PHYSICAL REVIEW D VOLUME 32, NUMBER 6 15 SEPTEMBER 1985

Quantum cosmological model of the inflationary universe

Ursula Carow and Satoshi Watamura
Research Institute for Fundamental Physics, Kyoto University, Kitashirakawa, Sakyo-ku, Kyoto 606, Japan
(Received 18 April 1985)

A quantum cosmological model of the inflationary universe is investigated by solving the Wheeler-DeWitt equation. We consider a model with a minimally coupled scalar field, the potential of which is a simple double well. By applying the boundary condition of "no boundary," we calculate the wave function of our model universe. We find that in a certain parameter range a big peak is formed near the maximum of the double-well potential of the scalar field, accompanied by a recession of the exponential behavior of the wave function. We show that this peak can be consistently interpreted as representing a high density of classical paths of generalized oscillating universes, and as a consequence of the constructive interference of quantum states corresponding to these classical paths by the WKB approximation. The cosmological scenario with nonvanishing, nearly critical "velocity" of the vacuum expectation value in the early universe, which is suggested by the behavior of the wave function, is discussed.

I. INTRODUCTION

A well-known difficult problem in cosmology is to find a principle with which we can specify the initial conditions of our universe, i.e., the initial conditions at the big bang. The inflationary scenario could improve the understanding of the very early universe;[1,2] however, the question of the initial values is still open. To study this initial-value problem and also the initial-singularity problem, recently, the quantum origin or quantum creation of the inflationary universe has been discussed by several authors.[3–6]

The attempt to formulate a quantum model of cosmology is also interesting from the point of view of the quantum theory of gravity, since it sets up a different approach to quantum gravity and reveals a different aspect of this theory.[7–11]

One of the ways to study quantum cosmology, inaugurated by DeWitt, is to investigate the wave function which obeys the Wheeler-DeWitt equation.[7,8] In this formulation of quantum gravity, the initial-condition problem appears as the problem of the boundary conditions of the wave function. Recently, Hawking proposed a quite appealing specification of the boundary conditions of the universe.[12] The boundary conditions in his proposal are that the universe has no boundary. This idea has been formulated by Hartle and Hawking describing the wave function of the universe as a path integral over compact Euclidean metrics and matter fields, which gives a rather natural and self-contained model of quantum cosmology.[13,14]

Several different approaches to quantum cosmological models, relating to the picture of the creation of the universe from "nothing," have been proposed, based on different types of boundary conditions.[4,5] However, the self-contained picture of the universe obtained by the boundary condition of "no boundary" is very attractive. Therefore, we investigate the quantum state of the inflationary universe, applying these boundary conditions.

As for the inflationary model, a calculation of the wave function was presented by Moss and Wright.[15] Their model includes a scalar field which is conformally coupled to gravity. Generally, it is known that a scalar field with a conformal coupling has the special feature that it completely decouples from the gravitational field in the minisuperspace.[13] Although gravity and matter do not decouple in the model by Moss and Wright owing to a logarithmic term in the potential, the nonzero curvature gives an effective scalar mass term through the conformal coupling. This mass term becomes, as they pointed out, effectively too large and therefore the resulting model is not very satisfying.

The model with a minimally coupled massive scalar field, which is the simplest case without decoupling of matter from gravity in the minisuperspace, was investigated.[16,17] The result led to the picture of an eternally oscillating universe which could also describe inflation.[16,18] This type of model seems to give a rather promising approach on the way to construct a model of the very early universe. The next ingredient on this line would be the phenomenon of the spontaneous symmetry breaking, which is considered to play an important role in particle physics.

The purpose of this paper is to investigate the quantum model of the universe with a minimally coupled scalar field, requiring our potential to reflect the properties of the spontaneous symmetry breaking. We shall discuss here the simplest case of such a kind of model: The potential of the scalar field consists of a negative mass term and a Φ^4 coupling, i.e., a double-well potential. The general features of the quantum states specified by the boundary condition of "no boundary" and the cosmological scenario which emerges from the wave function of this theory are investigated.

In the following, we shall solve the Wheeler-DeWitt equation numerically on a minisuperspace,[8–11,13,14] the degrees of freedom of which are the scale of the universe and the time-dependent vacuum expectation value (VEV)

 ©1985 The American Physical Society

of the scalar field. To determine the boundary condition of no boundary in our model using the semiclassical approximation, we first examine the Euclidean paths. On this level we find that in a certain parameter range all Euclidean paths between the minima of the double-well potential start to oscillate around the maximum and cross each other. Applying the resulting boundary conditions, we plot the wave function of our model. We shall see that the structure of the wave function depends strongly on the parameters of the potential. In particular, we shall observe the formation of a nontrivial peak in the oscillating region of the wave function, accompanied by a recession of the region with the exponential behavior. The recession of the exponential behavior suggests bounce solutions with nonvanishing "velocity" of the VEV at the bounce point. An analysis of the Lorentzian equations of motion shows that this class of bounce solutions gathers with a high density at the place where the peak occurs. This leads us to the interpretation of the peak as to be formed by the constructive interference of the quantum states corresponding to these types of classical Lorentzian paths.

