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Effects of quantum fields on singularities and particle horizons in the early universe. III. The conformally coupled massive scalar field

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The behaviors of solutions to the semiclassical backreaction equations are investigated for conformally invariant free quantum fields and a conformally coupled massive scalar field in spatially flat homogeneous and isotropic spacetimes containing classical radiation. With one exception, only solutions that begin with the scale factor equal to zero are considered. It is found that in the limit that the scale factor vanishes the presence of the massive scalar field does not result in any new types of solutions, nor does it eliminate any. Thus the results of the first paper of this series on the existence of particle horizons are also valid for solutions beginning with zero scale factor if a conformally coupled massive scalar field is present. For intermediate values of the scale factor the massive scalar field can significantly affect the behaviors of specific solutions. Nevertheless, no new types of behaviors are observed and no old ones are eliminated. For large values of the scale factor it is uncertain what the behaviors of all solutions are, but asymptotically de Sitter solutions and asymptotically classical solutions continue to exist. Particle production causes the latter to expand like classical-matter-dominated universes at late times.

I. INTRODUCTION

It is well established that quantum fields play a significant role in most models of the early universe. Their possible dynamical effects include the damping of anisotropy,[1–3] the removal of particle horizons and singularities,[4–8] and inflation.[9–11] Through particle production they can also account for the matter in the universe.

Because quantum fields have such an important influence on models of the early universe it is important to determine all of the effects which can occur and to determine under what circumstances they do occur. To date, most of the work that has been done examines either the effects of complicated fields such as interacting fields in a given background spacetime or the backreaction of simple fields such as free fields on the spacetime geometry. Even for free fields in homogeneous and isotropic spacetimes, the complete backreaction problem has only been solved for conformally invariant fields. For conformally noninvariant fields simplifying assumptions have been made. For example, Parker and Fulling[4] investigated the effects of a massive minimally coupled scalar field in an otherwise empty universe while Hu and Parker[12] studied the effects of gravitons in a universe containing a perfect fluid with equation of state $p=\gamma\rho$. In both calculations vacuum polarization effects were assumed small. Hartle[13] considered a massless scalar field in a universe containing classical radiation and dust. He assumed that the field was nearly conformally invariant.

In this paper, we investigate the backreaction problem for a conformally coupled massive scalar field along with conformally invariant free fields in spatially flat homogeneous and isotropic spacetimes containing classical radiation. Unlike previous calculations, we take all quantum effects, that is, both vacuum polarization and particle production, into account and we do not assume the conformal noninvariance to be small. One of our main limitations is that, with one exception, we only consider solutions which begin with the scale factor equal to zero. However, this limitation has more to do with the difficulty involved in determining starting values for the variables in our numerical computations than anything else. Given these starting values, our numerical methods would work for all solutions to the equations.

We investigate effects due to the conformally coupled massive scalar field because the conformal coupling makes it easier to work with than other conformally noninvariant fields. There is also a similarity between this field and massive spinor fields in that both become conformally invariant in the limit that the mass or the scale factor vanishes. Thus, study of the massive scalar field should provide insight into the effects of massive spinor fields in the early universe.

The neglect of minimally coupled scalar fields and thus gravitons[14] is perhaps our primary limitation. Preliminary work shows that many of the initial behaviors found for solutions in Ref. 7 cannot occur if minimally coupled scalar fields are present.[15] It is not known what behaviors they are replaced with. We hope that this work may be a step towards the very difficult problem of solving the semiclassical backreaction equations when minimally coupled scalar fields are present.

We choose spatially flat homogeneous and isotropic spacetimes for their simplicity and because in Ref. 8 it was shown that, for the most part, the spatial curvature has little effect on the early-time behaviors of solutions when conformally invariant fields are present.

We include classical radiation to support the expansion at late times. Since particle production does occur for the massive scalar field it is possible that the produced particles can support the expansion at late times without the help of classical radiation. We shall investigate this interesting question in the next paper of this series.

In Refs. 7 and 8, hereafter referred to as Papers I and II, we saw that the study of conformally invariant quantum fields in homogeneous and isotropic spacetimes allows one to address the questions of whether the universe began with an initial singularity and whether it has particle horizons. It was found that solutions both with and without particle horizons and singularities exist for various values of the regularization parameters α and β. The results are summarized in Table I of each paper. All values of α and β were considered in Paper I and most values were considered in Paper II because α and β for the early universe are unknown. This, in turn, is because their values depend on the number and types of fields present and, for spin-1 fields, on the regularization scheme used.

The massive scalar field makes a positive contribution to both α and β and adds new mass-dependent terms to the backreaction equations. Since the behaviors of solutions for all α and β are known when the mass-dependent terms are absent, it is the effects of these terms we shall be concerned with in this paper.

We show that, for universes which begin with the scale factor equal to zero, all of the initial behaviors found in Paper I still occur when a conformally coupled massive field is present and no new types of initial behaviors occur. Thus those results having to do with particle horizons go over unchanged. Since, with one exception, we consider only solutions which begin with the scale factor equal to zero and since the energy density of the classical radiation diverges in this limit, the solutions we investigate begin with initial singularities. The exception also begins with an initial singularity.

For intermediate values of the scale factor we find that the massive field can affect the behavior of solutions although it does not always do so. However, all of the behaviors found for $m=0$ still occur and no new types of behaviors have been observed.

For large values of the scale factor, when only conformally invariant fields are present, various authors have found several types of behaviors. Fischetti, Hartle, and Hu[5] found that asymptotically classical solutions (ACS) exist, for all α and β, which at late times expand like radiation-dominated universes. However, these are unstable to small perturbations in their initial conditions. In Paper I it was shown that if $\alpha,\beta>0$ then asymptotically de Sitter solutions which are stable to perturbations in their initial conditions occur. Starobinski[6] showed that for $\alpha<0$, $\beta>0$ unstable asymptotically de Sitter solutions exist along with stable solutions which at late times expand like matter-dominated universes and stable solutions which expand so fast that the scale factor becomes infinite in a finite amount of proper time.

Our results for the massive field are less certain in the limit that the scale factor becomes large than they are for small and intermediate values. Nevertheless, we find that the ACS still exist, and that for $\alpha>0$ they are still unstable. However for $\alpha<0$ they are stable. It has been speculated[16] that particle production may prevent "runaway" solutions such as asymptotically de Sitter solutions from occurring. We find that such solutions still occur and that for $\alpha,\beta>0$ they are probably still asymptotically de Sitter.

To compute the backreaction of the quantum fields on the spacetime geometry, which we treat as a classical field, we use canonical quantization and the semiclassical approximation to quantum gravity. To regularize the expectation value of the stress-energy tensor operator for the massive scalar field we use adiabatic regularization. These topics are reviewed in Sec. II. In Sec. III we derive and discuss the semiclassical backreaction equations. In Sec. IV we discuss solutions to the wave equation for the massive scalar field as well as solutions to the semiclassical backreaction equations which begin with the scale factor equal to zero. The behaviors of solutions in the limits that the scale factor vanishes, has intermediate values, and has large values are discussed separately.

II. REVIEW OF CANONICAL QUANTIZATION AND ADIABATIC REGULARIZATION

In this section we present brief reviews of canonical quantization in curved space and adiabatic regularization. For more thorough reviews see Ref. 17 and references included therein. We begin with canonical quantization and tailor our remarks to the specific case of a conformally coupled massive scalar field in a spatially flat homogeneous and isotropic spacetime.

The starting point for canonical quantization is the wave equation which for the conformally coupled scalar field is[18]

$$\Box\phi-(m^2+\tfrac{1}{6}R)\phi=0\ . \tag{2.1}$$

Here m is the mass of the field and R is the scalar curvature. In general, a complete set of mode solutions to (2.1) can be found and the field ϕ can be expanded in terms of them.

For spatially flat homogeneous and isotropic spacetimes the metric can be written

$$ds^2=a^2(\eta)(-d\eta^2+d\mathbf{x}^2)\ , \tag{2.2}$$

where $a(\eta)$ is the scale factor. Then the mode solutions of (2.1) are of the form

$$u_{\mathbf{k}}(\eta)=(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}a^{-1}(\eta)\psi(\eta) \tag{2.3}$$

and the field is given by

$$\phi=\int d^3k[a_{\mathbf{k}}u_{\mathbf{k}}(\eta)+a_{\mathbf{k}}^{\dagger}u_{\mathbf{k}}^{*}(\eta)]\ . \tag{2.4}$$

Substitution of (2.3) and (2.4) into (2.1) results in the following equation for ψ:

$$\frac{d^2\psi}{d\eta^2}+(k^2+m^2a^2)\psi=0\ . \tag{2.5}$$

The modes $u_{\mathbf{k}}$ are required to be orthonormal with respect to the conserved scalar product

$$(\phi_1,\phi_2)\equiv-i\int d^3x\,a^2(\eta)[\phi_1\partial_\eta\phi_2-(\partial_\eta\phi_1)\phi_2]\ . \tag{2.6}$$

That is,

$$(u_{\mathbf{k}},u_{\mathbf{k}'})=-(u_{\mathbf{k}}^{*},u_{\mathbf{k}'}^{*})=\delta^3(\mathbf{k}-\mathbf{k}')$$

and $(u_{\mathbf{k}},u^*_{\mathbf{k}'})=0$. These relations are satisfied by modes of the form (2.3) so long as ψ is chosen so that the condition

$$\psi\partial_\eta\psi^*-\psi^*\partial_\eta\psi=i \tag{2.7}$$

is satisfied.

