioned that in order to obtain SO(1,3) spinors upon compactification it is essential for ψ to be a spinor of $\mathrm{SO}(1,D-1)$. However, in general, it may transform according to any representation of $\mathrm{SO}(d-D)$.

It is fairly straightforward to generalize (26) for a spinor p-form

$$\psi = \frac{1}{p!}\psi_{M_1\cdots M_p} dz^{M_1}\wedge\cdots\wedge dz^{M_p} \,, \tag{28}$$

where $\psi_{M_1\cdots M_p}$ is a spinor of $\mathrm{SO}(1,D-1)\times\mathrm{SO}(d-D)$ and is completely antisymmetric in its covariant indices.

The appropriate $\mathscr{L}$ for such a field is a generalization of the usual Rarita-Schwinger Lagrangian,

$$\mathscr{L} = \bar\psi\Gamma\wedge{}^*D\psi + \mathrm{H.c.}\,, \tag{29}$$

where $D = d+\omega$ as in Eq. (27) and $\Gamma \equiv \Gamma_M dZ^M$ with $\Gamma_M \equiv e_M^A e_N^B \eta_{AB}\Gamma^N$ and Γ^N defined as in Eq. (25).

Generalization to the case of $G_T = \mathrm{SO}(1,D-1)\times G_T'$ with $G_T' \subset \mathrm{SO}(d-D)$ is not unique. One possible way of constructing such a Lagrangian is to fix an embedding of G_T' in $\mathrm{SO}(d-D)$ and thus decompose (29) into irreducible pieces under the branching of $\mathrm{SO}(d-D)\to G_T'$.

Alternatively we may consider a field $\psi^{a_1\cdots a_r}$ which is a spinor of $\mathrm{SO}(1,D-1)$ as before but carries the G_T' indices $a_1\cdots a_r$. Then defining Γ and D by

$$\Gamma = (\gamma_\alpha\otimes 1)e^\alpha \,, \tag{30}$$

$$D = d + \tfrac{1}{4}\omega_{[\alpha\beta]}\gamma^\alpha\gamma^\beta + \omega_I T^I \,, \tag{31}$$

where T^I are the generators of G_T' in the representation characterized by the indices $a_1,\ldots,a_r$, we may write

$$\mathscr{L} = \eta^{a_1\cdots a_r b_1\cdots b_r}\bar\psi_{a_1\cdots a_r}\Gamma\wedge{}^*D\psi_{b_1\cdots b_r} + \mathrm{H.c.}\,, \tag{32}$$

where η is an invariant tensor of G_T'.

Note that when $G_T = \mathrm{SO}(1,3)\times G_T'$ to leading order in the fluctuations around $(\mathrm{Minkowski})_4\times B$ background there will be no mass term since $\bar\psi\Gamma\wedge{}^*D\psi \simeq \bar\psi\gamma^\mu D_\mu\psi$. Thus the harmonic expansion will give an infinite number of zero-mass fermions. But only those belonging to complex representations (chiral fermions) of the isometry of G_T' will remain massless when radiative effects are taken into account.

Of course (32) can be generalized to include $\gamma^m D_m$ terms as well. For example, let $\psi_{a_1\cdots a_r}$ be a spinor of $\mathrm{SO}(-1,D-1)$, and be in some representation $\Sigma\otimes R\otimes R\otimes\cdots\otimes R$ of G_T' where a_i are R indices and the indices are omitted ($\psi_{a_1\cdots a_r}$ is a column matrix as a carrier of Σ). Then if the vielbein e^a is in a representation V of G_T' such that $V\subset\Sigma\otimes\bar\Sigma$ and C_a is the corresponding Clebsch-Gordan coefficient, then we may replace (30) by

$$\Gamma = \gamma_\alpha\otimes 1 e^\alpha + 1\otimes C_a e^a$$

and with the appropriate modification of D, (32) will be a fermion Lagrangian density. For the case $G_T' = \mathrm{SO}(d-D)$ of course $C_a = \gamma_a$.

