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General relativistic strings

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The properties of infinite-length cosmic strings are investigated using the full coupled equations for the metric and the scalar and gauge fields which make up the string. It is argued that there exists a class of static cylindrically symmetric solutions of these equations representing an isolated string. All of these solutions approach Minkowski spacetime minus a wedge. An exact formula is given for the angle of light bending by the string. These results are substantially in agreement with earlier treatments of cosmic strings, which used approximations for the stress energy of the string.

I. INTRODUCTION

There has been much interest in recent years in the subject of cosmic strings. Kibble *et al.*[1] have estimated the production of strings in a phase transition in the early universe. Zeldovich[2] has proposed that strings could be seeds for galaxy formation. Vilenkin[3] has proposed that strings could act as gravitational lenses.

An infinite-length cosmic string is a static cylindrically symmetric configuration of a self-interacting scalar field minimally coupled to a gauge field. The simplest case is that of a single scalar field coupled to a U(1) gauge field. The scalar field interacts with itself through the standard "Mexican hat" potential. Since different values of the field will minimize the potential, in a phase transition, the scalar field will take on different values in different regions of space. At the boundaries of these regions, the scalar field will arrange itself so as to minimize the energy. If the set of values of the scalar field which minimize the potential is not simply connected, this can result in a configuration of the field which has nonzero energy but for which there is no nearby state of lower energy. A string is such a field configuration.

Since strings have stress energy, they couple to the gravitational field. Thus one expects strings to have gravitational effects which should be calculable using Einstein's equation. Various approaches to this problem have been used. Vilenkin[3] approximated the stress energy of the string as that of an infinitely thin line with positive (δ function) energy density and equal negative pressure along the axis. He then used the linearized Einstein equation to find an approximation to the metric with this stress energy. Gott[4] approximated the stress energy as that of a cylinder of finite radius with uniform energy density and equal negative pressure along the axis. He then solved the exact Einstein equation for the metric. In both approaches, the spacetime in the exterior of the string is Minkowski spacetime minus a wedge, where the angular size of the wedge is equal to 8π times the mass per unit length of the string.

There are two difficulties with these approaches. The first has to do with the stress energy of the string. The actual stress energy of the string has other components besides the energy density and pressure along the axis. In addition, the energy density and pressure along the axis are not uniform over the interior of the string. Without a more exact treatment there is no way of knowing whether the terms neglected in these approximations are important. The second difficulty has to do with the scalar and gauge fields. The string is, after all, just a configuration of scalar and gauge fields. To find the gravitational field it is not enough simply to assume a stress energy and solve for the metric. The only consistent way to find the metric is simultaneously to solve the coupled Einstein-scalar-gauge field equations: i.e., the curved space equations for the scalar and gauge fields with a metric which is a solution of Einstein's equation with stress energy equal to the stress energy of the fields.

This approach will be taken in this paper. The field equations for the scalar and gauge fields in flat space will be treated in Sec. II. I will review what is known about the solutions of these equations and present numerical solutions for the fields and their stress energy. The full coupled Einstein-scalar-gauge equations will be treated in Sec. III. It will be argued that for sufficiently small η (the value of the magnitude of the scalar field which minimizes the potential) there exists a solution of these equations representing an isolated string. The properties of the solution will be discussed in Sec. IV. It will be shown that far from the axis the spacetime approaches Minkowski spacetime minus a wedge. Also an exact formula will be derived for the angle of light bending by the string. I will then argue that (a) the scalar and gauge fields are well approximated by their values in flat space, (b) the metric components are well approximated by certain simple integrals involving the stress energy of the flat space string solution, and (c) the angle of light bending by the string is well approximated by $8\pi\mu$ where μ is the proper mass per unit length of the string.

II. STRINGS IN FLAT SPACE

The string fields considered in this paper consist of a vector field A_a and a complex scalar field Φ which will be written $\Phi = Re^{i\psi}$ where R and ψ are real. In terms of these fields the Lagrangian is

$$\mathscr{L} = -\tfrac{1}{2}\nabla^a R \nabla_a R - \tfrac{1}{2}R^2(\nabla_a\psi + eA_a)(\nabla^a\psi + eA^a) - \lambda(R^2-\eta^2)^2 - \frac{1}{16\pi}F_{ab}F^{ab} , \qquad (1)$$

where $F_{ab} \equiv \nabla_a A_b - \nabla_b A_a$ and λ, η, and e are constants. (Note: we use units where $\hbar = c = G = 1$. We also use the abstract index notation as described by Wald.)[5] Variation of the Lagrangian with respect to the fields gives rise to the following equations of motion,

$$\nabla^a[R^2(\nabla_a \psi + eA_a)] = 0 , \tag{2}$$

$$\nabla^a \nabla_a R - R[4\lambda(R^2 - \eta^2) + (\nabla_a \psi + eA_a)(\nabla^a \psi + eA^a)] = 0 , \tag{3}$$

$$\nabla^a F_{ab} - 4\pi e R^2(\nabla_b \psi + eA_b) = 0 . \tag{4}$$

First we will examine these equations in Minkowski spacetime. Using the usual time and cylindrical coordinates (t,z,ρ,ϕ) the metric takes the form