To quantize the field one imposes the commutation relations

$$[a_{\mathbf{k}},a_{\mathbf{k}'}]=[a^\dagger_{\mathbf{k}},a^\dagger_{\mathbf{k}'}]=0\ ,$$
$$[a_{\mathbf{k}},a^\dagger_{\mathbf{k}'}]=\delta^3(\mathbf{k}-\mathbf{k}')\ . \tag{2.8}$$

The vacuum state is then defined as the state for which

$$a_{\mathbf{k}}\,|\,0\rangle=0 \tag{2.9}$$

for all $\mathbf{k}$. Other states are built up from this state by acting on it with various combinations of creation operators $a^\dagger_{\mathbf{k}}$.

In general, the field may be decomposed into many different complete sets of modes and each of these sets has its own vacuum state. Suppose we label one such set of modes as $\bar{u}_{\mathbf{k}}(\eta)$. Then, in terms of these modes the field is

$$\phi=\int d^3\mathbf{k}[\bar{a}_{\mathbf{k}}\bar{u}_{\mathbf{k}}(\eta)+\bar{a}^*_{\mathbf{k}}\bar{u}^*_{\mathbf{k}}(\eta)] \tag{2.10}$$

and the vacuum state is defined by $\bar{a}_{\mathbf{k}}\,|\,0\rangle=0$ for all $\mathbf{k}$. Because of completeness, the two sets of modes $u_{\mathbf{k}}$ and $\bar{u}_{\mathbf{k}}$ are related by the Bogolubov transformation

$$\bar{u}_{\mathbf{k}}(\eta)=\alpha_{\mathbf{k}}u_{\mathbf{k}}(\eta)+\beta_{\mathbf{k}}u^*_{\mathbf{k}}(\eta)\ , \tag{2.11}$$

where $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ are constants. One can compare the two vacuum states by noting that the number operator for the barred states is $\bar{N}\equiv\int d^3k\,\bar{a}^\dagger_{\mathbf{k}}\bar{a}_{\mathbf{k}}$. Taking its expectation value with respect to the unbarred vacuum, one finds

$$\langle 0\,|\,\bar{N}\,|\,0\rangle=\int d^3k\,|\,\beta_{\mathbf{k}}\,|^2\ . \tag{2.12}$$

Thus, the number of barred particles in the unbarred vacuum in the mode $\mathbf{k}$ is $|\,\beta_{\mathbf{k}}\,|^2$. Similarly, the number of unbarred particles in the barred vacuum in the mode $\mathbf{k}$ is $|\,\beta_{\mathbf{k}}\,|^2$.

There are two points which should be understood with respect to vacuum states in curved space. The first is that, in general, it is uncertain what criteria should be used in choosing a vacuum state. The problem is that many of the criteria used for Minkowski space such as Lorentz invariance and positive frequency with respect to a timelike Killing vector no longer apply in curved space. Second, if there is some "natural" choice of vacuum when the spacetime begins it does not in general correspond to the natural choice of vacuum when the spacetime ends. That is, the "in" vacuum state and the "out" vacuum state are different. This leads to particle production as can be seen from Eq. (2.12).

Once the field has been quantized, one uses a semiclassical approximation to quantum gravity to compute its backreaction on the spacetime geometry.[19] The approximation we use keeps the geometry classical and modifies Einstein's equations to read

$$G_{ab}=\frac{l^2}{2}(T^{\rm cl}_{ab}+\langle T^{\rm QM}_{ab}\rangle)\ . \tag{2.13}$$

Here G_{ab} is the Einstein tensor and $l\equiv(16\pi G)^{1/2}$ is the Planck length. $T^{\rm cl}_{ab}$ is the stress-energy tensor for any classical fields and $T^{\rm QM}_{ab}$ is the stress-energy tensor operator for the quantum fields. The expectation value is taken with respect to whatever states the fields are in.

For the massive scalar field, $T^{\rm QM}_{ab}$ is formed by substituting (2.4) into the classical expression for T_{ab} for a conformally coupled massive scalar field. If the field is in some n particle state then

$$\langle T^{\rm QM}_{ab}\rangle=\langle 0\,|\,T^{\rm QM}_{ab}\,|\,0\rangle+\int d^3k\,n_{\mathbf{k}}T_{ab}[u_{\mathbf{k}},u^*_{\mathbf{k}}]\ , \tag{2.14a}$$

$$\langle 0\,|\,T^{\rm QM}_{ab}\,|\,0\rangle=\int d^3k\,T_{ab}[u_{\mathbf{k}},u^*_{\mathbf{k}}]\ , \tag{2.14b}$$

where $n_{\mathbf{k}}$ is the number of particles in the $\mathbf{k}$th mode and $T_{ab}[u_{\mathbf{k}},u^*_{\mathbf{k}}]$ is the classical expression for the stress-energy tensor of the massive scalar field as a bilinear function of $u_{\mathbf{k}}$ and $u^*_{\mathbf{k}}$.

The quantity $\langle 0\,|\,T^{\rm QM}_{ab}\,|\,0\rangle$ is divergent and must be regularized. There are several ways to do this, but the one most amenable to numerical calculations is adiabatic regularization. We shall review it next.

One starts by assuming a solution to the wave equation, (2.5), of the form

$$\psi=(2W)^{-1/2}\exp\left[-i\int^{\eta}W(\eta')d\eta'\right]\ . \tag{2.15}$$

This means that W must satisfy the equation

$$W^2=\omega^2-\tfrac{1}{2}(W^{-1}\ddot{W}-\tfrac{3}{2}W^{-2}\dot{W}^2)\ , \tag{2.16}$$

where $\omega^2\equiv k^2+m^2a^2$ and dots denote derivatives with respect to η. For large values of ω and/or if $a(\eta)$ is slowly varying, Eq. (2.16) can be solved approximately by iteration. The result is

$$W=\omega-\tfrac{1}{4}\omega^{-3}m^2(a\ddot{a}+\dot{a}^2)+\tfrac{5}{8}\omega^{-5}m^4a^2\dot{a}^2+\cdots\ . \tag{2.17}$$

Clearly, each new iteration of (2.16) adds terms with two extra derivatives in them to W^2. The number of derivatives a term has gives its adiabatic order. Thus, in Eq. (2.17) we have explicitly shown terms up to and including adiabatic order two.

In a spacetime with the metric (2.2), $\langle 0\,|\,T_{ab}\,|\,0\rangle$ for the massive scalar field is given by[20,21]

$$\langle 0\,|\,T^0{}_0\,|\,0\rangle=(4\pi^2a^4)^{-1}\int_0^\infty dk\,k^2(\,|\,\dot{\psi}\,|^2+\omega^2\,|\,\psi\,|^2)\ , \tag{2.18a}$$

$$\langle 0\,|\,T\,|\,0\rangle=(2\pi^2a^2)^{-1}m^2\int_0^\infty dk\,k^2\,|\,\psi\,|^2\ , \tag{2.18b}$$

where the superscript QM has been dropped for simplicity. Homogeneity and isotropy allow one to deduce the other components of $\langle 0\,|\,T_{ab}\,|\,0\rangle$ from these. By substituting (2.15) and (2.17) into (2.18), approximate expressions for $\langle 0\,|\,T_{ab}\,|\,0\rangle$ can be obtained. We shall denote these by $\langle 0\,|\,T^{(A)}_{ab}\,|\,0\rangle$. Because (2.17) is exact in the limit $k\to\infty$, all the divergences in $\langle 0\,|\,T_{ab}\,|\,0\rangle$ also occur in $\langle 0\,|\,T^{(A)}_{ab}\,|\,0\rangle$ so long as terms up to adiabatic order four are kept. It is easy to show that all terms of higher adiabatic order give finite contributions to $\langle 0 | T_{ab}^{(A)} | 0 \rangle$.

Adiabatic regularization consists of truncating the expansion (2.17) at adiabatic order four, computing $\langle 0 | T_{ab}^{(A)} | 0 \rangle$ and subtracting it from $\langle 0 | T_{ab} | 0 \rangle$. Although this method may seem a bit *ad hoc*, Birrell[22] has shown that it is equivalent to point splitting. For the metric (2.2) Bunch[21] finds that

$$\langle 0 | T_0^{0(A)} | 0 \rangle = (4\pi^2 a^4)^{-1} \int_0^\infty dk\, k^2 \left[\omega + \frac{m^4 a^2 \dot{a}^2}{8\omega^5} - \frac{m^4}{32\omega^7}(2a^2 \dot{a}\dddot{a} - a^2 \ddot{a}^2 + 4a\dot{a}^2\ddot{a} - \dot{a}^4) + \frac{7m^6}{16\omega^9}(a^3\dot{a}^2\ddot{a} + a^2\dot{a}^4) - \frac{105 m^8 a^4 \dot{a}^4}{128\omega^{11}} \right], \tag{2.19a}$$

$$\langle 0 | T^{(A)} | 0 \rangle = (2\pi^2 a^2)^{-1} m^2 \int_0^\infty dk\, k^2 \left[\frac{1}{2\omega} + \frac{m^2}{8\omega^5}(a^3\ddot{a} + a^2\dot{a}^2) - \frac{5m^4}{16\omega^7} a^4 \dot{a}^2 + \frac{m^2}{32\omega^7}(a^3 \ddddot{a} + 4a^2\dot{a}\dddot{a} + 3a^2\ddot{a}^2) + \frac{m^4}{64\omega^9}(28a^4\dot{a}\dddot{a} + 126a^3\dot{a}^2\ddot{a} + 21a^4\ddot{a}^2 + 21a^2\dot{a}^4) - \frac{231m^6}{64\omega^{11}}(a^5\dot{a}^2\ddot{a} + a^4\dot{a}^4) + \frac{1155m^8}{256\omega^{13}} a^6 \dot{a}^4 \right]. \tag{2.19b}$$

The first term on the right in each of these expressions is divergent while the rest are finite. If the integration is performed then it is seen that the second term on the right in each expression is proportional to $G^0{}_0$ and $-R$, respectively. Such a term also occurs in the adiabatic stress tensor for Bianchi type-I spacetimes. Fulling *et al.*[20] suggest that it corresponds to a finite renormalization of the gravitational constant. We shall treat it as such, and therefore will leave this term out of the backreaction equations.