The field $\psi_{a_1\cdots a_r}$ is very similar to a field of spin $\geq\frac{3}{2}$ (for $r\geq 1$). The crucial difference is that the indices $a_1,\ldots,a_r$ transform according to the subgroup G_T' of the full tangent space group. This enables us to avoid the usual problems associated with higher spin fields.[12] Witten has already shown that this type of spinor leads to chiral fermions upon compactification,[13] for example, when $r=2$.

VI. CONCLUSION

In this paper we constructed action integrals for Weinberg's quasi-Riemannian theories of gravity. We studied the possibilities of spontaneous compactification into a product space $(\mathrm{Minkowski})_4\times G/H$ and exhibited explicit solutions in which the tangent space group of the d-dimensional manifold is $G_T = \mathrm{SO}(1,3)\times H$. This solution may be generalized to the case where $G_T = \mathrm{SO}(1,D-1)\times G_T'$, with $G_T\subset\mathrm{SO}(1-D)$. We also dealt with fermionic fields and constructed Lagrangians compatible with the invariance requirements of the theory.

It was remarked in the final paragraph of the last section that the possibility of having spinor tensor fields $\psi_{a_1\cdots a_r}$ with the tensor indices transforming only under G_T' can, in general, lead to chiral fermions upon compactification without the usual unitarity problems of the standard higher spin fields.

Although a reduced tangent space group G_T avoids some of the usual problems of the Riemannian theory, it also loses its uniqueness features. For a general $G_T = \mathrm{SO}(1,D-1)\times G_T'$, $G_T'\subset\mathrm{SO}(d-D)$, the invariant Lagrangian depends on many arbitrary parameters. For example, for $G_T' = \mathrm{SU}(3)$ with the vielbein in the octet representation of SU(3) we can write down at least 40 independent terms compatible with all invariances of the theory. Some of these terms may be excluded by the requirement of the absence of ghost and tachyon excitations. However the number of the remaining terms will still be large. Obviously a more fundamental principle such as supersymmetry is needed to reduce the number of the independent parameters. The requirement of anomaly cancellation may put further restrictions on the possible forms of the Lagrangians.

ACKNOWLEDGMENTS

We are grateful to Professor Abdus Salam and Professor J. A. Strathdee for useful discussions at earlier stages of this work. We are also appreciative of useful discussions with Professor I. G. Koh, Professor H. Leutwyler, Dr. S. Mallik, and Dr. C. Wetterich. We would also like to thank Professor S. Weinberg for pointing out an error in the counting of the number of independent invariant densities for the $G_T = \mathrm{SO}(1,D-1)\times\mathrm{SO}(d-D)$ case in an earlier unpublished version of this paper and for sending us a copy of his unfinished draft on constructing Lagrangians for quasi-Riemannian gravity. This work was supported in part by Schweizerischer Nationalfonds.

APPENDIX A: SOME DEFINITIONS AND USEFUL RESULTS

In an n-dimensional manifold with vielbein one-forms

$$e^a = e^a_\mu dx^\mu\,, \quad a=1,\ldots,n\,,\ \mu=1,\ldots,n \tag{A1}$$

we define p-forms

$$e^{a_1\cdots a_p} = e^{a_1}\wedge\cdots\wedge e^{a_p}\,. \tag{A2}$$

The Hodge duality operation $*$ is defined by

$$*e^{a_1\cdots a_p} = \frac{1}{(n-p)!}\epsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_n} e^{a_{p+1}\cdots a_n}\,, \tag{A3}$$

where $\epsilon_{a_1\cdots a_n}$ is completely antisymmetric in $a_1\cdots a_n$ and $\epsilon_{12\cdots n} = +1$.

