Cosmological compactification

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Assuming that all infinitesimal Killing vectors integrate to form a group of finite isometries, cosmological compactification of a fifth dimension is discussed. It requires a positive cosmological constant, while supporting both the big-bang singularity and the open character of ordinary space. The constancy of the fine-structure constant is correlated with the smallness of the cosmological constant.

The geometrical origin of local gauge invariance is quite an old[1] yet extremely provocative idea. Its renaissance stems from the popularity of electronuclear unifying theories,[2] whose characteristic mass scale is in an intriguing vicinity to the Planck mass. The realization of such an idea requires the compactification of extra dimensions. Possible scenarios for this to happen involve antisymmetric tensor fields[3] or sophisticated scalars.[4] Our expanding Universe, and the theoretical role played by grand unification in explaining its evolution, provide the motivation for linking[5,6] the Kaluza-Klein idea with cosmology. It is the static Kaluza-Klein scheme that we challenge, suggesting a possible cosmological origin for compactification.

We start by discussing an extended five-dimensional Robertson-Walker (RW) metric

$$ds^2=-dt^2+R^2(t)\frac{dx^i dx^i}{(1+\frac{1}{4}kr^2)^2}+a^2(t)dy^2 , \tag{1}$$

compatible with the ordinary cosmological principle. Given the spatial 3+1 split, what we want to understand is why does the y coordinate compactify? In other words, can the geometry give us, under reasonable physical assumptions, some clues concerning the underlying topology? The scale functions $R(t)$ and $a(t)$ are determined by the following Einstein equations:

$$\frac{\dot R^2+k}{R^2}+\frac{\dot R\dot a}{Ra}=\tfrac{1}{3}(\rho+\Lambda) , \tag{2a}$$

$$\frac{\dot R^2+k}{R^2}+2\frac{\ddot R}{R}+2\frac{\dot R\dot a}{Ra}+\frac{\ddot a}{a}=-P+\Lambda , \tag{2b}$$

$$\frac{\dot R^2+k}{R^2}+\frac{\ddot R}{R}=\tfrac{1}{3}(-Q+\Lambda) . \tag{2c}$$

$\rho(t)$, $P(t)$, and $Q(t)$ are the normalized energy density and pressures defined via

$$T^m{}_n\sim\mathrm{diag}(\rho,-P,-P,-P,-Q) ,$$

and Λ denotes the cosmological constant. For pedagogical reasons, we first solve the above equations in an empty ($\rho=P=Q=0$) five-dimensional space-time. The solution is

$$R^2(t)=\begin{cases}\xi\cosh 2\omega t+\eta\sinh 2\omega t+\dfrac{3k}{\Lambda} & \text{for } \Lambda>0 ,\\ -kt^2+\xi t+\eta & \text{for } \Lambda=0 ,\\ \xi\cos 2\omega t+\eta\sin 2\omega t+\dfrac{3k}{\Lambda} & \text{for } \Lambda<0 ,\end{cases} \tag{3}$$

$$a(t)\sim\frac{d}{dt}R(t) , \tag{4}$$

where $\omega^2=\frac{1}{6}|\Lambda|$, and ξ,η are fixed by initial conditions. Two remarks are in order. (1) There exists a choice of parameters for which the Kasner metric[6] is a limiting form of the above solution, and (2) for $\Lambda=0$ and $k\le 0$, $a(t)$ exhibits asymptotic constancy.[7]

We now analyze the local isometries of the above solutions. For the sake of simplicity, we momentarily content ourselves with $\mathbf{x}$=const surfaces, dealing with the residual 1+1 metric

$$ds^2=-dt^2+a^2(t)dy^2 . \tag{5}$$

The form invariance of Eq. (5) is respected by the infinitesimal transformations $y\to y+\epsilon\alpha(y,t)$ and $t\to t+\epsilon\beta(y,t)$ provided

$$\frac{\partial\beta}{\partial t}=\dot a\beta+a\frac{\partial\alpha}{\partial y}=\frac{\partial\beta}{\partial y}-a^2\frac{\partial\alpha}{\partial t}=0 . \tag{6}$$