The other terms in Eqs. (2.19) are all independent of the mass once the integration over k has been performed. Thus, they are exactly the same terms as occur for the conformally invariant scalar field. This ends our review of canonical quantization and adiabatic regularization.

III. DERIVATION AND DISCUSSION OF THE BACKREACTION EQUATIONS

The backreaction equations for the semiclassical approximation to quantum gravity are given by Eq. (2.13). To find their specific form for universes with the metric (2.2) which contain classical radiation, conformally invariant free quantum fields, and the conformally coupled massive scalar field one must find expressions for the quantities on the right-hand side of (2.13).

The stress-energy tensor for classical radiation in spacetimes with the metric (2.2) is

$$T^a{}_b{}^{\mathrm{CR}} = \tilde{\rho}_r a^{-4}(\eta)\mathrm{diag}(-1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}), \tag{3.1}$$

where $\tilde{\rho}_r$ is a constant. For conformally invariant fields $\langle T_{ab} \rangle$ breaks up into two pieces for any n-particle state. As in (2.14a) for the massive scalar field, one of these pieces is $\langle 0 | T_{ab} | 0 \rangle$ and the other, call it $T_{ab}^{(2)}$, depends on the number and distribution of the particles. In homogeneous and isotropic spacetimes the second piece has the same form as T_{ab}^{CR} in (3.1).[23] The proof of this is as follows: Homogeneity and isotropy require that in a comoving frame $T_{ab}^{(2)}$ be diagonal and have the components $-T^0{}_0 = \rho$, $T^1{}_1 = T^2{}_2 = T^3{}_3 = p$, where ρ and p are functions only of time. Conformal invariance implies that $T^a{}_a = 0$, so $p = \frac{1}{3}\rho$. Conservation implies that $T^{ab}{}_{;b} = 0$ and this gives the relation $(\rho a^3)^{\cdot} = -p(a^3)^{\cdot}$. Together these relations uniquely fix the form of $T_{ab}^{(2)}$ to be that given in Eq. (3.1).

Thus without loss of generality, we can take the conformally invariant fields to be in their vacuum states. The preferred choice of vacuum for conformally invariant fields in conformally flat spacetimes, such as those with the metric (2.2), is the conformal vacuum. It is obtained by conformally transforming the standard Minkowski-space modes to the curved spacetime and using them to define the vacuum state. For the conformally invariant scalar field in our metric, these conformally transformed modes are given by Eq. (2.3) with

$$\psi = \frac{e^{-ik\eta}}{(2k)^{1/2}}. \tag{3.2}$$

Finally, for the conformal vacuum in conformally flat spacetimes the regularized expression for $\langle 0 | T_{ab} | 0 \rangle$ is well known to be[24,25]

$$\langle 0 | T_{ab} | 0 \rangle = \frac{\alpha}{3}(g_{ab} R^{;c}{}_{;c} - R_{;ab} + RR_{ab} - \tfrac{1}{4} g_{ab} R^2) + \beta(\tfrac{2}{3} RR_{ab} - R_a{}^c R_{bc} + \tfrac{1}{2} g_{ab} R_{cd} R^{cd} - \tfrac{1}{4} g_{ab} R^2). \tag{3.3}$$

Here g_{ab} is the metric tensor and R_{ab} is the Ricci tensor. For the scalar field $\beta=\alpha=(2880\pi^2)^{-1}$, for four-component spinor fields

$$\beta=\tfrac{11}{6}\alpha=11(2880\pi^2)^{-1} ,$$

and for the vector field dimensional regularization gives

$$\beta=\tfrac{31}{6}\alpha=62(2880\pi^2)^{-1}$$

while zeta-function regularization gives

$$\beta=-\tfrac{31}{9}\alpha=62(2880\pi^2)^{-1} .$$

Note that regularization breaks the conformal invariance so $\langle 0\,|\,T\,|\,0\rangle\neq 0$.

To obtain a regularized expression for $\langle 0\,|\,T_{ab}\,|\,0\rangle$ for the conformally coupled massive scalar field, we can subtract Eqs. (2.19) from (2.18). The resulting expression along with the modes (2.3) can be substituted into (2.14a) to give $\langle T_{ab}\rangle$.

Before writing down the set of coupled equations that we shall solve it is useful to define scale-invariant variables. As in Paper I these allow us to scale the amount of classical radiation, $\tilde{\rho}_r$, out of the equations. The variables which accomplish this are

$$\begin{aligned} b&=l^{-1}\tilde{\rho}_r^{\,-1/4}a \ , \\ \chi&=6^{-1/2}\tilde{\rho}_r^{\,1/4}\eta \ , \\ \hat{k}&=\tilde{\rho}_r^{\,1/4}k \ , \\ \hat{\psi}&=\tilde{\rho}_r^{\,-1/8}\psi \ . \end{aligned} \tag{3.4}$$

Combining Eqs. (2.3), (2.5), (2.13), (2.14a), (2.18), (2.19), (3.1), (3.3), and (3.4) and dropping the carets one arrives at the following set of coupled equations which determine the backreaction of the fields on the spacetime geometry:

$$\psi''+6(k^2+m^2l^2b^2)\psi=0 \ , \tag{3.5a}$$

$$\frac{b'^2}{b^4}=\frac{1}{b^4}+\frac{\alpha}{3}\left[\frac{b'''b'}{2b^6}-\frac{b''b'^2}{b^7}-\frac{1}{4}\left[\frac{b''}{b^3}\right]^2\right]+\frac{\beta}{12}\left[\frac{b'}{b^2}\right]^4+I_1 \ , \tag{3.5b}$$

$$-\frac{b''}{b^3}=\alpha\left[-\frac{b''''}{12b^5}+\frac{1}{3}\frac{b'''b'}{b^6}+\frac{1}{4}\left[\frac{b''}{b^3}\right]^2-\frac{1}{2}\frac{b''b'^2}{b^7}\right]+\frac{\beta}{6}\left[\left[\frac{b'}{b^2}\right]^4-\frac{b''b'^2}{b^7}\right]-I_2 \ , \tag{3.5c}$$

where primes denote derivatives with respect to χ and

$$I_1\equiv(4\pi^2b^4)^{-1}\int_0^\infty dk\,k^2\{[\tfrac{1}{6}\,|\,\psi'\,|^2+(k^2+m^2l^2b^2)\,|\,\psi\,|^2]\times(1+2n_k)-(k^2+m^2l^2b^2)^{1/2}\} \ ,$$

$$I_2\equiv\frac{m^2l^2}{4\pi^2b^2}\int_0^\infty dk\,k^2[\,|\,\psi\,|^2(1+2n_k)-\tfrac{1}{2}(k^2+m^2l^2b^2)^{-1/2}] \ .$$

The values of α and β in these equations are equal to the sum of α and β for each of the conformally invariant fields plus a contribution of $(2880\pi^2)^{-1}$ to both from the massive scalar field. Thus the massive scalar field adds terms to the backreaction equations which are the same as those contributed by a conformally invariant scalar field. The massive field also contributes the terms I_1 and I_2. If $m=0$, $I_2=0$ and if $n_k=0$ as well, $I_1=0$. If $m=0$ but $n_k\neq 0$, $I_1\propto b^{-4}$ as was shown in the beginning of this section.

Before discussing the behaviors of solutions to Eqs. (3.5) it is useful to say a few words about the number of solutions we expect to find. Orthonormality of the modes implies that ψ must satisfy Eq. (2.7). Since (3.5a) is a second-order equation, this effectively leaves a one-parameter family of solutions. The specification of this parameter for each value of k corresponds to a choice of vacuum state. The specification of n_k in I_1 and I_2 for each k then gives the state that the field is in.

All three equations are explicitly independent of χ so that solutions are always invariant under the translation $\chi\rightarrow\chi+\chi_0$, where χ_0 is an arbitrary constant. The equations are also invariant under the transformation $\chi\rightarrow-\chi$, although their solutions in general are not. Equation (3.5c) is a fourth-order equation so one expects a four-parameter family of solutions to it for a given choice of vacuum and of n_k. However, the constraint equation, (3.5b), takes care of one of these parameters and the invariance of solutions under time translation takes care of another. Thus, one effectively has a two-parameter family of solutions to Eq. (3.5c). When we talk about families of solutions in the next section, it is this effective two-parameter family that we shall be referring to.