Indices are raised and lowered with the metric tensor $\eta_{ab} = \mathrm{diag}(-1,\ldots,1)$ (Minkowskian) or $\eta_{ab} = \delta_{ab}$ (Euclidean). η_{ab} and $\epsilon_{a_1\cdots a_n}$ are invariant $\mathrm{SO}(1,n-1)$ [or $\mathrm{SO}(n)$] tensors and hence G_T [$\mathrm{SO}(1,n-1)$] invariants. The invariant (i.e., invariant under general coordinate transformations) volume element is

$$dV = \det e\, d^n z = e^{12\cdots n}\,. \tag{A4}$$

Torsion is defined to be

$$T^a \equiv de^a + \omega^a{}_b \wedge e^b\,. \tag{A5}$$

The following identity is useful in translating from form language to tensor notation:

$$\Phi\wedge *\Sigma = \Phi_{a_1\cdots a_p}\Sigma^{b_1\cdots b_p}\delta^{a_1\cdots a_p}_{b_1\cdots b_p}\,, \tag{A6}$$

where

$$\delta^{a_1\cdots a_p}_{b_1\cdots b_p} = \sum_P (-1)^P \delta^{b_1}_{a_{P_1}}\delta^{b_2}_{a_{P_2}}\cdots\delta^{b_p}_{a_{P_p}} \det e\,, \tag{A7}$$

P being a permutation. Thus, for example,

$$R_{ab}\wedge * e^{ab} = \tfrac{1}{2}R^{ab}{}_{cd}\delta^{cd}_{ab}\det e = R\det e\,, \tag{A8}$$

where

$$R_{ab} = \tfrac{1}{2}R_{abcd}e^{cd}\,. \tag{A9}$$

Let G be a Lie group and $g^a{}_b$ a representation matrix of an element g of G in an arbitrary representation R (not necessarily the fundamental). Let

$$\omega = \omega^i T^i = \omega^i_\mu T^i dx^\mu \tag{A10}$$

be a connection (on the associated G bundles) valued in the Lie algebra of G, T^i being generators of G in the representation R. Under the action of G, $\omega\to\omega^g = g^{-1}(\omega+d)g$.

Consider a tensor p-form Φ which belongs to a representation which is a direct product of R's,

$$\phi^{a_1\cdots a_n}_{b_1\cdots b_m} \to g^{-1a_1}{}_{a_1'}\cdots g^{-1a_n}{}_{a_n'}\phi^{a_1'\cdots a_n'}_{b_1'\cdots b_m'} g^{b_1'}{}_{b_1}\cdots g^{b_m'}{}_{b_m} \tag{A11}$$

or more schematically,

$$\Phi\to\Phi^g = \prod_n g^{-1}\cdot\Phi\cdot\prod\nolimits_m g\,. \tag{A12}$$

Let Λ be a matrix-valued q-form. Define the inner products

$$(\Lambda\cdot\Phi)^{a_1\cdots a_n}_{b_1\cdots b_m} = \sum_{r'}\Lambda^{a_r}{}_{a_r'}\wedge\Phi^{a_1\cdots a_r'\cdots a_n}_{b_1\cdots b_m} \tag{A13}$$

and

$$(\Phi\cdot\Lambda)^{a_1\cdots a_n}_{b_1\cdots b_m} = \sum_{r'}\Phi^{a_1\cdots a_n}_{b_1\cdots b_r'\cdots b_m}\wedge\Lambda^{b_r'}{}_{b_r} \tag{A14}$$

and the commutator

$$[\Lambda,\Phi] = \Lambda\cdot\Phi - (-1)^{qp}\Phi\cdot\Lambda\,. \tag{A15}$$

From (A11) we have for an infinitesimal transformation $g = 1+v$,

$$\Phi\to\Phi^g = \Phi - [v,\Phi]\,. \tag{A16}$$

It is easily seen that

$$D\Phi \equiv d\Phi + [\omega,\Phi] \tag{A17}$$

is a G covariant derivative

$$(D\Phi)^g = \prod g^{-1}D\Phi\prod g\,. \tag{A18}$$

Now let $\Phi = \Sigma\cdot\Delta$ be a suitably contracted product of two tensor-valued forms Σ and Δ of degrees r and s, respectively ($p = r+s$). We see then that

$$[\Lambda,\Phi] = [\Lambda,\Sigma]\cdot\Delta + (-1)^{qr}\Sigma\cdot[\Lambda,\Delta]\,. \tag{A19}$$