This paper is organized in the following way. In Sec. II we present our model. The solutions of the Euclidean equations of motion are examined and the boundary conditions are determined in Sec. III. In Sec. IV, integrating the Wheeler-DeWitt equation, we derive the wave function and discuss its behavior. In Sec. V the cosmological model which is suggested by the wave function is analyzed. Section VI is devoted to discussions and conclusions.

II. THE MODEL

Our model is the Einstein theory with a minimally coupled scalar field Φ in four-dimensional space-time. The action is given by

$$S(g_{\mu\nu},\Phi)=\int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R+(\text{surface term})-\tfrac{1}{2}[g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi+W(\Phi)]\right], \tag{1}$$

where G is the gravitational constant, R is the scalar curvature, $g_{\mu\nu}$ is the metric, and $g=\det g_{\mu\nu}$. Our signature is $(-,+,+,+)$. The surface term is added in order to cancel the second derivatives of the metric in the scalar curvature.[8,19] $W(\Phi)$ is the potential of the scalar field. In order to deal with a more realistic model we require that the cosmological constant is zero at the minimum. From this it follows that $W(\Phi)\geq 0$.

We are going to apply the canonical quantization, so we formulate our model in the Arnowitt-Deser-Misner (ADM) parametrization.[20] The Wheeler-DeWitt equation resulting from our action (1) is a functional differential equation on the superspace with an infinite number of degrees of freedom.[7,8] To decrease this number to a finite one, we restrict ourselves to a minisuperspace,[8–11,13,14] imposing the metric of $R\times S^3$,

$$ds^2=\frac{2G}{3\pi}[-N^2dt^2+a^2(t)\tilde{g}_{ij}dx^idx^j], \tag{2}$$

where N is the lapse function,[21] $\tilde{g}_{ij}$ is the metric of the unit three-sphere ($i,j=1,2,3$), and $a(t)$ is the scale of our universe in units of $\sqrt{2G/3\pi}$. We also restrict the scalar field into the time-dependent VEV $\phi(t)$,

$$\phi(t)=\sqrt{4\pi G/3}\langle\Phi(x^\mu)\rangle,$$

being consistent with the metric (2).

Under the ansatz given above our action becomes

$$S(a,\phi)=\tfrac{1}{2}\int dt\left[\frac{N}{a}\right]\left[-\left[\frac{a}{N}\dot{a}\right]^2+a^4\left[\frac{\dot{\phi}}{N}\right]^2-V(a,\phi)\right] \tag{3}$$

with

$$V(a,\phi)=-a^2+a^4W(\phi), \tag{4}$$

where an overdot denotes a time derivative. The scalar potential is given by

$$W(\phi)=\frac{\lambda}{M^4}(\phi^2-M^2)^2. \tag{5}$$

λ and M are constants defining the value of $W(\phi)$ at the maximum and the value of ϕ at the minimum, respectively. The equations of motion and the Hamiltonian H are

$$\ddot{a}=\frac{-1}{2a}(\dot{a}^2+1)-\tfrac{3}{2}a[\dot{\phi}^2-W(\phi)], \tag{6}$$

$$\ddot{\phi}=-3\frac{\dot{a}}{a}\dot{\phi}-\frac{1}{2}\frac{\partial}{\partial\phi}W(\phi), \tag{7}$$

$$H=\frac{1}{2}\left[\frac{1}{a}\right]\left[-\Pi_a{}^2+\frac{1}{a^2}\Pi_\phi{}^2+V(a,\phi)\right], \tag{8}$$

where $\Pi_a=-(a/N)\dot{a}$ and $\Pi_\phi=a^3\dot{\phi}/N$ are the conjugate momenta of a and ϕ, respectively.

Canonical quantization leads us immediately to the Wheeler-Dewitt equation for the wave function $\Psi(a,\phi)$ on the minisuperspace:

$$\left[a^{-p}\frac{\partial}{\partial a}\left[a^p\frac{\partial}{\partial a}\right]-\frac{1}{a^2}\frac{\partial^2}{\partial\phi^2}+V(a,\phi)\right]\Psi(a,\phi)=0. \tag{9}$$

The p indicates some ambiguity in the factor ordering but we leave aside the ordering problem and choose, for convenience $p=1$. To get a simpler wave equation which allows more insight into the "causal" structure, we introduce new variables:

$$x=a\sinh\phi,\quad y=a\cosh\phi. \tag{10}$$

In these variables the Wheeler-DeWitt equation (9) reads

$$\left[\frac{\partial}{\partial y^2}-\frac{\partial^2}{\partial x^2}+V(x,y)\right]\Psi(x,y)=0. \tag{11}$$

The potential $V(x,y)$ can be obtained immediately by substituting

$$a=(y^2-x^2)^{1/2}\quad\text{and}\quad\phi=\tfrac{1}{2}\ln\left[\frac{y+x}{y-x}\right]$$

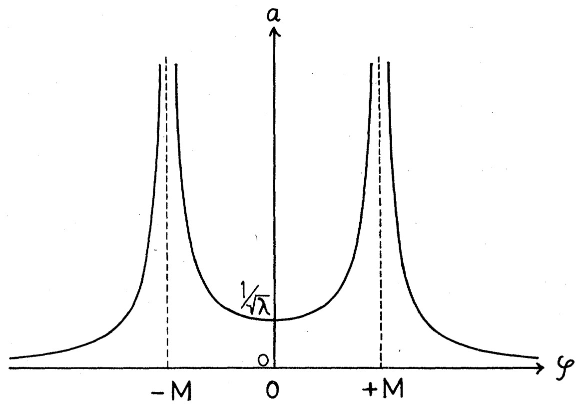

FIG. 1. The potential zero line $a=1/[W(\phi)]^{1/2}$ in (a,ϕ) variables.

into the previous definition of $V(a,\phi)$ in Eq. (4).