$$ds^2 = -dt^2 + dz^2 + d\rho^2 + \rho^2 d\phi^2 . \tag{5}$$

We are looking for solutions of Eqs. (2)—(4) whose stress energy has cylindrical symmetry. We will assume that the scalar and gauge fields have the form

$$R = R(\rho) , \tag{6}$$

$$\psi = \phi , \tag{7}$$

$$A_a = \frac{1}{e}[P(\rho) - 1]\nabla_a \phi . \tag{8}$$

With these choices Eq. (2) is automatically satisfied and Eqs. (3) and (4) become

$$\rho \frac{d}{d\rho}\left[\rho \frac{dR}{d\rho}\right] = R[4\lambda\rho^2(R^2 - \eta^2) + P^2] , \tag{9}$$

$$\rho \frac{d}{d\rho}\left[\rho^{-1}\frac{dP}{d\rho}\right] = 4\pi e^2 R^2 P . \tag{10}$$

These equations have three arbitrary constants λ, η, and e. We can reduce the number of arbitrary constants to one by the following definitions:

$$X \equiv \frac{R}{\eta} , \tag{11}$$

$$r \equiv \sqrt{\lambda}\eta\rho , \tag{12}$$

$$\alpha \equiv \frac{4\pi e^2}{\lambda} . \tag{13}$$

Then Eqs. (9) and (10) become

$$r\frac{d}{dr}\left[r\frac{dX}{dr}\right] = X[4r^2(X^2 - 1) + P^2] , \tag{14}$$

$$r\frac{d}{dr}\left[r^{-1}\frac{dP}{dr}\right] = \alpha X^2 P . \tag{15}$$

These equations have been studied extensively[6] and though no closed-form solutions have been found, it has been proven that there are solutions for X and P which are smooth and for which $P \to 0$ and $X \to 1$ as $r \to \infty$ faster than any power of r. For each value of α Eqs. (14) and (15) can be numerically integrated to give X and P as functions of r. The results of such a numerical integration for $\alpha = 1$ are shown in Fig. 1.

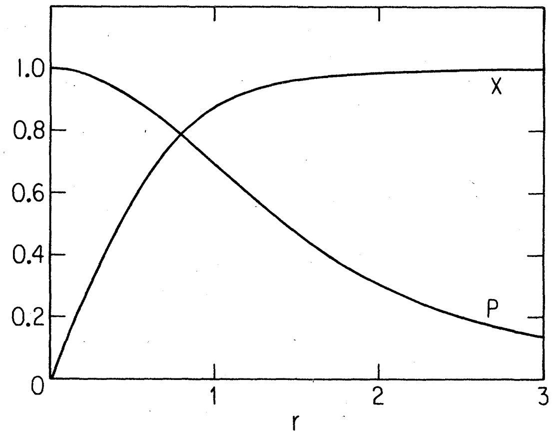

FIG. 1. Magnitudes of the scalar and gauge fields.

The stress energy is given in terms of the fields by

$$T_{ab} = \nabla_a R \nabla_b R + R^2(\nabla_a \psi + eA_a)(\nabla_b \psi + eA_b) + \frac{1}{4\pi} F_a{}^c F_{bc} + L g_{ab} . \tag{16}$$

For fields of the form given by Eqs. (6)—(8), T_{ab} is diagonal in the coordinates (t,z,ρ,ϕ),

$$T_{ab} = \sigma \nabla_a t \nabla_b t + P_z \nabla_a z \nabla_b z + P_\rho \nabla_a \rho \nabla_b \rho + \rho^2 P_\phi \nabla_a \phi \nabla_b \phi , \tag{17}$$

where

$$\sigma = -P_z = \frac{1}{2}\left[\left[\frac{dR}{d\rho}\right]^2 + \rho^{-2}R^2P^2 + 2\lambda(R^2 - \eta^2)^2 + \frac{1}{4\pi e^2}\rho^{-2}\left[\frac{dP}{d\rho}\right]^2\right] , \tag{18}$$

$$P_\rho = \frac{1}{2}\left[\left[\frac{dR}{d\rho}\right]^2 - \rho^{-2}R^2P^2 - 2\lambda(R^2 - \eta^2)^2 + \frac{1}{4\pi e^2}\rho^{-2}\left[\frac{dP}{d\rho}\right]^2\right] , \tag{19}$$

$$P_\phi = \frac{1}{2}\left[-\left[\frac{dR}{d\rho}\right]^2 + \rho^{-2}R^2P^2 - 2\lambda(R^2 - \eta^2)^2 + \frac{1}{4\pi e^2}\rho^{-2}\left[\frac{dP}{d\rho}\right]^2\right] . \tag{20}$$

Thus the stress energy is given in terms of the solutions to Eqs. (14) and (15). Numerical results for the components of the stress energy for $\alpha = 1$ are shown in Fig. 2.