IV. SOLUTIONS TO THE BACKREACTION EQUATIONS

In this section we discuss those solutions to the backreaction equations, (3.5), which begin with $b=0$ for $\alpha\neq 0$. We also discuss the solutions that occur for $\alpha=0$, $\beta>0$; these begin at nonzero values of b. Because different techniques are used to analyze the behavior of solutions for small, intermediate, and large values of b, we discuss each of these cases separately. For intermediate and large values of b, we restrict ourselves to solutions that occur if $\beta>0$. This is because all regularization schemes for the fields give $\beta>0$. For small values of b our results are general.

A. Solutions in the limit $b\rightarrow 0$

The chief difficulty in finding solutions to Eqs. (3.5) lies in the fact that analytical solutions to Eq. (3.5a) are difficult to find for arbitrary functions of the scale factor and when found tend to be nonlocal. Thus, one is led to do background-field calculations or numerical computations. The limit $b\rightarrow 0$ is an exception in that, for this case, analytical solutions to (3.5a) can be found and the initial behaviors of solutions to (3.5c) can be determined.

From the initial behaviors one obtains starting values which can be used for numerical integrations of the equations. We describe these numerical integrations and the results obtained from them in the next subsection.

In the limit $b \to 0$, Eq. (3.5a) has the general solution

$$\psi = A\exp(-i6^{1/2}k\chi)+B\exp(i6^{1/2}k\chi)\ , \tag{4.1}$$

where A and B are arbitrary constants. This allows (3.5a) to be written as the Volterra equation:

$$\psi = A\exp(-i6^{1/2}k\chi)+B\exp(i6^{1/2}k\chi) - m^2l^26^{1/2}k^{-1}\int_w^\chi du\, b^2(u)\sin[6^{1/2}k(\chi-u)]\psi(u)\ , \tag{4.2}$$

where $b(w)=0$. Substitution of (4.2) into (2.7) at $\chi=w$ gives the condition

$$|A|^2-|B|^2=(2k)^{-1}\ . \tag{4.3}$$

Equation (4.2) can be solved iteratively with the result that

$$\begin{aligned}\psi = {} & A\exp(-i6^{1/2}k\chi)+B\exp(i6^{1/2}k\chi) \\ & +\sum_{n=1}^{\infty}(-1)^n(ml)^{2n}6^{n/2}k^{-n} \\ & \times\int_w^\chi du_1 b^2(u_1)\sin 6^{1/2}k(\chi-u_1) \\ & \times\int_w^{u_1}du_2 b^2(u_2)\sin 6^{1/2}k(u_1-u_2)\cdots\int_w^{u_{n-1}}du_n b^2(u_n)\sin 6^{1/2}k(u_{n-1}-u_n) \\ & \times[A\exp(-i6^{1/2}ku_n)+B\exp(i6^{1/2}ku_n)]\ .\end{aligned} \tag{4.4}$$

By bounding all of the sines and complex exponentials by unity we find that

$$|\psi| < 2(|A|+|B|)\exp\left[m^2l^26^{1/2}k^{-1}\int_w^\chi du\, b^2(u)\right]\ . \tag{4.5}$$

So the series in (4.4) converges if $\int_w^\chi du\, b^2(u)$ is finite. This is the case for all the solutions to Eq. (3.5) that begin with $b=0$.

The vacuum state for the field is specified by choosing for each value of k values of A and B such that (4.3) is satisfied. In the limit $b \to 0$, the wave equation (2.1) is conformally invariant. Thus an obvious choice of vacuum is the one which reduces to the conformal vacuum, (3.2), in the limit $b \to 0$. It is given by

$$A=(2k)^{-1/2},\quad B=0\ . \tag{4.6}$$

This vacuum has several properties that make it particularly attractive as an "in" vacuum state. First, it is the only one which reduces to the conformal vacuum in the limit $b \to 0$. Second, it is the only one for which the expectation value of the stress tensor does not have a piece that behaves like classical radiation as $b \to 0$. To see this one can substitute Eqs. (4.4), (2.3), and the difference of Eqs. (2.18a) and (2.19a) into Eq. (2.14a). The result is

$$\begin{aligned}-\langle T^0{}_0\rangle = {} & (4\pi^2l^4b^4)^{-1}2\int_0^\infty dk\, k^4\{[|A|^2+|B|^2+AB^*\exp(-i24^{1/2}k\chi)+A^*B\exp(i24^{1/2}k\chi)] \\ & \times(1+2n_k)-(2k)^{-1}+O(m^2l^2b^2)\}+\text{terms which are independent of } m\ .\end{aligned} \tag{4.7}$$

Only if $|A|=(2k)^{-1/2}$ and $B=n_k=0$ is there no term in (4.7) that is proportional to b^{-4}. Thus for all other states the stress energy of the field behaves as though particles were present in the limit $b \to 0$.

There are two other reasons (4.6) is an attractive choice of vacuum. First, for de Sitter space where $b \propto (\chi_0-\chi)^{-1}$, (4.6) gives the standard de Sitter-invariant vacuum.[17] Second, for spacetimes with $b \propto e^{c\chi}$, one finds that (4.6) is identical to the Chitre and Hartle vacuum defined by computing the propagator using a path integral and summing over paths only to the future of the initial singularity.[26]

For all of these reasons, this is the vacuum state that we chose for the numerical work described in the next subsection. However, the behavior of solutions in the limit $b \to 0$ is qualitatively the same regardless of what initial state is chosen for the field, so long as $\langle T^0{}_0\rangle$ is finite for $b>0$. This is because, except for the terms which are independent of the mass, the most divergent terms in (4.7) go like b^{-4}. This means that the dominant effect of the mass terms is to effectively change the amount of classical radiation in the universe. This changes the quantitative behavior of solutions in the limit $b \to 0$, but it does not lead to new types of solutions as was shown in Paper II.[27] As already pointed out, the terms independent of the mass simply increase the values of α and β in Eq. (3.5b).

Thus the presence of the conformally coupled massive scalar field does not lead to any new types of behaviors in the limit $b \to 0$ and it does not rule out any old ones, regardless of what state the field is in. So all of the initial behaviors found in Paper I for spacetimes beginning with $b=0$ still occur. This means that the results of Paper I on the numbers of solutions with and without particle horizons that begin with $b=0$ are unchanged by the presence of a conformally coupled massive scalar field.

B. Intermediate values of b

To determine what effects, if any, the mass terms in (3.5) have on solutions for intermediate values of b, we used the solutions in Paper I along with (4.4) and (4.6) to obtain starting values for numerical integrations of Eqs. (3.5a) and (3.5c) using (3.5b) as a constraint.

Because we were interested in particle production effects and because it is a case of obvious interest we assumed the massive field was in the "in" vacuum state given by (4.6). We assumed $\beta > 0$ because all regularization schemes give positive values of β for the fields we are considering. For intermediate values of b, the mass terms I_1 and I_2 tend to be small compared to other terms in (3.5b) and (3.5c). To maximize the effects of these terms, we limited ourselves to the values $|\alpha| = (2880\pi^2)^{-1}, 0$ because these are the smallest possible values for $|\alpha|$ that any regularization scheme gives. Although smaller masses were investigated, most of the numerical work done was for $ml = 100$ because this was the smallest value for which significant changes in the solutions were regularly observed. We display some of our results in Figs. 1–5.

The structure of this subsection is as follows. First we describe the numerical scheme used to solve Eqs. (3.5) and estimate its accuracy. Then we discuss the solutions which begin with $b=0$, for $\alpha \neq 0$, $\beta > 0$. Finally we discuss the solutions for $\alpha = 0$, $\beta > 0$.

1. Description of the numerical methods

To solve Eqs. (3.5a) and (3.5c) one must have initial conditions for $\chi, \psi, \psi', b, b', b'', b'''$. Those for χ, b, b', and

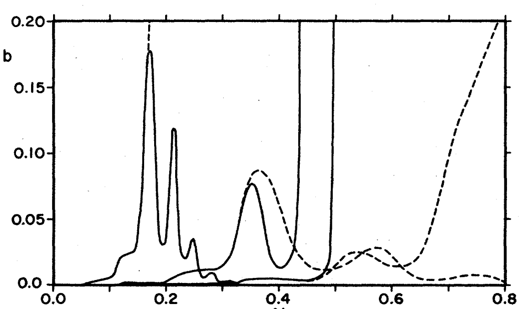

FIG. 1. This figure shows three typical solutions to Eqs. (3.5) when $\beta = 6\alpha = 6(2880\pi^2)^{-1}$. The dashed curves are solutions for $m=0$ while the solid curves are solutions with the same initial values for $ml = 100$. The dashed curve on the top right is an ACS. Each solution undergoes an infinite number of multiple bounces.