As is described by Hawking, we shall interpret the wave function as corresponding to Euclidean or Lorentzian geometry according to whether the wave function has exponential or oscillating behavior in a.[14,16] In the case where the momentum of ϕ is negligible, we can see approximately from the sign of the potential whether the wave function is oscillating or has exponential behavior. In the following we call, for convenience, the region where $V(a,\phi)<0$ the Euclidean and where $V(a,\phi)>0$ the Lorentzian area, respectively. The graph of $V(a,\phi)=0$ of our model is shown in the variables (a,ϕ) and (x,y) in Figs. 1 and 2. The light cone of the origin $y=|x|$ in Fig. 2 is the surface $a=0$, $\phi=\pm\infty$. The line which separates the oscillating region from the exponential region will be shifted from the line $V=0$, if the momentum of the scalar field is not negligible.

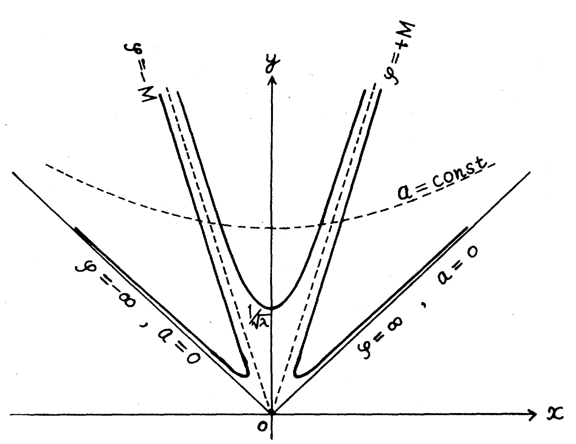

FIG. 2. The potential zero line in (x,y) variables. The lines $\phi=$const and $a=$const are shown by dashed lines. The physical region is mapped into the area above the light cone $y=|x|$, i.e., $a=0$ and $\phi=\pm\infty$. The area which includes the light cone is called the Euclidean area, $V(x,y)<0$.

III. EUCLIDEAN PATHS AND BOUNDARY CONDITIONS

To solve the Wheeler-DeWitt equation on the minisuperspace of our model, we must specify the boundary conditions on the light cone. The direct application of the boundary condition of no boundary on the light cone is difficult to perform. Instead, we estimate the form of the wave function which obeys these boundary conditions, near the light cone (on lines of constant large positive or negative ϕ) and use the result to find the proper boundary conditions.[16]

Hawking's prescription of the wave function and the boundary conditions is based on the Euclidean formulation of the path integral. Using the Euclidean action $I(g_{\mu\nu},\Phi)$ which is obtained from the Lorentzian action $S(g_{\mu\nu},\Phi)$ in Eq. (1) by a rotation of the time $t=-i\tau$, the wave function is given by

$$\Psi(h_{ij},\tilde{\Phi})=\int_C [dg_{\mu\nu}][d\Phi]\exp[-I(g_{\mu\nu},\Phi)] \ . \tag{12}$$

C is the domain over which the path integral has to be evaluated, h_{ij} is the metric of a three-manifold, and $\tilde{\Phi}$ is the matter field configuration regular on it. The boundary condition of no boundary is formulated by specifying the domain C of the path integral as the class of all regular, compact, and Euclidean four-manifolds which possess the three-manifold given by h_{ij} as a boundary and regular configurations of the matter fields on these four-manifolds. In the WKB approximation, the wave function is written as

$$\Psi(h_{ij},\tilde{\Phi})=\sum_j A_j\exp(-I_j) \ . \tag{13}$$

The exponents I_j are the actions of the solutions of the Euclidean equations of motion deduced from the Euclidean action $I(g_{\mu\nu},\Phi)$. The boundary condition of the wave function can be estimated in this approximation by fixing the initial conditions of these solutions, which can be derived from the regularity required by the condition of no boundary.

In the minisuperspace, we apply this semiclassical approximation to the wave function of the Euclidean area to define the boundary conditions and hence, we need the information about the solutions of the Euclidean equations of motion. The equations are derived from the Euclidean minisuperspace action $I(a,\phi)$, which is defined from the Lorentzian action $S(a,\phi)$ in Eq. (3) by rotating the time. From the requirement of regularity, the boundary conditions for the Euclidean paths which relate to our wave function are

$$a=0,\quad \frac{da}{d\tau}=1,\quad \phi=\phi_0,\quad \frac{d\phi}{d\tau}=0 \quad \text{at} \quad \tau=0 \ . \tag{14}$$

An analytical estimation of the solution of the equations of motion shows that

$$a=\tau \quad \text{and} \quad \phi=\phi_0+\tfrac{1}{2}k\tau^2 \quad \text{for} \quad \tau \ll \left[\frac{1}{W(\phi_0)}\right]^{1/2} \tag{15}$$

with

$$k=\frac{1}{8}\frac{dW}{d\phi}\bigg|_{\phi=\phi_0}$$

and $\tau \ll 1$ in the region $1/W(\phi) \gtrsim 1$.