It is instructive to compare the stress energy of the flat space string with the approximate stress energy assumed by Gott. Gott's stress-energy has σ constant inside the string, $\sigma = 0$ outside the string and $P_\rho = P_\phi = 0$ everywhere. In contrast, the flat-space string stress energy has a nonconstant σ, and P_ρ and P_ϕ about $\frac{1}{5}$ the magnitude of σ.

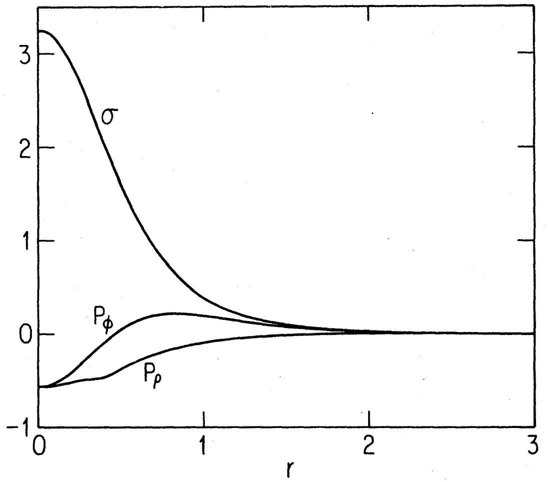

FIG. 2. Components of the stress-energy tensor in units of $\lambda\eta^4$.

III. STRINGS IN CURVED SPACE

To find strings in curved space we will once again look for static scalar and gauge fields whose stress energy has cylindrical symmetry. In addition, we will assume that the metric is static and has cylindrical symmetry. Thus we seek a spacetime with three commuting Killing fields all orthogonal to each other and hypersurface orthogonal. One of the Killing fields is timelike; the other two are spacelike. One of the spacelike Killing fields has closed orbits. In addition, we will assume the existence of an axis, i.e., a set of points where the Killing field with closed orbits vanishes. The normalization of the Killing fields is arbitrary and will be chosen as follows: the Killing field with closed orbits will be chosen so that the parameter along a closed integral curve goes from 0 to 2π. The other two Killing fields will be chosen to have norm 1 or -1 on the axis.

We will choose coordinates t, z, ρ, and ϕ as follows: ρ is geodesic distance from the axis in a direction orthogonal to all the Killing fields. $(\partial/\partial t)^a$ is the timelike Killing field. $(\partial/\partial\phi)^a$ is the spacelike Killing field with closed orbits. $(\partial/\partial z)^a$ is the spacelike Killing field orthogonal to the others. Note that our choice of normalization implies that $0\le\phi\le 2\pi$ and the points at $\phi=0$ and $\phi=2\pi$ are identified. With this choice of coordinates the metric takes the form

$$ds^2=-e^A dt^2+e^B dz^2+e^C d\phi^2+d\rho^2\ , \tag{21}$$

where A, B, and C are functions of ρ only. Note that our choice of the normalization of the Killing fields implies that $A(0)=B(0)=0$ and smoothness of the metric and the normalization of $(\partial/\partial\phi)^a$ implies that as $\rho\to 0$ we have $e^C/\rho^2\to 1$.

We define the following four orthonormal vector fields:

$$\hat{t}^a\equiv e^{-A/2}\left[\frac{\partial}{\partial t}\right]^a\ , \tag{22}$$

$$\hat{z}^a\equiv e^{-B/2}\left[\frac{\partial}{\partial z}\right]^a\ , \tag{23}$$

$$\hat{\phi}^a\equiv e^{-C/2}\left[\frac{\partial}{\partial \phi}\right]^a\ , \tag{24}$$

$$\hat{\rho}^a\equiv \left[\frac{\partial}{\partial \rho}\right]^a\ . \tag{25}$$

The stress energy is given by Eq. (16) and the scalar and gauge fields satisfy Eqs. (2)—(4). We will once again assume that the scalar and gauge fields have the form given in Eqs. (6)—(8). With this choice for the fields, once again Eq. (2) is automatically satisfied. Equations (3) and (4) become

$$\frac{d^2R}{d\rho^2}+\frac{1}{2}\frac{d}{d\rho}(A+B+C)\frac{dR}{d\rho}=R\,[4\lambda(R^2-\eta^2)+e^{-C}P^2]\ , \tag{26}$$

$$\frac{d^2P}{d\rho^2}+\frac{1}{2}\frac{d}{d\rho}(A+B-C)\frac{dP}{d\rho}=4\pi e^2R^2P\ . \tag{27}$$