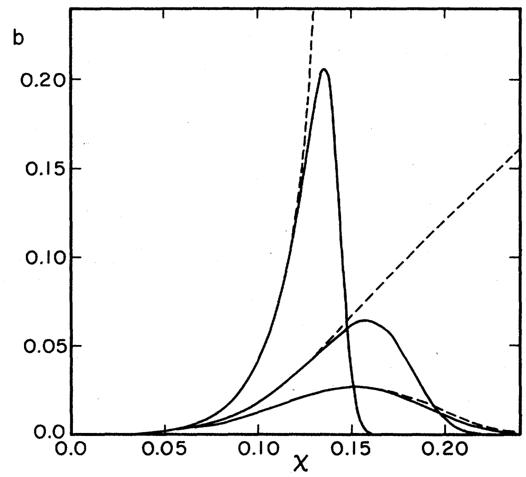

FIG. 2. This figure shows three typical solutions to Eqs. (3.5) when $\beta = 3\alpha = 3(2880\pi^2)^{-1}$. The dashed curves are solutions for $m=0$ while the solid curves are solutions with the same initial values for $ml = 100$. The dashed curve on the top right is an ACS.

b'' are provided by the solutions found in Paper I. That for b''' is supplied by Eq. (3.5b) and those for ψ and ψ' are supplied by Eqs. (4.4) and (4.6) except for the case $\alpha = 0$ which will be discussed separately.

Given the starting values, one solves the equations in the usual manner by breaking them up into a series of first-order equations and numerically solving them. The only difficulties lie in computing the integral I_2. To compute it numerically one must know ψ for several values of k, which means that (3.5a) must be solved for several values of k. Further, as $k \to \infty$, the $|\psi|^2$ term in I_2 comes closer to canceling the $(k^2 + m^2 l^2 b^2)^{-1/2}$ term, so that for large k the value of $|\psi|^2$ must be known very ac-

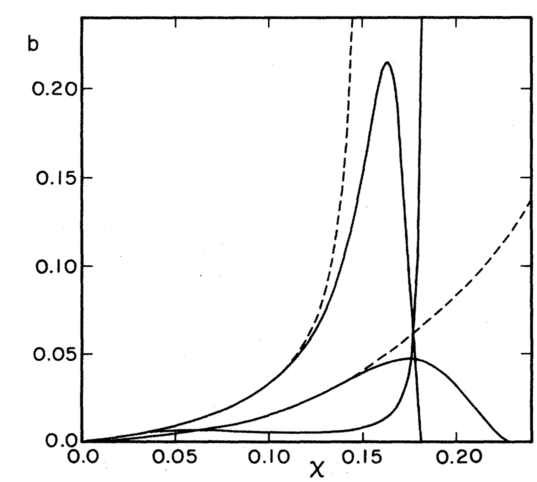

FIG. 3. This figure shows three typical solutions to Eqs. (3.5) when $\beta = \alpha = (2880\pi^2)^{-1}$. The dashed curves are solutions for $m=0$ while the solid curves are solutions with the same initial values for $ml = 100$. The solid curve on the top right is an example of a solution for which, on the scale of this plot, the $m=0$ and $ml = 100$ curves coincide.

curately. Finally one must put some kind of ultraviolet cutoff on I_2 since a finite number of values for k must be used.

These difficulties put severe restraints on solutions that can be obtained numerically. First, one must simultaneously solve a large number of equations to get an accurate value for I_2. Second, as can be seen by examination of the wave equation (3.5a), only for $k \gg mlb$ does the $|\psi|^2$ term in I_2 approximately cancel the $(k^2+m^2l^2b^2)^{-1/2}$ term. Thus, if one wishes to evolve the spacetime to large values of b or consider a large value of m, the cutoff must be correspondingly large. However, a large cutoff means that for small values of b, more equations must be used to compute I_2 accurately. A large cutoff also means that the system must be evolved more slowly because the solutions to the wave equation for large k oscillate with a frequency of $6^{1/2}k(2\pi)^{-1}$ so smaller time steps must be used. Thus, a factor of 10 increase in the mass or in the maximum value of b that a solution is integrated out to usually results in a factor of 100 increase in the computing time.

We were able to greatly improve the accuracy of I_2 by noting that for large k, the series in Eq. (4.4) is really an expansion in powers of k^{-1}. By substituting the first few terms of (3.5c) into I_2 with a lower limit cutoff λ and an upper limit cutoff of the form $e^{-\sigma k}$, one finds in the limit $\sigma \rightarrow 0$ that

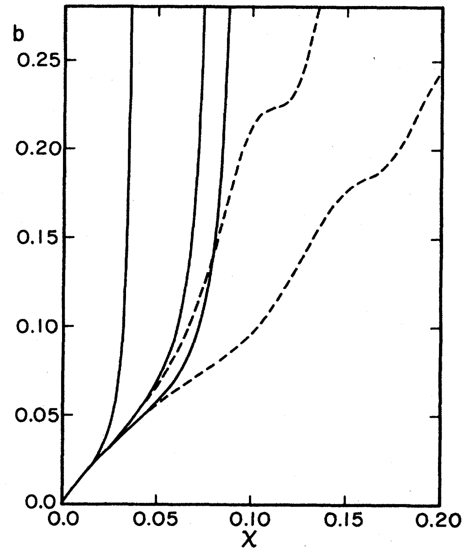

FIG. 4. This figure shows three solutions to Eqs. (3.5) for $\beta=-\alpha=(2880\pi^2)^{-1}$. The dashed curves are solutions for $m=0$ while the solid curves are solutions with the same initial values for $ml=100$. The solid curve on the left is an example of a solution for which, on the scale of this plot, the $m=0$ and $ml=100$ curves coincide.

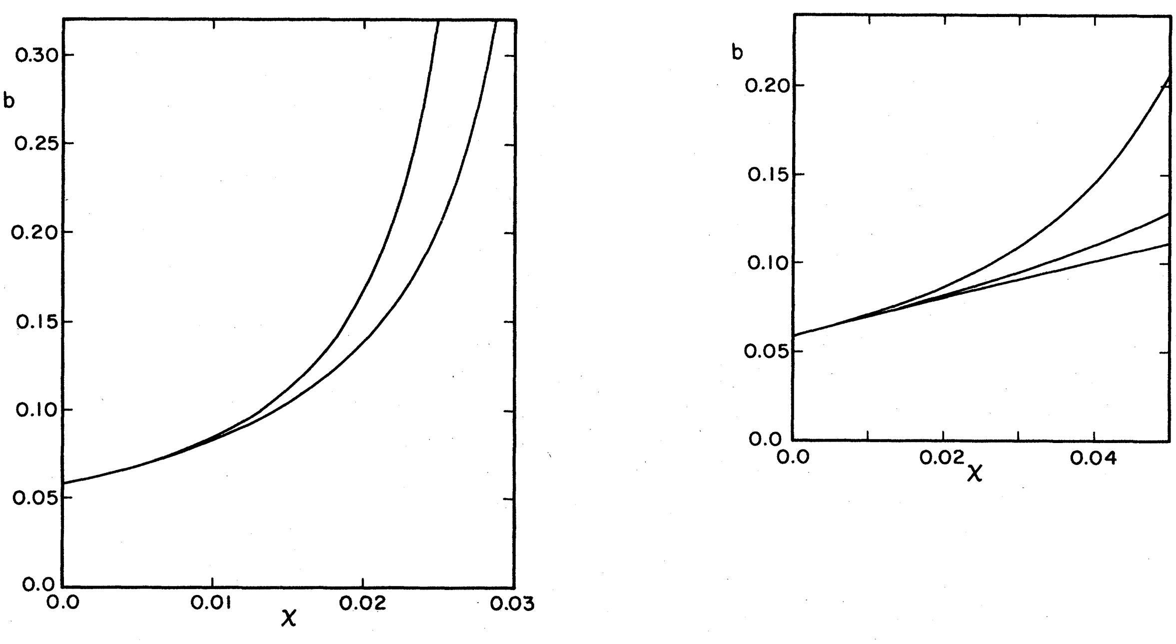

FIG. 5. All solutions in this figure begin at $X=0$ with a finite value of b and b' and an initial singularity. The plot on the left shows two solutions to Eqs. (3.5) for $\alpha=0$ which occur when the plus sign in Eq. (4.9) is chosen. The solution on the left is that for $m=0$ and is asymptotically de Sitter. That on the right is for $ml=72$ and is probably also asymptotically de Sitter. For $ml \gtrsim 75$, solutions end at finite values of b and b' with a final singularity. The plot on the right shows three solutions to Eqs. (3.5) for $\alpha=0$ which occur when the minus sign in Eq. (4.9) is chosen. From top to bottom the solutions are for $ml=70$, 50, 0. The solution for $ml=0$ is an ACS and the solution for $ml=50$ is probably also an ACS. For $ml=70$ it is uncertain whether the solution is an ACS, asymptotically de Sitter or something in between. For $ml \gtrsim 80$ solutions end at a finite value of b and b' with a final singularity.

$$I_{2,\text{excess}}=-\frac{m^4l^4}{16\pi^2}\left[\gamma+24^{1/2}b^{-2}\int_w^{\chi}du\,\ln[24^{1/2}\lambda(\chi-u)]\{6^{-1/2}b(u)b'(u)\cos[24^{1/2}\lambda(\chi-u)]+2\lambda b^2(u)\sin[24^{1/2}\lambda(\chi-u)]\}\right]+\cdots\,, \tag{4.8}$$

where γ is Euler's constant. By bounding higher-order terms in $I_{2,\text{excess}}$ one can also obtain an estimate of the error in I_2.