Under the initial conditions described above, we calculated classical Euclidean paths numerically for different initial values of ϕ and several sets of the parameters M and λ. We have found that the behavior of the paths is rather insensitive to the value of λ but it differs qualitatively, depending on whether M is "large," i.e., larger than a critical value M_c ($\cong 0.7$) or "small," which means $M<M_c$. We shall see later that M_c does not depend on λ. Some typical Euclidean paths corresponding to "large" M and "small" M are shown in Figs. 3(a) and 3(b).

The qualitative difference appears for the Euclidean paths in the region $-M<\phi<+M$. For large M, we see from Fig. 3(a) that in this region the paths run parallel until near to the maximal value of a. Thus the semiclassical approximation dominated by a single path will work well and we can expect an exponential behavior of the wave function there.[14,16]

A rather new behavior arises for small M. The Euclidean paths with initial values of ϕ between $-M$ and $+M$ start to oscillate and gather around the maximum of the potential $W(\phi)$. For small $\phi_0 \ll M$, the analytic approximation shows that $a(\tau)$ behaves as in the Euclidean four-sphere case,

$$a(\tau)=\frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\tau)\,, \tag{16}$$

and ϕ oscillates with a period $\sqrt{2}M\pi/\sqrt{\lambda}$. We define the critical value of M from the condition that a half of the period of ϕ is equal to the time that the Euclidean universe needs to reach its maximum. We get $M_c=\sqrt{1/2}\cong 0.7$, which is independent of λ.

Since in this case the Euclidean paths cross each other frequently, we cannot say anything definite about the behavior of the wave function from the level of the classical paths in the region $-M<\phi_0<+M$. A remarkable point is that all paths which start off in this region run through the area which corresponds to the location of the local maximum of the potential $W(\phi)$. If we adopt the picture of the Euclidean paths as a tunneling process, our results suggest a tendency to "tunnel" via this maximum.[22]

To determine the boundary conditions along the light cone we estimate the behavior of the wave function for large ϕ, $\phi \gg M$, and $a^2<1/W(\phi)\ll 1$. In this case we have the same behavior as in the case of the simple massive scalar, i.e., near the light cone the Euclidean paths run parallel for large M as well as for small M. Therefore, we can apply the WKB approximation dominated by a single path to determine the boundary conditions for the wave function and the approximation of constant ϕ is justified, i.e., the momentum of ϕ is negligible.[14,16] Hence we neglect the kinetic term of the scalar field. The Wheeler-DeWitt equation in this approximation describes the pure de Sitter case with the cosmological constant, $\Lambda=W(\phi)$ (Refs. 13, 14, and 16);

$$\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left[a^p\frac{\partial}{\partial a}\right]-a^2+\Lambda a^4\right]\Psi(a)=0\,. \tag{17}$$

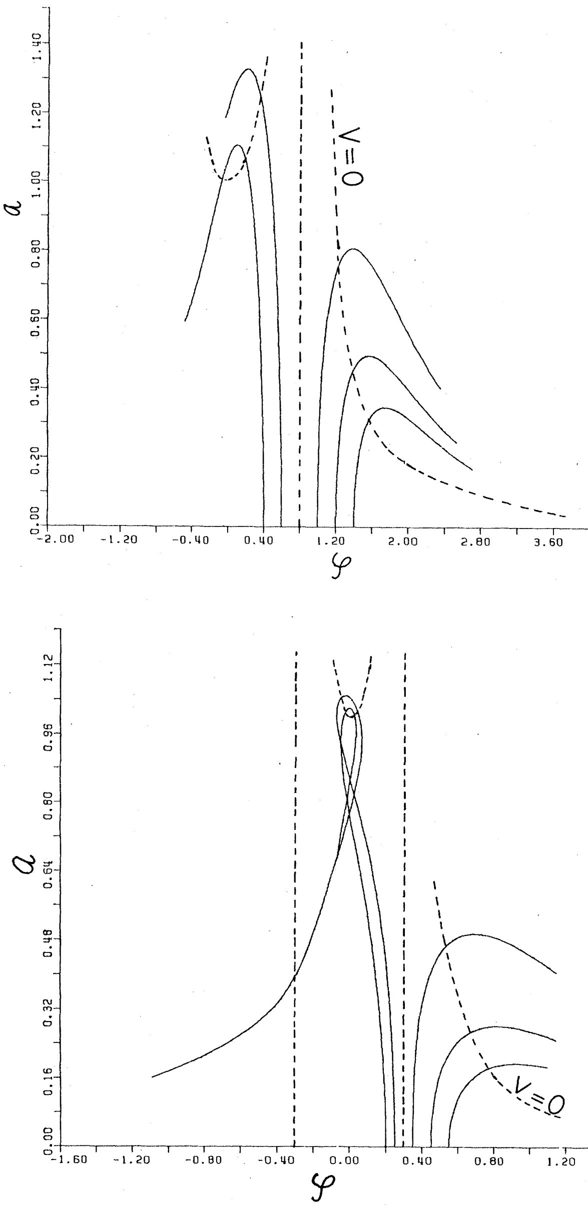

FIG. 3. (a) The Euclidean paths for $M=0.8$ and $\lambda=1$ in (a,ϕ) variables. The potential zero line is also drawn by dashed lines, for clarity. (b) The Euclidean paths for $M=0.3$ and $\lambda=1$.