The stress energy of the fields is

$$T_{ab}=\sigma\hat{t}_a\hat{t}_b+P_z\hat{z}_a\hat{z}_b+P_\rho\hat{\rho}_a\hat{\rho}_b+P_\phi\hat{\phi}_a\hat{\phi}_b\ , \tag{28}$$

where

$$\sigma=-P_z=\frac{1}{2}\left[\left[\frac{dR}{d\rho}\right]^2+e^{-C}R^2P^2+2\lambda(R^2-\eta^2)^2+\frac{1}{4\pi e^2}e^{-C}\left[\frac{dP}{d\rho}\right]^2\right]\ , \tag{29}$$

$$P_\rho=\frac{1}{2}\left[\left[\frac{dR}{d\rho}\right]^2-e^{-C}R^2P^2-2\lambda(R^2-\eta^2)^2+\frac{1}{4\pi e^2}e^{-C}\left[\frac{dP}{d\rho}\right]^2\right]\ , \tag{30}$$

$$P_\phi=\frac{1}{2}\left[-\left[\frac{dR}{d\rho}\right]^2+e^{-C}R^2P^2-2\lambda(R^2-\eta^2)^2+\frac{1}{4\pi e^2}e^{-C}\left[\frac{dP}{d\rho}\right]^2\right]\ . \tag{31}$$

Einstein's equation is

$$R_{ab}=8\pi(T_{ab}-\tfrac{1}{2}Tg_{ab})\ . \tag{32}$$

The only nonzero components of Eq. (32) are

$$2R_{ab}\hat{t}^a\hat{t}^b=\frac{d^2A}{d\rho^2}+\frac{1}{2}\frac{d}{d\rho}(A+B+C)\frac{dA}{d\rho}=8\pi(P_\rho+P_\phi)\ , \tag{33}$$

$$-2R_{ab}\hat{z}^a\hat{z}^b=\frac{d^2B}{d\rho^2}+\frac{1}{2}\frac{d}{d\rho}(A+B+C)\frac{dB}{d\rho}=8\pi(P_\rho+P_\phi)\ , \tag{34}$$

$$-2R_{ab}\hat{\phi}^a\hat{\phi}^b=\frac{d^2C}{d\rho^2}+\frac{1}{2}\frac{d}{d\rho}(A+B+C)\frac{dC}{d\rho}=8\pi(-2\sigma+P_\rho-P_\phi)\ , \tag{35}$$

$$-2R_{ab}\hat{\rho}^a\hat{\rho}^b=\frac{d^2}{d\rho^2}(A+B+C)+\frac{1}{2}\left[\left[\frac{dA}{d\rho}\right]^2+\left[\frac{dB}{d\rho}\right]^2+\left[\frac{dC}{d\rho}\right]^2\right]=8\pi(-2\sigma-P_\rho+P_\phi)\ . \tag{36}$$

The full set of equations that we must solve for the metric and the string fields are Eqs. (26), (27), and (33)–(36). Actually Eq. (36) is superfluous. We will now demonstrate that Eqs. (26), (27), and (33)–(35) along with the existence of an axis imply Eq. (36): define the tensor Q_{ab} by

$$Q_{ab}\equiv R_{ab}-8\pi(T_{ab}-\tfrac{1}{2}Tg_{ab})\ . \tag{37}$$

Then Eqs. (33)–(35) imply

$$Q_{ab}=J\nabla_a\rho\nabla_b\rho\ , \tag{38}$$

where J is a function of ρ. Equations (26) and (27) imply conservation of stress energy. Conservation of stress energy and the Bianchi identity imply

$$\nabla^a Q_{ab}=\tfrac{1}{2}\nabla_b(Q_a{}^a)\ . \tag{39}$$

Using Eq. (38) we obtain

$$\nabla^a Q_{ab}=\left[\frac{dJ}{d\rho}+\frac{1}{2}J\frac{d}{d\rho}(A+B+C)\right]\nabla_b\rho\ , \tag{40}$$

$$\nabla_b(Q_a{}^a)=\frac{dJ}{d\rho}\nabla_b\rho\ . \tag{41}$$

Thus we obtain

$$\frac{dJ}{d\rho}+J\frac{d}{d\rho}(A+B+C)=0\ , \tag{42}$$

$$J=\kappa e^{-(A+B+C)}\ , \tag{43}$$

where κ is a constant. Boundary conditions at the axis then imply $\kappa=0$. Thus $Q_{ab}=0$ everywhere so Eq. (36) is automatically satisfied given Eqs. (26), (27), and (33)–(35).

We will now show that $B=A$. Subtracting Eq. (34) from Eq. (33) we obtain

$$\frac{d^2}{d\rho^2}(A-B)+\frac{1}{2}\frac{d}{d\rho}(A-B)\frac{d}{d\rho}(A+B+C)=0\ , \tag{44}$$

$$\frac{d}{d\rho}(A-B)=\hat{\kappa}e^{-(A+B+C)/2}\ , \tag{45}$$

where $\hat{\kappa}$ is a constant. Boundary conditions at the axis imply $\hat{\kappa}=0$. Thus since $A(0)=B(0)$ we obtain

$$B=A \tag{46}$$

everywhere. Equations (33) and (34) now give the same information. Thus we now have only two independent equations for the metric (34) and (35). Introducing the quantity $H\equiv e^{A+C/2}$ the metric equations, equations (34) and (35), are equivalent to

$$\frac{d}{d\rho}\left[H\frac{dA}{d\rho}\right]=8\pi H(P_\rho+P_\phi)\ , \tag{47}$$