Hence we have the distributive property of the covariant derivative

$$D\Phi = (D\Sigma)\cdot\Delta + (-1)^r\Sigma\cdot(D\Delta)\,. \tag{A20}$$

If Φ is an invariant (scalar) of G, then

$$D\Phi = d\Phi\,. \tag{A21}$$

If Φ is globally defined, then integrating over a p-dimensional manifold m with $\partial m = 0$ yields from Stokes's theorem and (A20)

$$\int_m (D\Sigma)\cdot\Delta = -(-1)^r\int_m \Sigma\cdot(D\Delta)\,. \tag{A22}$$

Note also that an invariant tensor is covariantly constant; i.e., if $\eta^{a_1\cdots a_n}{}_{b_1\cdots b_m}$ is such that

$$g^{-1a_1}{}_{a'_1}\cdots g^{-1a_n}{}_{a'_n}\eta^{a'_1\cdots a'_n}{}_{b'_1\cdots b'_m}g^{b'_1}{}_{b_1}\cdots g^{b'_m}{}_{b_m}=\eta^{a_1\cdots a_n}{}_{b_1\cdots b_m} \quad (A23)$$

and $d\eta=0$, then

$$D\eta=0 . \quad (A24)$$

Another useful result is the following. Let D be a G_T covariant derivative. Then

$$D^*e^{a_1\cdots a_p}=D\frac{1}{(n-p)!}\epsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_n}e^{a_{p+1}\cdots a_n}$$
$$=\frac{1}{(n-p)!}\epsilon^{a_1\cdots a_p}{}_{a_{p+1}\cdots a_n}De^{a_{p+1}\cdots a_n}$$
$$=T_{a_{p+1}}\wedge {}^*e^{a_1\cdots a_p a_{p+1}} . \quad (A25)$$

The second equality in the above follows from the SO(n) [and hence $G_T\subset$SO(n)] invariance of $\epsilon^{a_1\cdots a_n}$, η_{ab}, (A20), and (A24), and the third from (A15) and the definition of torsion (A5).

By making use of Eqs. (A25) and (A26) we can demonstrate that any terms involving $D\varpi$ in the action can be reexpressed as a sum of bilinears in ϖ. This may be seen from the fact that a $D\varpi$ term in the action is necessarily of the form

$$\int_m D\varpi^A{}_B\wedge {}^*e^{CD}\eta^B{}_{ACD} , \quad (A26)$$

where $\eta^B{}_{ACD}$ is a G-invariant tensor.

In deriving field equations from Lagrangians written in form language, the following results are useful. Let δ_e denote the variation of a p-form with respect to the one-form e, i.e.,

$$\delta_eF(\Phi,e)=F(\Phi,e+\delta e)-F(\phi,e) , \quad (A27)$$

where Φ is a form which is taken to be independent of e. Then

$$\delta_e{}^*e^{a_1\cdots a_p}=\delta e_{p+1}\wedge {}^*e^{a_1\cdots a_{p+1}} . \quad (A28)$$

So, for example,

$$\delta_eR_{ab}(\omega)\wedge {}^*e^{ab}=\delta e_c\wedge R_{ab}(\omega)\wedge {}^*e^{abc}$$
$$=(\delta_{ab}R-2R_{ab})\Delta^{ab}dV , \quad (A29)$$

where we put $\delta e_c=\Delta_c{}^de_d$. Also from

$$R^a{}_b(\omega)=d\omega^a{}_b+\omega^a{}_c\wedge\omega^c{}_b \quad (A30)$$

we have

$$\delta_\omega R^a{}_b(\omega)=d\delta\omega^a{}_b+\delta\omega^a{}_c\wedge\omega^c{}_b+\omega^a{}_c\wedge\delta\omega^c{}_b$$
$$=D\delta\omega^a{}_b . \quad (A31)$$