To check the accuracy of our numerical scheme we computed I_2 for de Sitter space where the answer is known.[17] We found that without $I_{2,\text{excess}}$ and with 20 equations, we could always come within an order of magnitude of the correct answer and were often much closer. Without $I_{2,\text{excess}}$ and with 100 equations we could come within about 10% of the correct answer and with 100 equations and $I_{2,\text{excess}}$ we could come within at least 5% of the correct answer. Unfortunately, for more than 2000 time steps the integrals in $I_{2,\text{excess}}$ require a great deal of computer time since they must be evaluated at each time step. So for the runs used for Figs. 1—4, $I_{2,\text{excess}}$ was not computed. Nevertheless, both runs with 20 values of k and 100 or 200 values of k were done for each solution plotted and with the exception of the solution on the top left in Fig. 3 the differences are not apparent on the scale of the plots.

Having discussed the numerical scheme used to solve Eqs. (3.5) we now discuss our results.

2. *Solutions for $\alpha\neq 0$.*

As shown in Paper I, if $\alpha\neq 0$ then there is always a two-parameter family of solutions which begin with $b=0$. We showed in Sec. IV A that their behaviors in the limit $b\rightarrow 0$ are completely unaffected by the mass terms in (3.5c) if the field is in the vacuum state specified by Eq. (4.6).

For larger values of b, our numerical work shows that I_2 can be large enough to significantly affect the behaviors of solutions, but that for most solutions it is not. This is because for most solutions b'''' is observed to be relatively large and if it does pass through zero it does so very quickly. However, solutions are significantly affected if b'''' is relatively small for some period of time or if the solutions are unstable. In particular, for $\alpha>0$ the ACS are unstable and I_2 changes their behaviors even for relatively small values of m.

Even though the mass term in (3.5c) does change the behavior of some solutions, the types of behaviors which can occur are left unchanged. Thus, for $\beta>3\alpha>0$ solutions beginning with $b=0$ still undergo multiple bounces. For $0<\beta\leqq 3\alpha$, solutions continue to either expand monotonically, reach a maximum and collapse to a singularity, or reach one maximum followed by a minimum and then expand monotonically. For $\alpha<0$, $\beta>0$ solutions continue to either diverge quickly or undergo an infinite number of phase plane oscillations[28] while expanding monotonically. Some examples of the ways that particular $m\neq 0$ solutions differ from $m=0$ solutions with the same initial conditions, are shown in Figs. 1—4.

From these figures one can see that $m\neq 0$ can cause a solution to either diverge more quickly than it was or to diverge less quickly and even turn over. This is dependent on the sign of α and on whether I_2 is positive or negative. In the limit $b\rightarrow 0$, I_2 is positive and it is observed to remain so much of the time. Thus, for $\alpha>0$ solutions are more likely to be turned over and for $\alpha<0$ they are more likely to be made to diverge more quickly.

3. $\alpha=0$

As for the case $\alpha\neq 0$, we shall only consider values of $\beta>0$ in this case. To maximize the effects of I_1 and I_2 in Eqs. (3.5b) and (3.5c) we chose $\beta=(2880\pi^2)^{-1}$ for our numerical work since this is the smallest value of β any regularization scheme gives for any field.

When $\alpha=0$, the higher derivative terms in (3.5b) and (3.5c) vanish and one is left with two solutions rather than a two-parameter family of solutions. To see this one simply solves (3.5b) algebraically for b' with the result:

$$b'=(6/\beta)^{1/2}\{1\pm[1-(\beta/3)(b^{-4}+I_1)]^{1/2}\}^{1/2}\,. \tag{4.9}$$

For $m=n_k=0$, $I_1=0$ and the solutions begin at $b=(\beta/3)^{1/4}$ with $b'=(6/\beta)^{1/2}$. Substitution into (3.5c) shows that $|b''|$ is initially infinite so these somewhat pathological solutions begin with an initial singularity. By expanding (4.9) in powers of b^{-1} it is seen that the solution with the plus sign is asymptotically de Sitter while that with the minus sign is an ACS. Thus, despite the pathology of these solutions they give us an ideal chance to see what effects the massive field has on ACS and asymptotically de Sitter solutions when the instabilities which result from the b'''' and b''' terms in (3.5b) and (3.5c) are not present. However, because of the square roots in (4.9) one must be careful about the generalization of these results to the case $\alpha\neq 0$. In particular, if enough particle production occurs soon enough or if vacuum polarization effects get too large then I_1 becomes large causing b' to become imaginary. If this happens then b'' becomes infinite and the solution ends at a finite value of b and b' with a final singularity.

Examination of (4.9) shows that only if $-I_1>b^{-4}$ as $b\rightarrow 0$, do the solutions begin at $b=0$. In fact, in the limit $b\rightarrow 0$, one can see from (4.7) that for most states $I_1>0$. Thus the solutions begin at $b=b_0$ for some $b_0>0$. For such solutions Eq. (3.5a) can again be written as a Volterra equation and solved iteratively with the result:

$$\psi = Ce^{-i6^{1/2}\omega\chi} + De^{i6^{1/2}\omega\chi} + \sum_{n=1}^{\infty}(-1)^n(ml)^{2n}6^{n/2}\omega^{-n}\int_0^{\chi}du_1 \cdots \int_0^{u_{n-1}}du_n\{[b^2(u_1)-b_0{}^2]\cdots[b^2(u_n)-b_0{}^2] \times \sin 6^{1/2}\omega(\chi-u_1)\cdots \sin 6^{1/2}\omega(u_{n-1}-u_n)\} \times (Ce^{-i6^{1/2}\omega u_n}+De^{i6^{1/2}\omega u_n}) . \tag{4.10}$$

Here $\omega^2 \equiv k^2+m^2l^2b_0{}^2$.

The obvious choice of vacuum state is that which corresponds to the Minkowski vacuum in the limit $b \to b_0$. It is given by

$$C=(2\omega)^{-1/2} , \quad D=0 . \tag{4.11}$$

If the field is in this state initially then at $b=b_0$, $I_1=I_2=0$, and the solution starts off at the same value of b_0 as it does for $m=0$, with the same initial value of b'.

To maximize the effects of particle production and vacuum polarization due to the massive scalar field, we chose the field to be in the vacuum state (4.11). The results of our numerical work for $\beta=(2880\pi^2)^{-1}$ are shown in Fig. 5. From that figure one can see that the asymptotically de Sitter solution still expands rapidly for $m\neq 0$, but the expansion is slowed. For $ml>75$, b' decreases to the point that it becomes imaginary and the solution ends with a final singularity. In a less pathological situation the solution would probably continue to slow its expansion; however, if it reached the de Sitter phase, then as we shall argue in the next subsection, it would probably still be asymptotically de Sitter. For $ml \lesssim 1$, the solution is essentially unaffected and I_2 is observed to always be several orders of magnitude smaller than the leading terms in (3.5c).

For the ACS, one sees from Fig. 5 that the mass terms make the solution expand faster. For $ml \gtrsim 80$, I_1 becomes too large and the solution ends with a final singularity. For $ml \sim 70$, I_2 quickly dominates the other terms in (3.5c) and the expansion is faster than for a matter-dominated universe, for which $b=\text{const}(\chi-\chi_0)^2$, but slower than for a de Sitter universe, for which $b=\text{const}(\chi_0-\chi)^{-1}$. For smaller masses, the matter dominates later and one has a situation such as occurs for the ACS if $\alpha\neq 0$, where the universe smoothly goes from being radiation dominated to matter dominated. It becomes matter dominated because of particle production due to the massive scalar field.

C. Large values of b

We have seen that for intermediate values of b, the massive scalar field can drastically alter the behavior of individual solutions to Eqs. (3.5) but that it does not give rise to any new types of behaviors. For large values of the scale factor our results are not as conclusive, but we shall argue that all of the large-b behaviors found for $m=0$ probably still occur for $m\neq 0$ although they may be modified somewhat.

1. ACS

The one type of behavior that occurs for all values of α and β if $m=0$ is that exhibited by the ACS. At late times they expand like radiation-dominated universes, so $b=\chi+\chi_0$, where χ_0 is an arbitrary constant.

If $m\neq 0$ we shall show that for $\alpha>0$ a one-parameter family of ACS exist and for $\alpha<0$ a two-parameter family of ACS exist for a large number of states that the massive scalar field can be in. Unfortunately, our argument gives no insight into which initial behaviors and which initial states result in ACS. It is possible, for example, that there are ACS for all reasonable initial states that the massive scalar field can be in. The numerical work described in Sec. IV B indicates there are ACS for the vacuum state given by (4.6) if $\alpha<0$ and that there are likely to be ACS for this state if $\alpha>0$ as well.

For $\alpha>0$ there is a one-parameter family of ACS with the late-time behavior

$$b=b_c+A\tau b_c{}^{-1/2}\exp\left[-2\tau^{-1}\int^{\chi}b_c\,d\chi\right] , \tag{4.12a}$$

where A is an arbitrary constant and $\tau\equiv|\alpha/3|^{1/2}$. Substituting (4.12a) into (3.5b) and assuming that $b_c \gg |b_c'| \gg |b_c''| \gg \cdots$, one finds that to leading order the equation satisfied by b_c is

$$(b_c' b_c{}^{-2})^2=b_c{}^{-4}+I_1 . \tag{4.12b}$$

We shall show that to leading order

$$I_1=cb_c{}^{-3}+m^2l^2(576\pi^2)^{-1}(b_c'b_c{}^{-2})^2 \tag{4.13}$$

for some positive constant c which depends on the state the field is in and the past behavior of the solution. Thus in the limit $b\to\infty$, b_c is a solution to the backreaction equation for a classical Friedmann universe containing radiation and matter. This is why the solutions in (4.12a) are ACS.