We estimate the wave function in the region $a^2 \ll 1/W(\phi)$. Recalling the nature of our boundary condition we look for a regular solution at $a=0$ and use a power expansion in a. Generally, there are two solutions $\Psi\sim$const and $\Psi\sim\ln(a)$, for $a\cong 0$ satisfying Eq. (17) under this condition.[13] Thus the wave function is approximated by a constant near the light cone.

The WKB approximation to the corresponding wave function leads us to[14,16]

$$\Psi(a,\phi)=A\exp\left[\frac{1}{3W(\phi)}\{1-[1-a^2W(\phi)]^{3/2}\}\right]\,. \tag{18}$$

The prefactor A is giving the effect of the quantum fluctuations around the classical solution and to connect it with the above regular solution we have

$$A=C_0(1-\tfrac{1}{2}a^2) \tag{19}$$

in the region $a^2W(\phi)\ll 1$. C_0 gives us the normalization constant of the wave function and here we choose it to be unity. Hence the value of the wave function on the light cone is one.

In principle, all these considerations are sufficient to determine the boundary condition of the wave function, since the light cone lies entirely in the Euclidean area. However, for the numerical calculation we have to estimate the wave function in the Lorentzian area near the light cone as well, for technical reasons (see the Appendix). To estimate the wave function for large a, $a^2>1/W(\phi)$ we use the solution which is the analytic continuation of Eq. (18):

$$\Psi(a,\phi)=B\exp\left[\frac{1}{3W(\phi)}\right]\cos\left[\frac{[a^2W(\phi)-1]^{3/2}}{3W(\phi)}-\frac{\pi}{4}\right], \tag{20}$$

where B is a slowly varying amplitude. This means that we use the wave function of the de Sitter universe with the cosmological constant $W(\phi)$ on the whole line $\phi=$const and large, i.e., along the light cone.

In the following calculation we used the numerical solution of the wave function of the de Sitter universe given by Eq. (17) to formulate the boundary conditions near the light cone. The boundary conditions of the wave function for this de Sitter universe are $\Psi(a)=1$ and $(\partial/\partial a)\Psi(a)=0$ at $a=0$.

IV. THE WAVE FUNCTION

Using these boundary conditions we numerically integrate the Wheeler-DeWitt equation by a standard technique (see the Appendix). The result is shown in Figs. 4(a)–4(d) for various values of M. A remarkable feature is the qualitative change in the behavior of the wave function around $M\cong M_c$ when the parameter M is varied, as we expected from the previous observation of the Euclidean paths. On the contrary, the wave function is rather insensitive with respect to the variation of the value of λ in the region under consideration, so we choose $\lambda=1$ in our numerical examples.

We observe three typical properties of the wave function: One is the exponential behavior near the light cone. Another feature is a recession of the exponential behavior together with a formation of a peak near the potential maximum in the oscillating region. The third observation is that the phase of our wave function on the line $x=0$ is comparable to the de Sitter universe, in a certain parameter region.

To draw conclusions regarding the properties of our model universe, we have to specify how the wave function has to be understood. Our interpretation of the wave function is the following.[7,8,13,14,16]

The wave function in the oscillating region is a superposition of quantum states peaked around a certain class of classical paths. The class of these paths is selected by the boundary conditions. The envelope of the square of the wave function, $|\Psi(x,y)|^2$, in the superspace can be interpreted as representing the relative probability of finding a classical universe.[23] Thus, if there is a peak in the superspace, it means that the probability of finding a universe evolving through this peak is high. Note that in the superspace, a universe corresponds to a line. The definition of the classical universe can be given through the WKB approximation of the wave function.

From this point of view, we can understand the behavior of our wave function as follows.

A. Bounce solutions

We find generally that the wave function has an exponential behavior in the Euclidean area near the light cone in the case of large M as well as of small M, which is consistent with the semiclassical approximation in Sec. III. We interpret this circumstance as follows: The class of the classical Lorentzian paths mentioned above consists of solutions which do not reach the light cone but bounce before, in analogy with the de Sitter case.[13] We shall see in the next section that such a universe generally has a maximal value of the scale a and contracts after reaching its maximum. Therefore, in our model, we can understand the behavior of our wave function described above in a consistent way by using the picture of the oscillating universes.[14,16]

In the Lorentzian area the wave function is oscillating rapidly with a small amplitude. This gives us the possibility of applying the WKB approximation there as well. The corresponding class of classical solutions of oscillating universes which forms the phase in this approximation should therefore exist. This will be confirmed in a detailed analysis of the classical Lorentzian paths given in the succeeding section.