$$\frac{d^2H}{d\rho^2}=4\pi H(-2\sigma+3P_\rho+P_\phi)\ . \tag{48}$$

Equations (26), (27), (47), and (48) are the full set of equations for the string. As in the flat-space case, no closed-form solutions of these equations have been found. However, I will now give a plausibility argument for the existence of solutions: First we define X, r, and α by Eqs. (11)–(13) and K by $K\equiv\sqrt{\lambda}\eta H$. Then Eqs. (47), (48), (26), and (27) become

$$\frac{d}{dr}\left[K\frac{dA}{dr}\right]-4\pi\eta^2\left[-4K(X^2-1)^2+2\alpha^{-1}e^{2A}K^{-1}\left[\frac{dP}{dr}\right]^2\right]=0\ , \tag{49}$$

$$\frac{d^2K}{dr^2}-4\pi\eta^2\left[-2e^{2A}K^{-1}P^2X^2-6K(X^2-1)^2+\alpha^{-1}e^{2A}K^{-1}\left[\frac{dP}{dr}\right]^2\right]=0\ , \tag{50}$$

$$K\frac{d}{dr}\left[K\frac{dX}{dr}\right]-X[4K^2(X^2-1)+e^{2A}P^2]=0\ , \tag{51}$$

$$e^{-2A}K\frac{d}{dr}\left[e^{2A}K^{-1}\frac{dP}{dr}\right]-\alpha X^2P=0\ . \tag{52}$$

We seek solutions of these equations representing an isolated string; that is, solutions where, as $\rho\to\infty$, the stress-energy goes to zero at an appropriate rate. Thus we impose the conditions $\lim_{r\to\infty}X=1$ and $\lim_{r\to\infty}P=0$. Smoothness of the metric and fields requires additional boundary conditions at the axis.

Note that for $\eta=0$ there is a solution: namely, $A=A_0$, $K=K_0$, $X=X_0$, $P=P_0$, where $A_0=0$, $K_0=r$, and X_0 and P_0 are X and P in the flat-space string solution. This is reasonable since the $\eta\to0$ limit of the rescaled equations (49)–(52) corresponds to the $G\to0$ limit of the Einstein-scalar-gauge equations. In this limit we expect gravity and the string fields to decouple yielding a flat-space metric and string solution.

We now use the implicit function theorem to argue that for sufficiently small η there is a solution of Eqs. (49)–(52). The implicit function theorem states the following: Let $\mathscr{F}$ and $\mathscr{G}$ be two Banach spaces, $\mathscr{A}$ an open subset of $\mathbb{R}\times\mathscr{F}$ and f a continuously differentiable mapping from $\mathscr{A}$ into $\mathscr{G}$. Let $(x_0,y_0)\in\mathscr{A}$ satisfy $f(x_0,y_0)=0$ and $D_2f(x_0,y_0)$ is a homeomorphism of $\mathscr{F}$ onto $\mathscr{G}$ where $D_2f(x_0,y_0)$ is the derivative of f with respect to y evaluated at (x_0,y_0). Then there is an open interval U containing x_0 in $\mathbb{R}$ and a continuous mapping u of U into $\mathscr{F}$ such that $u(x_0)=y_0$, $(x,u(x))\in\mathscr{A}$ and $f(x,u(x))=0$ for all x in U.

In our case let $\mathscr{F}$ be a Banach space of quadruples of functions $(\hat{A}(r),\hat{K}(r),\hat{X}(r),\hat{P}(r))$ where $\hat{A}(r)=A-A_0$ and A is a function satisfying the appropriate boundary conditions; and similarly for $\hat{K}$, $\hat{X}$, and $\hat{P}$. Note that we must find an appropriate norm on $\mathscr{F}$ in order to make it into a Banach space. Let f be the following map: given $(\hat{A},\hat{K},\hat{X},\hat{P})\in\mathscr{F}$, f produces the quadruple of functions given by the left-hand sides of Eqs. (49)—(52). Let $\mathscr{G}$ be a Banach space of these quadruples of functions. Note that the norms on $\mathscr{F}$ and $\mathscr{G}$ must be chosen so that boundary conditions are satisfied and so that f is continuously differentiable.

With these definitions of $\mathscr{F}$, $\mathscr{G}$, and f it follows that the map $D_2f(x_0,y_0)$ given $(\delta\hat{A},\delta\hat{K},\delta\hat{X},\delta\hat{P})\in\mathscr{F}$ produces a quadruple of functions which are the left-hand sides of the linearized forms of Eqs. (49)—(52) about the $\eta=0$ solution. It also follows that the map u given by the implicit function theorem is a one-parameter family of solutions to Eqs. (49)—(52).