To compute I_1 one needs expressions for the modes ψ. It might at first appear that the adiabatic approximation can be used, but this is not correct. Writing Eq. (4.12a) as $b=b_c+b_1$, one sees that although

$$b_c \gg |b_c'| \gg \cdots , \quad |b_1|<|b_1'|<|b_1''|<\cdots ,$$

so that the adiabatic approximation is not valid.

However, expressions for the modes can be obtained in the following way. Write $\psi=\psi_0(1+h_1+h_2+\cdots)$, where ψ_0 is a solution to the equation

$$\psi_0''+6(k^2+m^2l^2b_c^2)\psi=0 \tag{4.14}$$

and it is assumed that

$$1 \gg |h_1| \gg |h_2| \gg \cdots .$$

Then substitute this expression for ψ into (3.5a) and impose the boundary conditions $h_i, h_i' \to 0$ as $b\to\infty$. If only terms of $O(b_1)$ and $O(h_1)$ are kept then one has a linear equation for h_1 which can be solved with the result

$$h_1=c_1+\int_r^{X} du_1 \psi_0^{-2}(u_1)\left[c_2-12\int_s^{u_1} du_2 \psi_0^2(u_2) b_c(u_2) b_1(u_2)\right], \tag{4.15}$$

where c_1 and c_2 are chosen so that the lower limits of the integrals are canceled, that is, so that h_1 and h_1' have no constant pieces. Substituting this back into (3.5a) one next obtains an equation for h_2 and so on. Finally, the resulting expression for ψ is substituted into the expression for I_1 which appears below Eqs. (3.5).

One still needs an expression for ψ_0, but since

$$b_c \gg |b_c'| \gg |b_c''| \gg \cdots ,$$

the adiabatic approximation is valid for ψ_0. Since ψ_0 will not be in an adiabatic vacuum state one finds from Eqs. (2.11) and (2.15) that, in general,

$$\psi_0 \approx (2W)^{-1/2}\left[\alpha_k \exp\left[-i6^{1/2}\int^{X} W\,dX\right]+\beta_k \exp\left[i6^{1/2}\int^{X} W\,dX\right]\right], \tag{4.16}$$

where W is given by Eq. (2.17) and Eq. (2.7) implies that $|\alpha_k|^2-|\beta_k|^2=1$.

Substituting (4.14) and (4.15) into I_1 one can show that to zeroth order in h_1 and b_1 the leading-order contributions to I_1 are of the form

$$(2\pi^2 b_c^4)^{-1}\int_0^\infty dk\, k^2 W |\beta_k|^2+m^2l^2(576\pi^2)^{-1}(b_c' b_c^{-2})^2 .$$

After some calculation one finds that to first order in h_1 and b_1, the leading-order term can be found by setting $\beta_k=0$ and can be bounded by $ml(96\pi)^{-1}b_c^{-3}b_c' b_1$. Thus, for large b, the zeroth-order terms dominate and I_1 is given by Eq. (4.13).

For $\alpha<0$, the ACS have the late-time behavior

$$b=b_c+B\tau b_c^{-1/2}\cos\left[2\tau^{-1}\int^{X} b_c dX+\delta\right], \tag{4.17a}$$

where B and δ are arbitrary constants. Substitution of (4.17a) into (3.5b) shows that to leading order b_c is a solution of the equation

$$(b_c' b_c^{-2})^2=b_c^{-4}+I_1+4B^2 b_c^{-3} . \tag{4.17b}$$

The calculation of I_1 is just the same as in the $\alpha>0$ case with the same result, so in the limit $b\to\infty$, b_c is a solution to the backreaction equation for a classical Friedmann universe containing matter and radiation. Note that in this case, however, the conformally invariant fields add a term to this "classical" backreaction equation which is of the same form as the energy density for classical matter and which is present even if the massive scalar field is not.

In Papers I and II we only found one ACS for $\alpha<0$ rather than the two-parameter family shown in (4.17a). This is because massive fields were not included in these models so we restricted the definition of ACS to mean solutions which at late times expand like classical radiation-dominated Friedmann universes. If massive fields are present then the ACS expand like classical matter-dominated universes at late times.

If the ACS in Eqs. (4.12) and (4.17) were integrated backwards in time one would find a variety of initial behaviors and initial states even for solutions with the same values of c. To determine if ACS occur for a given type of initial behavior and state one can attempt to numerically integrate solutions with that initial behavior and state forward in time. For $\alpha>0$, the ACS are unstable so this does not work very well. However, for $ml \lesssim 1$, $b \lesssim (ml)^{-1}$, and the initial vacuum state (4.6) our numerical work shows that I_1 and I_2 are usually not large enough to significantly affect the behaviors of solutions. Since for $m=0$ a one-parameter family of ACS exists for $\alpha>0$, we expect a one-parameter family of solutions to begin expanding like ACS for $ml \lesssim 1$. For solutions expanding like ACS, the adiabatic approximation for ψ_0 in (4.14) is valid for $b \gtrsim 1$. Thus, it is reasonable to assume that these solutions will be ACS and thus that a one-parameter family of ACS exists for $\alpha>0$ when the massive field is in the initial vacuum state (4.6). For $\alpha<0$, the ACS are stable and our numerical work shows that a two-parameter family exists when $ml \lesssim 1$ and the field is in the vacuum state (4.6).

To get some idea of how much particle production occurs for the ACS, we have computed the energy density of the massive scalar field in a classical radiation-dominated universe. We chose the initial state of the field to be the "in" vacuum state given by Eq. (4.6). For intermediate values of b, I_1, which is equal to the energy density, ρ_m, when vacuum polarization effects are small, was computed numerically for $ml=1$. Our results are shown in Fig. 6. For larger values of b it is shown in the Appendix that

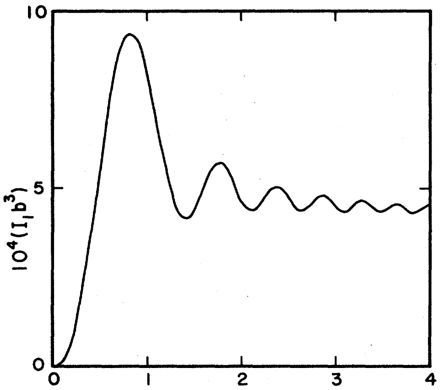

FIG. 6. This figure shows the time evolution of the quantity I_1 which appears in Eq. (3.5b) for a radiation-dominated universe. I_1 is the part of $-\langle 0|T^0{}_0|0\rangle$ for the conformally coupled massive scalar field that vanishes in the limit $m\to 0$. Note that because of particle production $I_1 \propto b^{-3}$ at late times when vacuum polarization effects are small.

$$\rho_m l^4 \approx 4.2\times 10^{-4}(ml)^{5/2}\chi^{-3}+O(\chi^{-4}), \quad \chi\to\infty \ . \tag{4.18}$$

Comparison with Fig. 6 shows that for large χ, (4.18) agrees with our numerical results.

Thus, at late times the energy density of the field behaves as though classical massive particles have been created. Further, these dominate the classical radiation, which from (3.1) has the density

$$\rho_r l^4 = \chi^{-4} \ , \tag{4.19}$$

when

$$\begin{aligned} b &= \chi \approx 2\times 10^3 (ml)^{-5/2} \ , \\ l^{-1}t &\approx 7\times 10^6 (ml)^{-5} \ . \end{aligned} \tag{4.20}$$

Nucleosynthesis occurs at $l^{-1}t \sim 5\times 10^{44}$, so if the universe is to remain radiation dominated until after nucleosynthesis, the scalar field must have $ml \lesssim 10^{-8}$, i.e., $m \lesssim 10^{11}$ GeV. Of course, this is only true for stable particles. For more realistic theories, the particles will generally decay into lighter particles so this constraint would not occur. Nevertheless, there appears to be enough particle production to easily account for the matter in the universe if the field has a mass of 10^{10} GeV or greater.

2. *Asymptotically de Sitter solutions*

For $\alpha,\beta>0$, $m=0$, it was shown in Paper I that a two-parameter family of stable solutions are asymptotically de Sitter and that except for the ACS no other large-b behavior is possible. For $\alpha<0$, $\beta>0$, $m=0$, Starobinski[6] showed that an unstable one-parameter family of asymptotically de Sitter solutions exists. For all these solutions the effective cosmological constant is $\Lambda_{\text{eff}}=6l^{-2}\beta^{-1}$.