B. Recession of the exponential behavior and formation of the peak

Let us now consider the typical properties of the case where M is bigger than M_c [Fig. 4(a)]. The wave function expands exponentially in the Euclidean area near the line $\phi=\pm M$. This behavior can be interpreted as corresponding to a flat Euclidean space.[16]

A completely new property arises when we decrease the value of M below M_c, as we already expected from the previous analysis of the Euclidean paths. The exponential behavior recedes from the line $V=0$. Simultaneously a large peak appears above the local maximum of $W(\phi)$ on the line $x=0$, inside the Lorentzian area, and its height exceeds unity. This big peak is shifted towards the line $V=0$ accompanied by a receding exponential behavior. Besides this, the global structure of the wave function experiences a modification and a wavelet structure is forming in the Lorentzian area which contains the peak. Contrary to this, the structure of the wave function in the other oscillating regions as well as the behavior along the lines $\phi=\pm M$ are qualitatively the same as that of the massive scalar field.[16,17] The development of these struc-

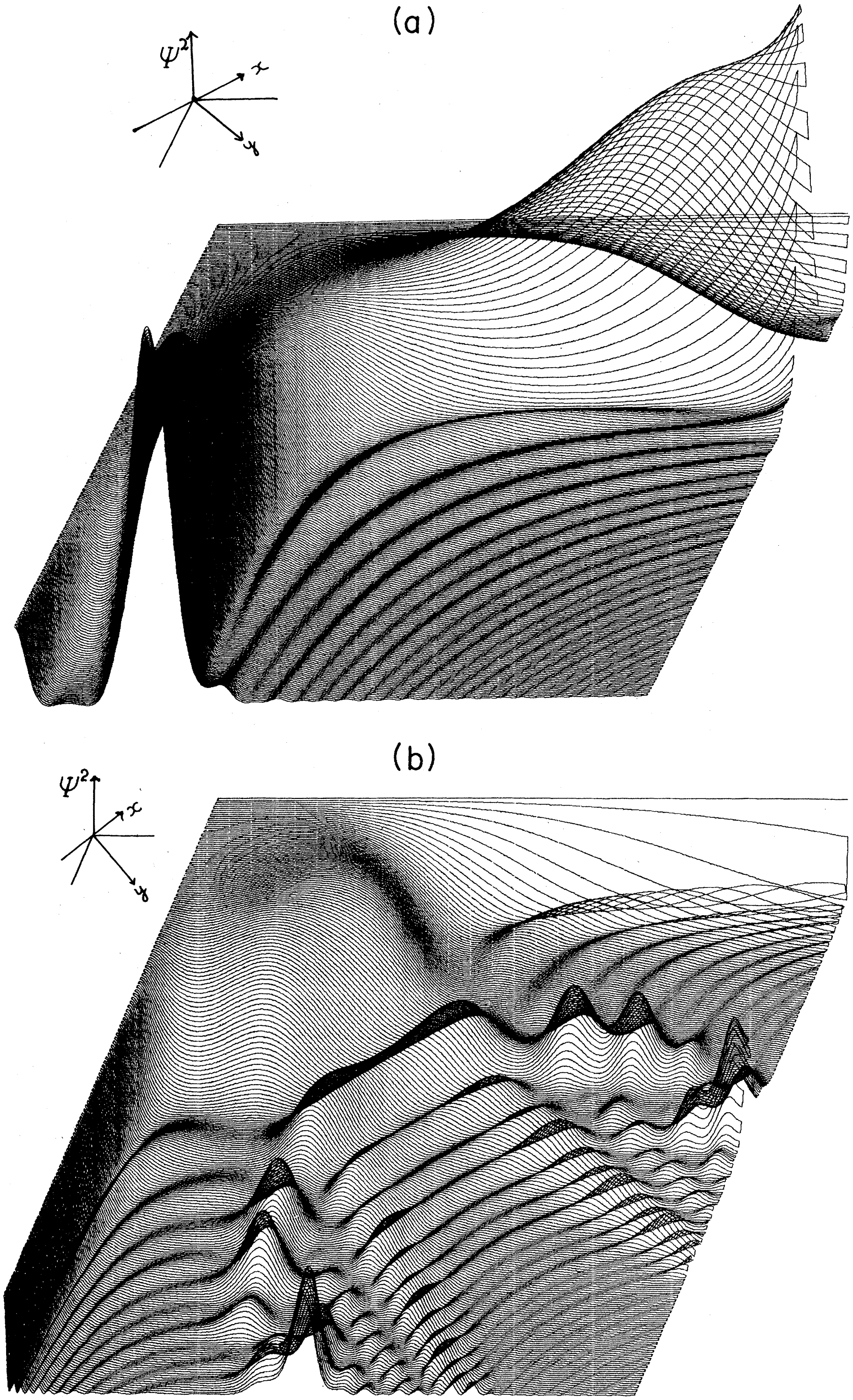

FIG. 4. (a) The square of the wave function, $|\Psi(x,y)|^2$, for $M=0.8$ and $\lambda=1$. In all the following diagrams of the wave function, the section $0 \le y+x \le 6$ and $0 \le y-x \le 6$ is presented. On the line $x=0$, the value of y is equal to a. In all diagrams we can see until $a=6$. (b) The square of the wave function for $M=0.4$ and $\lambda=1$. (c) The square of the wave function for $M=0.3$ and $\lambda=1$. (d) The square of the wave function for $M=0.25$ and $\lambda=1$.