To apply the implicit function theorem we must show that $D_2f(x_0,y_0)$ is a homeomorphism, i.e., that it is one to one, onto, and bicontinuous. An actual proof of continuity would involve constructing the Banach spaces $\mathscr{F}$ and $\mathscr{G}$ and showing that $D_2f(x_0,y_0)$ is bicontinuous in the induced topologies. We shall not attempt to do this here. However, we will argue that $D_2f(x_0,y_0)$ should be one to one and onto. To show that $D_2f(x_0,y_0)$ is one to one we must show that solutions of the linearized equations are unique. To show that $D_2f(x_0,y_0)$ is onto we must show that the linearized equations have solutions in $\mathscr{F}$ for all sources in $\mathscr{G}$. However, it is easy to show that the linearized equations corresponding to Eqs. (49) and (50) can be integrated to yield unique solutions for all sources. Furthermore, the linearized equations corresponding to Eqs. (51) and (52) are equivalent to a variational principle. Presumably the standard variational principle techniques[6] could be used to show existence and uniqueness of solutions to these equations. Thus it appears that $D_2f(x_0,y_0)$ is one to one and onto. Hence if we were able to define $\mathscr{F}$ and $\mathscr{G}$ so that $D_2f(x_0,y_0)$ is bicontinuous, the implicit function theorem would imply the existence of a one-parameter family of solutions to Eqs. (49)—(52).

Note that although we have argued that solutions exist, we have not shown that they are stable. A string might, for example, be unstable against collapse in the ρ direction or against collapse or expansion in the z direction. This issue is presently under investigation.

IV. PROPERTIES OF THE STRING SOLUTION

For the remainder of the paper I will assume that string solutions exist and will investigate some of their properties. We will make some fairly weak assumptions about the behavior of the stress-energy as $\rho\to\infty$: We will assume that $\int_0^\infty H\sigma\,d\rho$ converges. We will also assume that

$$\lim_{\rho\to\infty} H^2\sigma=0\ . \tag{53}$$

It follows from Eqs. (29)—(31) that $\sigma>|P_\rho|$ and $\sigma>|P_\phi|$. Thus our assumptions imply that if σ is replaced by P_ρ or P_ϕ then the integral still converges and Eq. (53) still holds.

Our first step will be to examine the behavior of the metric as $\rho\to\infty$. We will then demonstrate that far from the axis the metric approaches Minkowski space minus a wedge. We will then derive a formula for the bending of light by the string.

It will be useful to have the following consequence of the conservation of stress energy:

$$\begin{aligned}0&=\hat{\rho}_b\nabla_a T^{ab}\\ &=\nabla_a(T^{ab}\hat{\rho}_b)-T^{ab}\nabla_a\hat{\rho}_b\\ &=H^{-1}\frac{d}{d\rho}(HP_\rho)-\left[H^{-1}P_\phi\frac{dH}{d\rho}-(\sigma+P_\phi)\frac{dA}{d\rho}\right]\ .\end{aligned} \tag{54}$$

Thus we obtain

$$\frac{d}{d\rho}(HP_\rho)-P_\phi\frac{dH}{d\rho}+(\sigma+P_\phi)H\frac{dA}{d\rho}=0\ . \tag{55}$$

We introduce the quantities θ_1 and θ_2 defined by

$$\theta_1\equiv H\frac{dA}{d\rho}\ , \tag{56}$$

$$\theta_2\equiv\frac{dH}{d\rho}\ . \tag{57}$$

Then Eqs. (47), (48), and (55) can be written as

$$\frac{d\theta_1}{d\rho}=8\pi H(P_\rho+P_\phi)\ , \tag{58}$$

$$\frac{d\theta_2}{d\rho}=4\pi H(-2\sigma+3P_\rho+P_\phi)\ , \tag{59}$$

$$\frac{d}{d\rho}(HP_\rho)=P_\phi\theta_2-(\sigma+P_\phi)\theta_1\ . \tag{60}$$

Integrating Eqs. (58) and (59) we obtain

$$\theta_1=\int_0^\rho 8\pi H(P_\rho+P_\phi)d\rho'\ , \tag{61}$$

$$\theta_2=1+\int_0^\rho 4\pi H(-2\sigma+3P_\rho+P_\phi)d\rho'\ . \tag{62}$$

By our assumptions θ_1 and θ_2 must approach constant values as $\rho\to\infty$. Using Eqs. (58)—(60) we obtain the following:

$$\begin{aligned}\frac{d}{d\rho}[\theta_1(\theta_2-\tfrac{3}{4}\theta_1)]&=\frac{d\theta_1}{d\rho}(\theta_2-\tfrac{3}{4}\theta_1)+\theta_1\left[\frac{d\theta_2}{d\rho}-\frac{3}{4}\frac{d\theta_1}{d\rho}\right]\\ &=\theta_2\frac{d\theta_1}{d\rho}+\theta_1\left[\frac{d\theta_2}{d\rho}-\frac{3}{2}\frac{d\theta_1}{d\rho}\right]\\ &=\theta_2 8\pi H(P_\rho+P_\phi)+\theta_1[4\pi H(-2\sigma+3P_\rho+P_\phi)-\tfrac{3}{2}8\pi H(P_\rho+P_\phi)]\\ &=\theta_2 8\pi H(P_\rho+P_\phi)+\theta_1[4\pi H(-2\sigma-2P_\phi)]\end{aligned}$$

$$=8\pi H[\theta_2 P_\rho+\theta_2 P_\phi-\theta_1(\sigma+P_\phi)]$$

$$=8\pi H\left[P_\rho\frac{dH}{d\rho}+\frac{d}{d\rho}(HP_\rho)\right]$$

$$=\frac{d}{d\rho}(8\pi H^2P_\rho)\ . \tag{63}$$

Boundary conditions at the axis imply

$$\theta_1(\theta_2-\tfrac{3}{4}\theta_1)=8\pi H^2P_\rho\ . \tag{64}$$

Dividing both sides by H we obtain

$$\frac{dA}{d\rho}\left[\frac{dH}{d\rho}-\frac{3}{4}H\frac{dA}{d\rho}\right]=8\pi HP_\rho\ . \tag{65}$$