APPENDIX B: THE FIELD EQUATIONS

It is convenient to use the Lagrange multiplier method and the Cartan calculus to derive the field equations. Let us consider first a theory with any tangent space group $G_T\subset$SO($1,d-1$) (with the vielbein e^A in some real representation of it), a connection $\omega^A{}_B$, and a torsion (two-form) field T^A. e^A, $\omega^A{}_B$, and T^A are related by

$$De^A=de^A+\omega^A{}_B\wedge e^B=T^A . \quad (B1)$$

Let us, however, consider the action as a functional of the independent fields e, ω, and T:

$$S=S[e^A,\omega^A{}_B,T^A] .$$

Now introduce a Lagrange "parameter" two-form ϕ^A and define the new action

$$\bar S[e,\omega,T,\phi]\equiv S[e,\omega,T]+\int\phi_A\wedge {}^*(De^A-T^A) . \quad (B2)$$

Defining $\delta_e\bar S\equiv\bar S[e^A+\delta e^A,\omega,T,\phi]-\bar S[e,\omega,T,\phi]$ and similarly $\delta_\omega\bar S$, $\delta_T\bar S$, and $\delta_\phi\bar S$, we have (treating e^A, $\omega^A{}_B$, T^A, ϕ as independent, i.e., unconstrained, forms) from the variational principle:

$$\delta_e\bar S=\delta_eS+\int\phi_A\wedge\delta_e{}^*(De^A-T^A)=0 , \quad (B3a)$$

$$\delta_\omega\bar S=\delta_\omega S+\int\phi_A\wedge {}^*\delta\omega^A{}_B\wedge e^B=0 , \quad (B3b)$$

$$\delta_T\bar S=\delta_TS-\int\phi_A\wedge {}^*\delta T^A=0 , \quad (B3c)$$

$$\delta_\phi\bar S=\int\delta\phi_A\wedge {}^*(De^A-T^A)=0 . \quad (B3d)$$

The last equation gives us the constraint whilst the second and third enable us to eliminate ϕ_A, T_A, and some or all components of ω (depending on the dimension of G_T). Substituting into the first we can get the field equations in terms of the independent fields of the theory. Let us illustrate the above method for pure Riemannian gravity. Here $T^A=0$ and

$$S[e,\omega]=\int R\det e\, d^nx=\int R_{AB}\wedge {}^*e^{AB} . \quad (B4)$$

From (A31) and (A22)

$$\delta_\omega S[e,\omega]=\int\delta\omega^A{}_B\wedge D^*e^{AB} . \quad (B5)$$

Since $T^a=0$, from (A25) $D^*e^{AB}=0$, which yields $\delta_\omega S=0$ giving us $\phi_A=0$ from (B3b) so that (B3a) gives

$$\delta_eS[e,\omega]=0 \quad (B6)$$

which is Einstein's equation [see (A20)].

For the case of $G_T=$SO($1,D-1$)$\times G'_T$ with $T^A=-\varpi^A{}_B\wedge e^B$, out of the plethora of terms involving ϖ [see Eq. (11) or Eq. (20)] we will just keep one term for illustrative purposes, namely, $\varpi_{AB}\wedge\varpi^A{}_C\wedge {}^*(e^B\wedge e^C)$. This will in fact be a linear combination of terms in (10) and (20), but in the case $D=4$ with $G'_T\subset$SO($d-4$) it will act as a source term for compactification. Thus we write

$$S[e,\omega,\varpi]=c_1\int R_{\alpha\beta}(\omega)\wedge {}^*e^{\alpha\beta}+c_2\int R_{ab}(\omega)\wedge {}^*e^{ab}$$
$$+c_{11}\int\varpi_{AB}\wedge\varpi^A{}_C\wedge {}^*e^{BC} \quad (B7)$$

and

$$\bar S[e,\omega,\varpi,\phi]=S[e,\omega,\varpi]$$
$$+\int\phi_A\wedge {}^*[de^A+\omega^A{}_B\wedge e^B$$
$$+\varpi^A{}_B\wedge e^B] . \quad (B8)$$

In the above we have used $\eta^{AB}=(\eta^{\alpha\beta},\delta^{ab})$ to raise and lower the indices. Note also that the c_{11} term [in Eq.