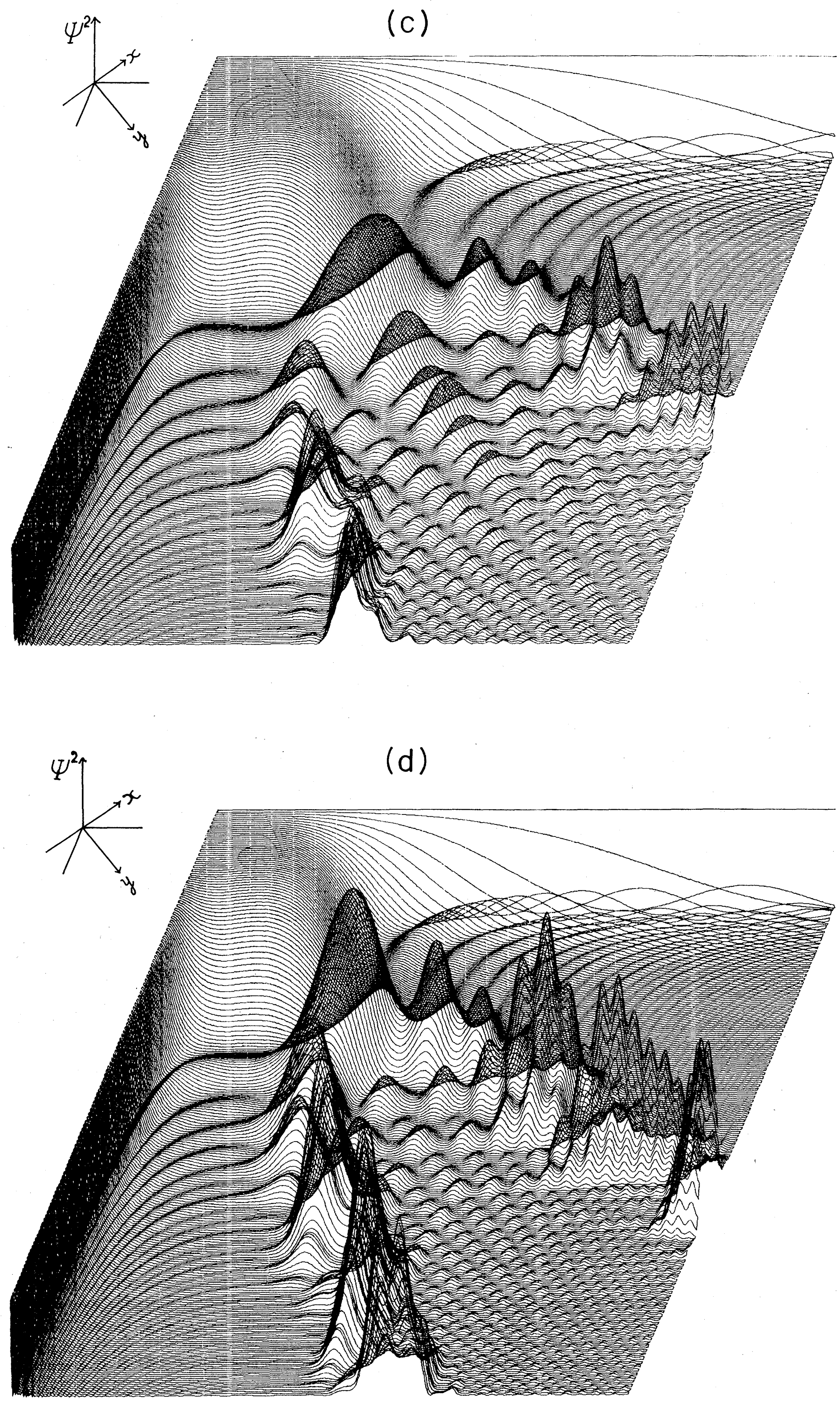

FIG. 4. (*Continued*).

tures is shown in Figs. 4(b), 4(c), and 4(d).

The large peak in the Lorentzian area is to be interpreted as a high probability for the existence of classical universes around there. It is remarkable that this large peak appears at a significant point: Below that local maximum of the potential $W(\phi)$ in the Euclidean area, all Euclidean paths out of the range $-M<\phi<+M$ gather.

The recession of the exponential behavior around this maximum can also be understood as a consequence of the frequent crossing of the Euclidean paths observed in Sec. III.

C. The phase

To see this development from a different perspective, we give the value of the wave function on the line $x=0$, i.e., $\phi=0$ (Fig. 5). Note that $y=a$ on this line. We also plot a wave function of the de Sitter universe with cosmological constant $\Lambda=1$ and $\Psi(a)\,|_{a=0}=1$, in order to compare the phases. [We choose $p=1$ for this de Sitter example, Eq. (17).] In the parameter range $M>M_c$, the phase of the wave function of our model and that of the de Sitter universe are nearly the same in the oscillating region. The same is true for the range of the exponential region. It is worth noting these properties because they tell us that in the case of large M, the approximation of constant ϕ works very well on the line $x=0$. Furthermore, the similarity with the de Sitter case supports our interpretation of the bouncing Lorentzian solutions.