Let us call the constant values approached by θ_1 and θ_2 as $\rho\to\infty$ k_1 and k_2, respectively. By our assumptions and Eq. (64) we obtain

$$k_1(k_2-\tfrac{3}{4}k_1)=0\ . \tag{66}$$

Thus $k_1=0$ or $k_1=\frac{4}{3}k_2$. Thus as $\rho\to\infty$ the string must approach either a vacuum metric which has $\theta_1=0$ or one which has $\theta_1=\frac{4}{3}\theta_2$. This result has been obtained previously by Vilenkin[3] under the assumption that the metric is boost symmetric, $B=A$ (which we have now proved). As we will show later $\theta_1=0$ corresponds to a flat metric. $\theta_1=\frac{4}{3}\theta_2$ corresponds to a nonflat metric which is an analog of a Kasner metric. This metric has the property that as $\rho\to\infty$ the length of a closed integral curve of $(\partial/\partial\phi)^a$ approaches zero. Thus this metric does not represent an isolated system and it is unlikely that this metric has anything to do with any strings which may have formed in our universe.

As I will argue later, for small η, H and A should be near their flat-space values. Thus k_1 should be near zero and k_2 near 1. This rules out the $k_1=\frac{4}{3}k_2$ possibility. For the remainder of this section we will focus on the case where $k_1=0$. Denote by $\overline{H}$ and $\overline{A}$ the values of H and A, respectively, for the vacuum metric which the string metric approaches far from the axis. Then

$$\frac{d\overline{A}}{d\rho}=0\ , \tag{67}$$

$$\frac{d\overline{H}}{d\rho}=k_2\ . \tag{68}$$

Solving these equations we obtain

$$\overline{A}=a_0\ , \tag{69}$$

$$\overline{H}=k_2\rho+a_1\ , \tag{70}$$

where a_0 and a_1 are constants. Writing the metric explicitly we obtain

$$ds^2=-e^{a_0}dt^2+e^{a_0}dz^2+d\rho^2+e^{-2a_0}(k_2\rho+a_1)^2d\phi^2\ . \tag{71}$$

Changing variables to

$$t'=e^{a_0/2}t,\quad z'=e^{a_0/2}z\ ,$$

$$\rho'=\rho+\frac{a_1}{k_2},\quad \phi'=k_2e^{-a_0}\phi$$

we obtain

$$ds^2=-dt'^2+dz'^2+d\rho'^2+\rho'^2d\phi'^2\ . \tag{72}$$

Equation (72) is just the metric of flat space. Note however that since ϕ ranges from 0 to 2π, ϕ' will have a different range. Thus the metric of equation (72) is not Minkowski spacetime but Minkowski spacetime minus a wedge. Thus initially parallel null geodesics which pass on opposite sides of the string will cross and the angle of light bending will be equal to the angular deficit $\Delta\phi$ of the metric. Ford and Vilenkin[7] give a formula for the angular deficit for a class of static translation symmetric spacetimes which are asymptotically Minkowski spacetime minus a wedge. We will derive a different formula for $\Delta\phi$ as follows: Denote by l the length of an orbit of $(\partial/\partial\phi)^a$. In a flat spacetime with angular deficit $\Delta\phi$ we have

$$l(\rho_2)-l(\rho_1)=(2\pi-\Delta\phi)(\rho_2-\rho_1)\ . \tag{73}$$

Since the string metric is approaching the flat metric as $\rho\to\infty$ we obtain

$$2\pi-\Delta\phi=\lim_{\rho\to\infty}\frac{dl}{d\rho}\ . \tag{74}$$

However, by the definition of l we have

$$l=\int_0^{2\pi}\left[g_{ab}\left[\frac{\partial}{\partial\phi}\right]^a\left[\frac{\partial}{\partial\phi}\right]^b\right]^{1/2}d\phi$$

$$=2\pi e^{C/2}$$

$$=2\pi(e^{-A}H)\ . \tag{75}$$

Thus we obtain

$$2\pi-\Delta\phi=\lim_{\rho\to\infty}\frac{d}{d\rho}(2\pi e^{-A}H)\ , \tag{76}$$

$$\Delta\phi=2\pi\left[1-\lim_{\rho\to\infty}\frac{d}{d\rho}(e^{-A}H)\right]\ . \tag{77}$$