(B4)] is actually a linear combination of the terms in Eqs. (10) and (20). Variation of $-S$ with respect to ϕ gives the constraint, and variations with respect to ω and ϖ will determine ϕ and variation with respect to e^{α},e^{a} will give the field equations. The latter are as follows (after substituting the constraint):

$$c_1 R_{\alpha\beta}\wedge {}^*e^{\alpha\beta\gamma}+c_2 R_{ab}\wedge {}^*e^{ab\gamma}+c_{11}\varpi_{AB}\wedge\varpi^{A}{}_{C}\wedge {}^*e^{BC\gamma}+D^*\phi^{\gamma}+\varpi^{\gamma B}\wedge {}^*\phi_B=0\ , \qquad (B9)$$

$$c_1 R_{\alpha\beta}\wedge {}^*e^{\alpha\beta d}+c_2 R_{ab}\wedge {}^*e^{abd}+c_{11}\varpi_{AB}\wedge\varpi^{A}{}_{C}\wedge {}^*e^{BCd}+D^*\phi^{d}+\omega^{dB}\wedge {}^*\phi_B=0\ . \qquad (B10)$$

In varying with respect to ω and ϖ we should write

$$\omega^{A}{}_{B}=\omega^{I}Q^{IA}{}_{B}\ ,\quad \varpi^{A}{}_{B}=\varpi^{\hat I}Q^{\hat I A}{}_{B} \qquad (B11)$$

and vary with respect to the independent one-forms ω^{I} and $\varpi^{\hat I}$. Here Q^{I} and $Q^{\hat I}$ are the subsets of the generators of SO$(1,d-1)$ in the subgroup G_T and the complementary space SO$(1,d-1)/G_T$. Then we get the following equations:

$$e^{[\beta}\wedge {}^*\phi^{\alpha]}=c_1\varpi_{CD}\wedge e^{D}\wedge {}^*e^{\alpha\beta C}\ , \qquad (B12a)$$

$$Q^{I}{}_{ab}e^{b}\wedge {}^*\phi^{a}=c_2 Q^{I}{}_{ab}\varpi_{CD}\wedge e^{D}\wedge {}^*e^{abC}\ , \qquad (B12b)$$

$$Q^{\hat I}{}_{AB}e^{B}\wedge {}^*\phi^{A}=-2c_{11}Q^{\hat I}{}_{AB}\varpi^{A}{}_{C}\wedge {}^*e^{BC}\ . \qquad (B12c)$$

In the first of these we have used the fact that $(Q^{\hat I})_{\alpha\beta}=0$. These equations determine ϕ in terms of e^{A} and ϖ $[=\varpi(e)$ on using the constraint] and substitution into (B9), (B10) gives the field equations for the vielbeins.

We note first that if ϖ is zero, ϕ is zero and only the curvature terms are left in the field equations. Thus in the SO$(1,D-1)\times$SO$(d-D)$ case we have the vacuum field equations (19a), (19b) when we require the internal manifold to be G/H. This conclusion is obviously unchanged even if all possible ϖ terms are included in (B7) since they must all be necessarily quadratic in ϖ.

Next consider the case of $G_T'=H$. Substituting the vacuum values Eqs. (22a), (22b) for ϖ_{CD} into Eqs. (B12a)–(B12c) we find

$$\phi^{\alpha}=0 \qquad (B13)$$

and

$$\phi^{a}=c_{11}c^{a}{}_{bc}e^{bc}\ . \qquad (B14)$$

To derive (B14) we have used the completeness relation

$$Q^{I}{}_{ab}Q^{I}{}_{cd}+Q^{\hat I}{}_{ab}Q^{\hat I}{}_{cd}=\tfrac{1}{2}(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc})\ . \qquad (B15)$$

From the H (and hence $G_T'=H$) invariance of C_{abc} it follows that

$$D^*\phi^{a}+\varpi^{a}{}_{c}\wedge {}^*\phi^{c}=0\ . \qquad (B16)$$

Upon substituting Eqs. (22a), (22b), (B13), (B14), (B16) into Eqs. (B9), (B10) we get

$$R_{\alpha\beta}=0\ , \qquad (B17)$$

$$R_{ab}=\theta_{ab}\ , \qquad (B18)$$

where

$$\theta_{ab}=\frac{c}{4}C_{cda}C^{cd}{}_{b}\ . \qquad (B19)$$

---

*Present address: Department of Physics, The University of Texas at Austin, Austin, Texas 78712.