The solution can be written as $\alpha=-(\dot a/a)Y(y)+T(t)$ and $\beta=Y'(y)$, subject to the constraint

$$Y''+(\ddot a a-\dot a^2)Y-a^2\dot T=0 . \tag{7}$$

If $\ddot a a-\dot a^2\neq$const, $Y(y)$ must be y independent. Consequently, only one Killing vector survives, corresponding to trivial translations in y. But if $a(t)$ is special, in the sense that $\ddot a a-\dot a=$const, the situation is completely different. Equation (7) then splits into

$$\ddot a a-\dot a^2=p \Longrightarrow Y''+pY=q,\ \ \dot T=q/a^2 , \tag{8}$$

with p,q being constant parameters, so that the metric (5) becomes maximally symmetric. The crucial observation is that the $p>0$ case, with its corresponding scale

$$a(t)=a_0\cosh(\omega t+\phi), \quad p=a_0^2\omega^2 , \tag{9}$$

exhibits Killing vectors periodic in y. It is this periodicity which is hereby advocated as the possible origin of the extra dimension compactification.

At this stage, we find it necessary to assume that *all* infinitesimal Killing vectors integrate to give a group of finite isometries. This is how local properties acquire global significance, allowing us to identify the underlying manifold as the surface

$$-u_1^2+u_2^2+u_3^2=1/\omega^2 , \tag{10}$$

living in a flat Minkowski space $(ds^2=-du_1^2+du_2^2+du_3^2)$. The parametrization associated with the proper scale (9) is found to be

$$\begin{aligned} u_1&=\frac{1}{\omega}\sinh\omega t ,\\ u_2&=\frac{1}{\omega}\cosh\omega t\cos\sqrt{p}\,y ,\\ u_3&=\frac{1}{\omega}\cosh\omega t\sin\sqrt{p}\,y . \end{aligned} \tag{11}$$

Attached to each cosmic time t there is a circle of radius $(1/\sqrt{p})a_0\cosh\omega t$, parametrized by the angle $\sqrt{p}y$, reflecting the SO(2) subgroup of the initial SO(1,2).

Notice that the definition of the cosmic time is crucial. Given the same manifold (10), a different parametrization, namely,

$$u_1=(1/\omega)\sinh\omega t'\cosh\sqrt{p'}y' ,$$

$$u_2=(1/\omega)\sinh\omega t'\sinh\sqrt{p'}y', \quad u_3=(1/\omega)\cosh\omega t' ,$$

gives rise to $ds^2=-dt'^2+a_0^2\sinh^2\omega t' dy'^2$. Consequently, we obtain $\ddot{a}a-\dot{a}^2=-p'=-a_0^2\omega^2<0$, so that the corresponding y' space turns out to be infinite for any t'. In a group-theoretical language, the space defined by the new cosmic time is associated with the noncompact SO(1,1) of our SO(1,2).

We return now to the complete $1+(3+1)$ space-time. Although the set of Killing vectors is more complicated now, our former conclusions persist. In particular the scale factor $a(t)$ must again be of the form (9) if y periodicity is to be a natural cosmological consequence. A straightforward investigation of the solutions (3) and (4) reveals that the desirable $a(t)$ can only live in the $\Lambda>0$ category, and furthermore requires an algebraic relation between ξ and η, namely,

$$\xi^2-\eta^2=\left[\frac{3k}{\Lambda}\right]^2 . \tag{12}$$

In turn, we find that $R(t)$ and $a(t)$ are of the same functional form, only with exchanged amplitudes

$$\begin{aligned} R(t)&=A\cosh wt+B\sinh wt ,\\ a(t)&=c(B\cosh wt+A\sinh wt) , \end{aligned} \tag{13}$$

where $w^2=\frac{1}{6}\Lambda$. The arbitrary parameter c can be as small as desired. Note that

$$p=\tfrac{1}{6}\Lambda c^2(B^2-A^2)=-kc^2 , \tag{14}$$

so that p and k are necessarily of opposite signs. If cosmological compactification is achieved, that is, $p>0$, the ordinary three-space must be open ($k<0$). Yet the hyperbolic structure of $a(t)$ is not a sufficient condition for compactification. The additional requirement $B^2-A^2>0$ will be shown to be related with the big-bang singularity of the ordinary space.