Using boundary conditions at the axis we obtain

$$\Delta\phi=-2\pi\int_0^\infty\frac{d^2}{d\rho^2}(e^{-A}H)d\rho\ . \tag{78}$$

Using Eqs. (56)–(59) and (64) we obtain

$$\frac{d^2}{d\rho^2}(e^{-A}H)=\frac{d}{d\rho}[e^{-A}(\theta_2-\theta_1)]$$

$$=e^{-A}\left[\frac{d\theta_2}{d\rho}-\frac{d\theta_1}{d\rho}\right]-H^{-1}\theta_1 e^{-A}(\theta_2-\theta_1)$$

$$=e^{-A}\left[\frac{d\theta_2}{d\rho}-\frac{1}{2}\frac{d\theta_1}{d\rho}-H^{-1}\theta_1(\theta_2-\tfrac{3}{4}\theta_1)+\tfrac{1}{4}H^{-1}\theta_1^2-\frac{1}{2}\frac{d\theta_1}{d\rho}\right]$$

$$=e^{-A}\left[4\pi H(-2\sigma+3P_\rho+P_\phi)-4\pi H(P_\rho+P_\phi)-8\pi HP_\rho-\frac{1}{2}\left[\frac{d\theta_1}{d\rho}-H^{-1}\theta_1^2\right]-\tfrac{1}{4}H^{-1}\theta_1^2\right]$$

$$=-8\pi e^{-A}H\sigma-\tfrac{1}{2}\left[e^{-A}\frac{d\theta_1}{d\rho}-e^{-A}H^{-1}\theta_1^2\right]-\tfrac{1}{4}e^{-A}H\left[\frac{dA}{d\rho}\right]^2$$

$$=-8\pi e^{-A}H\sigma-\frac{1}{2}\frac{d}{d\rho}(e^{-A}\theta_1)-\tfrac{1}{4}e^{-A}H\left[\frac{dA}{d\rho}\right]^2 . \tag{79}$$

Thus using equation (78) we obtain

$$\Delta\phi=8\pi\int_0^\infty 2\pi e^{-A}H\sigma\,d\rho+\pi[(e^{-A}\theta_1)]_0^\infty+\frac{\pi}{2}\int_0^\infty e^{-A}H\left[\frac{dA}{d\rho}\right]^2 d\rho$$

$$=8\pi\int_0^\infty 2\pi e^{-A}H\sigma\,d\rho+\frac{\pi}{2}\int_0^\infty e^{-A}H\left[\frac{dA}{d\rho}\right]^2 d\rho, \tag{80}$$

where the middle term dropped out since $\theta_1=0$ at $\rho=0$ and as $\rho\to\infty$. At this point it is useful to introduce the proper mass per unit length μ of the string. μ is defined as the integral of the energy density σ over a $t=$constant, $z=$constant two-surface. Using the form of the metric we obtain

$$\mu=\int_0^\infty 2\pi e^{-A}H\sigma\,d\rho . \tag{81}$$

Thus we obtain

$$\Delta\phi=8\pi\mu+\frac{\pi}{2}\int_0^\infty e^{-A}H\left[\frac{dA}{d\rho}\right]^2 d\rho . \tag{82}$$

In Sec. III we saw that for $\eta^2=0$ the string solution was given by the flat-space string solution for R and P, the scalar and gauge variables, and by $A=0$ and $H=\rho$ for the metric variables. This corresponds to the metric of Minkowski spacetime. Thus it seems reasonable that for small η^2 the string solution should be close to the flat-space solution for both the field and metric variables. In most models η is less than or of the order of the grand unified mass. Since in our units the Planck mass is the unit mass, η is a small number. In particular, $\eta^2<10^{-4}$ for most grand unified strings.

Let us see if this approximation is self-consistent. Using Eqs. (47) and (48) and the boundary conditions at the axis we obtain integral equations for A and H:

$$A=\int_0^\rho H^{-1}d\rho'\int_0^{\rho'} 8\pi H(P_\rho+P_\phi)d\rho'' , \tag{83}$$

$$H=\rho+\int_0^\rho d\rho'\int_0^{\rho'} 4\pi H(-2\sigma+3P_\rho+P_\phi)d\rho'' . \tag{84}$$

Using the flat-space values for the string and metric variables on the right side of Eqs. (83) and (84) we can obtain "corrected" values for H and A. For our approximation to be self-consistent, the corrections must be small. Using the flat-space solution, we find that the correction to A and the fractional correction to H are of order η^2. Thus our approximation is self-consistent.

Thus it seems likely that the string variables, X and P, are well approximated by their values in flat space and that the metric variables, H and A, are well approximated by the right-hand sides of Eqs. (84) and (83), respectively, where all quantities in the integrals take on their flat-space values. Using this approximation to evaluate the two terms in Eq. (82) we find that the first term is of order η^2 and the second term is of order η^4. Thus to a good approximation

$$\Delta\phi\approx 8\pi\mu . \tag{85}$$

This is the formula for the angular deficit found by Vilenkin and Gott using different methods. It is remarkable that such simple approximations for the stress energy should nonetheless give the correct answer for the angle of light bending.

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