[1]L. Palla, in *Proceedings of the 19th International Conference on High Energy Physics, Tokyo, 1978,* edited by S. Homma, M. Kawaguchi, and H. Miyazawa (Phys. Soc. of Japan, Tokyo, 1979), p. 629; E. Witten, Proceedings of the 1983 Shelter Island II Conference (unpublished); C. Wetterich, Nucl. Phys. **B223**, 109 (1983).

[2]H. Sarmadi, University of Trieste Report No. IC/84/3, 1984 (unpublished); S. Randjbar-Daemi and H. Sarmadi, Phys. Lett. **151B**, 343 (1985).

[3]A. Chodos and E. Myers, Yale Report No. YTP84-09, 1984 (unpublished); C. R. Ordonez and M. Rubin, University of Texas Report No. UTTG 18-84 (unpublished).

[4]Z. Horvath, L. Palla, F. Cremmer, and J. Scherk, Nucl. Phys. **B127**, 57 (1977); G. Chapline and R. Slansky, *ibid.* **B209**, 461 (1982); S. Randjbar-Daemi, Abdus Salam, and J. Strathdee, *ibid.* **B214**, 491 (1983); Phys. Lett. **132B**, 56 (1983); P. H. Frampton and K. Yamamoto, Phys. Rev. Lett. **52**, 2016 (1984); C. Wetterich, Nucl. Phys. **B244**, 359 (1984); K. Pilch and A. N. Schellekens, Stony Brook Report No. ITB-SB-84-69 (unpublished).

[5]C. Wetterich, Nucl. Phys. **B252**, 473 (1984); **B253**, 366 (1985); M. Gell-Mann and B. Zwiebach, Phys. Lett. **147B**, 111 (1984).

[6]S. Weinberg, Phys. Lett. **138B**, 47 (1984).

[7]The direct-product structure is necessitated by the requirement (a) of local Lorentz invariance in four dimensions, (b) that the vielbein is a single four-vector plus scalars under SO(1,3) (see Ref. 6).

[8]S. P. de Alwis and I. G. Koh, Phys. Lett. **148B**, 428 (1984).

[9]S. Weinberg, University of Texas Report No. UTTG-12-84, 1984 (unpublished).

[10]We use the following conventions: $\eta_{AB}=\mathrm{diag}(-1,1,\ldots,1)$, $A,B=1-d$, $M,N=1-d$; A,B are indices relative to a nonholomonic basis, and M,N are indices relative to a holonomic basis, $\alpha,\beta=1,\ldots,D$ and $a,b=D+1,\ldots,d$. For a Lie-algebra-valued one-form ω we define the components by the following: $\omega=\frac{1}{2}Q^{B}{}_{A}\omega^{A}{}_{BC}e^{C}$ where $Q^{B}{}_{A}$ are the generators of the Lie algebra and e^{C} are the basis one-forms.

[11]Abdus Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982).

[12]M. T. Grisaru, H. N. Pendleton, and P. Van Nieuwenhuizen, Phys. Rev. D **15**, 996 (1977); F. A. Berends, J. W. van Holten, B. de Wit, and P. Van Nieuwenhuizen, Nucl. Phys. **B154**, 261 (1979); Phys. Lett. **83B**, 188 (1979); J. Phys. A **13**, 1643 (1980); C. Aragone and S. Deser, Phys. Lett. **85B**, 161 (1979); Phys. Rev. D **21**, 352 (1980); K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) **13**, 126 (1961); G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1967).

[13]See Witten, Ref. 1.