Next we study the effective four-dimensional world. The Robertson-Walker cosmology can be extracted from the five-dimensional Kaluza-Klein model, being aware of the fact that now the four-dimensional action is

$$S_4=-\frac{1}{\kappa_4}\int d^4x\sqrt{g_4}\left[R_4+\frac{\Lambda}{a}+\frac{3}{2}\frac{\partial_\mu a\partial_\mu a}{a^2}\right] , \tag{15}$$

where R_4 is the Riemann curvature scalar associated with the metric

$$g_{\mu\nu}=a(t)\,\mathrm{diag}\left[-1,R^2(t)\frac{\delta_{ij}}{(1+\frac{1}{4}kr^2)^2}\right] . \tag{16}$$

From the action (15), an effective energy-momentum tensor $T_{\mu\nu}$ can be read off given by

$$\begin{aligned} \rho_{\mathrm{eff}}&=\frac{3}{2a}\left[\frac{\dot{a}}{a}\right]^2+\frac{\Lambda}{a} ,\\ P_{\mathrm{eff}}&=\frac{3}{2a}\left[\frac{\dot{a}}{a}\right]^2-\frac{\Lambda}{a} , \end{aligned} \tag{17}$$

so that the effective four-dimensional equation of state is

$$P+\frac{\Lambda}{a}=\rho-\frac{\Lambda}{a} . \tag{18}$$

In order to interpret the four-dimensional metric (16) in the usual Robertson-Walker way, i.e.,

$$ds^2=-d\tau^2+R_{\mathrm{RW}}^2(\tau)\frac{dx_idx_i}{(1+\frac{1}{4}kr^2)^2} , \tag{19}$$

a redefinition of proper time is needed:

$$\tau=\sqrt{a_0}\int_0^t[\cosh(\omega t+\phi)]^{1/2}dt' , \tag{20}$$

$$R_{\mathrm{RW}}^2=a(t(\tau))R^2(t(\tau)) . \tag{21}$$

In the limit $wt\ll 1$, $\tau=\sqrt{a_0}t$ and

$$R_{\mathrm{RW}}^2(\tau)=c\left[BA^2+\frac{\omega\tau}{\sqrt{a_0}}A(A^2+2B^2)+\frac{(\omega\tau)^2}{a_0}B(B^2+\tfrac{7}{2}A^2)\right] . \tag{22}$$

As we see, the scale function R_{RW} grows like $\sqrt{\tau}$.

The time is ripe now to ask how does the traditional Kaluza-Klein scheme fit into our discussion. Its associated vacuum is characterized by a static radius of compactification to account for the observed constancy of the fine-structure constant. But $a(t)=a_0$, that is $p=0$, would not support the kind of compactification we want. After all, unlike $ds^2=-dt^2+a_0^2\cosh^2 wt\,dy^2$, only two out of the three Killing vectors associated with $ds^2=-dt^2+a_0{}^2dy^2$ can be globally respected once y is forced to compactify (the boost is lost of course), exhibiting no built-in periodicity. Yet the gauge-coupling constancy is a most desirable feature. For our cosmological Kaluza-Klein version we find

$$\frac{\dot{R}\dot{a}}{Ra}=\tfrac{1}{6}\Lambda \Longrightarrow \frac{\dot{\alpha}}{\alpha}\sim\frac{\Lambda}{H} \tag{23}$$

(H is the Hubble constant), meaning that the age of the Universe must be small on the $\sqrt{1/\Lambda}$ scale, hence $\cosh wt \sim 1$. *The observed constancy of α is thus correlated with the fact that Λ is practically zero.*

If we attempt to extend the idea to still higher dimensions, our conclusions are likely to survive. Although Einstein equations are more complicated for $n>1$ (n denotes the total number of extra dimensions), a maximally symmetric solution that generalizes Eq. (13) always exists and is furthermore n independent save for

$$w^2=\frac{2\Lambda}{(n+3)(n+2)}\ :$$

the empty space-time limit is therefore straightforwardly traced.

Finally, we would like to summarize the main points of this paper. Following the assumption that all local Killing vectors integrate to give a group of global isometries, the geometry, represented by the metric tensor, is capable of probing the underlying topology. In particular we derive solutions of Einstein equations, isotropic and homogeneous in the usual sense, exhibiting cosmological compactification of the extra dimension. A positive cosmological constant appears as a necessary ingredient, while our three-space must be open and recover from the big-bang singularity. The constancy of the fine-structure constant is correlated with the smallness of the cosmological constant. The observation that an empty five-dimensional space-time can be interpreted as a four-dimensional Universe with a well-defined equation of state has been exposed but is to be discussed in more detail elsewhere.

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