For $m\neq 0$, there are still asymptotically de Sitter solutions for all α if $\beta>0$. For $\alpha<0$ it seems likely that an unstable one-parameter family of solutions continues to exist while for $\alpha>0$ we find it to be very likely that a two-parameter family of stable asymptotically de Sitter solutions continues to exist. Our reasoning is as follows:

If the field is in the standard de Sitter vacuum which is given by[17]

$$\psi_d = \tfrac{1}{2}[6^{1/2}\pi(\chi_0-\chi)]^{1/2} H_\nu^{(2)}[6^{1/2}k(\chi_0-\chi)] \ , \tag{4.21}$$

where $H_\nu^{(2)}$ is a Hankle function, χ_0 is an arbitrary constant and $\nu^2=\frac{1}{4}-3m^2/\Lambda_{\text{eff}}$, then the de Sitter solution

$$b = 6^{-1/2}(3/\Lambda_{\text{eff}})^{1/2}(\chi_0-\chi)^{-1}$$

is a solution to Eqs. (3.5) in the limit $b\to\infty$ if $\beta>0$. Further, if the field is in some other state, as it will be for the solutions of interest, then a Bogolubov transformation allows one to write

$$\psi = \alpha_k \psi_d + \beta_k \psi_d^* \tag{4.22}$$

with $|\alpha_k|^2-|\beta_k|^2=1$ following from orthonormality of the modes. Substituting (4.22) along with the de Sitter solution into (3.5b) and taking the limit $\chi\to\chi_0$ one finds

$$\begin{aligned} 2\Lambda_{\text{eff}} l^2 = {} & \frac{\beta}{3}\Lambda_{\text{eff}}{}^2 l^4 + \frac{m^4 l^4}{64\pi^2}[\psi(\tfrac{3}{2}+\nu)+\psi(\tfrac{3}{2}-\nu)-\ln(3m^2/\Lambda_{\text{eff}})] \\ & +\pi^{-2}\Lambda_{\text{eff}}{}^2 l^4(\chi_0-\chi)^4\int_0^\infty dk\, k^2\{\tfrac{1}{3}|\beta_k|^2|\psi_d'|^2+\tfrac{1}{6}\alpha_k\beta_k^*(\psi_d')^2+\tfrac{1}{6}\alpha_k^*\beta_k(\psi_d^{*\prime})^2 \\ & +[k^2+\tfrac{1}{2}m^2\Lambda_{\text{eff}}{}^{-1}(\chi_0-\chi)^{-2}][2|\beta_k|^2|\psi_d|^2+\alpha_k\beta_k^*(\psi_d)^2+\alpha_k^*\beta_k(\psi_d^*)^2]\} \ , \end{aligned} \tag{4.23}$$

where $\psi(\frac{3}{2}\pm\nu)$ is a digamma function.

In the limit $\chi\to\chi_0$, $\psi_d\to\text{const}(\chi_0-\chi)^{1/2-\nu}$. So the last term in (4.23) is small in this limit which means that particle production does not occur fast enough, once the de Sitter phase of the expansion is reached, for the stress energy of the particles to significantly affect the expansion.

Thus, for any state that the field is in, asymptotically de Sitter solutions to Eqs. (3.5) can exist for $\beta>0$. The question then is how many of them do exist and are they stable? The answer for small m is that solutions should not be affected much by the mass terms in (3.5) before reaching the de Sitter phase and upon reaching it they will still not be affected much since $\Lambda_{\rm eff}$ is nearly the same as for $m=0$. To see this, note that $\Lambda_{\rm eff}$ is given implicitly by Eq. (4.23) with $\beta_k=0$. Thus for small m a two-parameter family of stable asymptotically de Sitter solutions should exist for $\alpha>0$ and a one-parameter family of unstable asymptotically de Sitter solutions should exist for $\alpha<0$.

For large m we found in Sec. IV B 2 that for intermediate values of b, all the types of behaviors found for $m=0$ still occur. Thus solutions exist which are divergent enough to approach the de Sitter solution at large b. Given that asymptotically de Sitter solutions can exist and for small m do exist, it seems likely that for large m a one-parameter family exists for $\alpha<0$ and a two-parameter family exists for $\alpha>0$.

3. Other solutions for $\alpha<0$

For $\alpha<0$, $\beta>0$ we have shown that a two-parameter family of ACS exist and we have argued that a one-parameter family of asymptotically de Sitter solutions probably exists. For $m=0$ Starobinski[6] found a two-parameter family of solutions with a third type of large-b behavior of the form

$$b=(\chi_0-\chi)^{-1/2\sigma},\ \ \chi\to\chi_0 \tag{4.24}$$

where χ_0 is an arbitrary constant and $\sigma\equiv\frac{1}{2}(1-\beta/3\alpha)^{1/2}$. These solutions end with a final curvature singularity at $b=\infty$ as can be seen by evaluating the scalar curvature

$$R=l^{-2}b^{-3}b'' \tag{4.25}$$

in the limit $\chi\to\chi_0$.

Our numerical work shows that a two-parameter family of solutions exists for $m\neq0$ which for intermediate values of b, diverge just as rapidly as those solutions for $m=0$ which have the behavior (4.24) at late times. However, we have not found a solution to the wave equation (3.5a) for this behavior so we do not know what the contributions of I_1 and I_2 to the backreaction equations are for large b. This, in turn, prevents us from knowing the late-time behavior of these solutions for $m\neq0$. It seems likely that it is some type of divergent behavior, though perhaps nonsingular at $b=\infty$.

This finishes our discussion of the effects of a massive scalar field on the dynamical evolution of the early universe. We have seen that the mass terms in the backreaction equations do not change the types of behaviors that can occur in the limit $b\to0$. For intermediate values of b the mass terms can significantly affect the behaviors of individual solutions but they do not eliminate any types of behaviors, nor are they observed to create any new ones.

For large values of b the behaviors of solutions are less well known. However, it is clear that the massive scalar field can have a significant effect on these behaviors, through both vacuum polarization effects, as occur for asymptotically de Sitter solutions and particle production as occurs for the ACS.

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APPENDIX

In this appendix we compute the vacuum expectation value of the energy density of a conformally coupled massive scalar field in a radiation-dominated universe in the limit that the scale factor is large. We choose as our "in" vacuum state that defined by Eq. (4.6).

For a radiation-dominated universe

$$b=\chi \tag{A1}$$

and Eq. (3.5a) becomes

$$\psi''+6(k^2+m^2l^2\chi^2)\psi=0\ . \tag{A2}$$

The solution to this equation which in the limit $\chi\to0$ reduces to Eq. (4.4) and (4.6) is[29]

$$\begin{aligned}\psi(x)=&\ 2^{-3/4}k^{-1/2}(G_3/G_1)^{1/2}[W(a,x)+W(a,-x)]\\&+i6^{1/4}(k/ml)^{1/2}2^{-7/4}(G_1/G_3)^{1/2}\\&\times[W(a,x)-W(a,-x)]\ ,\end{aligned}$$

$$\begin{aligned}D_{-1/2+ia}[(1-i)x/2]=&\ 2^{-1/2}\exp(-\tfrac{1}{4}\pi a-i\tfrac{1}{8}\pi-i\tfrac{1}{2}\phi_2)\\&\times[\kappa^{-1/2}W(a,x)\\&+i\kappa^{1/2}W(a,-x)]\ ,\end{aligned} \tag{A3}$$

$$\begin{aligned}D_{-1/2+ia}[(1+i)x/2]=&\ 2^{-1/2}\exp(-\tfrac{1}{4}\pi a+i\tfrac{1}{8}\pi+i\tfrac{1}{2}\phi_2)\\&\times[\kappa^{-1/2}W(a,x)\\&-i\kappa^{1/2}W(a,-x)]\ ,\end{aligned}$$

where $D_\mu(z)$ is a parabolic cylinder function,

$$a\equiv-6^{1/2}k^2/2ml,\ \ x\equiv6^{1/4}(2ml)^{1/2}\chi\ ,$$

$$G_1\equiv|\Gamma(\tfrac{1}{4}+i\tfrac{1}{2}a)|,\ \ G_3\equiv|\Gamma(\tfrac{3}{4}+i\tfrac{1}{2}a)|\ ,$$

$$\kappa\equiv(1+e^{2\pi a})^{1/2}-e^{\pi a},\ \ \phi_2\equiv\arg[\Gamma(\tfrac{1}{2}+ia)]\ .$$

For $x \gg |a|$,

$$W(a,x)=(2\kappa/x)^{1/2}[S_1(a,x)\cos(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2) -S_2(a,x)\sin(\tfrac{1}{4}x^2-a\ln x +\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2)]\ ,$$

$$W(a,-x)=(2/\kappa x)^{1/2}[S_1(a,x)\sin(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{4}\pi +\tfrac{1}{2}\phi_2) +S_2(a,x)\cos(\tfrac{1}{4}x^2-a\ln x +\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2)]\ , \tag{A4}$$

$$S_1(a,x)=1+ax^{-2}+O(x^{-4})\ ,$$

$$S_2(a,x)=(\tfrac{1}{2}a^2-\tfrac{3}{8})x^{-2}+O(x^{-4})\ .$$

Substituting (A4) into (A3) one obtains an expression for $\psi(\chi)$ at late times. Taking the derivative with respect to χ one then obtains an expression for $\psi'(\chi)$. Substituting these expressions in (2.18a) and subtracting (2.19a) gives to leading order in χ:

$$\rho_m=(4\pi^2\chi^3)^{-1}\int_0^\infty dk\,k^2[6^{-1/4}(ml)^{3/2}G_1{}^{-1}G_3k^{-1} +6^{1/4}(ml)^{1/2}\tfrac{1}{4}G_1G_3{}^{-1}k] \times[1+\exp(-6^{1/2}\pi k^2/ml)]\ . \tag{A5}$$

The mass can be scaled out of the integral and the integral can be computed numerically with the final result being

$$\rho_m=4.2\times10^{-4}(ml)^{5/2}\chi^{-3}+O(\chi^{-4})\ . \tag{A6}$$

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