On the other hand, when M becomes smaller, we can see clearly that the exponential behavior recedes from the line $V=0$ in the Euclidean area and the peak is formed near the maximum in the Lorentzian area. In contrast to this change, the phase of the wave function of our model remains nearly the same as that of the de Sitter universe until $M\sim0.3$. For smaller M, however, the phase as well starts to deviate remarkably.

Thus, the classical paths, which give the phase factor in the WKB approximation of the wave function around the line $x=0$, correspond to exponentially expanding universes evolving along this line in the parameter range $M\gtrsim0.3$.

We shall come back to these points in our discussion after analyzing the Lorentzian paths.

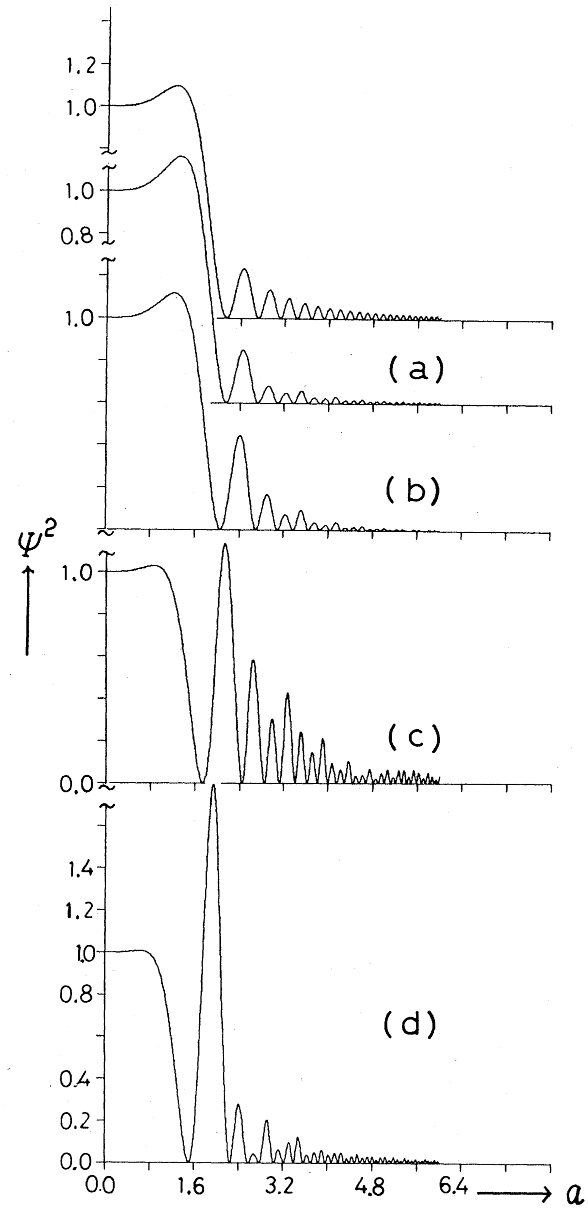

FIG. 5. The values of the square of the wave function on the line $x=0$. From top to bottom: The first graph is the square of the wave function of the de Sitter universe with cosmological constant $\Lambda=1$. The graphs following below are the values of the square of the wave function of our model corresponding to the cases of $M=0.8$ (a), $M=0.4$ (b), $M=0.3$ (c), and $M=0.25$ (d) on the line $x=0$. In all examples we took $\lambda=1$.

V. The Classical Universes

Keeping in mind the considerations made in the previous section, we investigated the classical solutions, integrating the Lorentzian equations of motion, (6) and (7). From the behavior of the wave function, we expect bounce solutions for the Lorentzian paths.

A. Large M

In the case of large M the wave function keeps its exponential behavior nearly until the line $V=0$. Thus the bounce of the classical Lorentzian solutions is located near or at $V=0$. From the constraint equation (8), it follows that the "velocity" of the VEV, $\dot{\phi}$, at the bounce point $\dot{a}=0$ is negligible.

The numerical analysis of the corresponding Lorentzian paths gives the following results: A universe which bounces near the local maximum of the potential $W(\phi)$ can become very large, which corresponds to the inflationary model starting with low temperature.[3,24] It can also become very large far outside beyond the minimum, which is to be interpreted as the chaotic inflationary scenario.[18] Universes which start their expansion phase with initial values around the local minimum of $W(\phi)$ cannot expand to a large size and soon collapse.

B. Small M

On the other hand, for small M the exponential behavior of the wave function near the line $V=0$ is shifted into the Euclidean area. This behavior indicates the existence of a certain class of Lorentzian paths of classical universes which cross over the line $V=0$ and bounce in the Euclidean area. To explain the bounce below $V=0$ we have to consider the possibility of a nonvanishing velocity of the VEV $\dot{\phi}$ at the bounce point, $\dot{a}=0$. In the following discussion we denote this nonvanishing velocity of the VEV by $\dot{\phi}_0$. From the constraint equation (8) we find at $\dot{a}=0$, the minimum radius of the corresponding universe as

$$a_{\text{minimum}}=1/[\dot{\phi}_0{}^2+W(\phi_0)]^{1/2}\ . \tag{21}$$

This equation tells us that the larger the velocity of the VEV, the smaller the minimal radius of the corresponding universe can become at the bounce point.

Furthermore, the recession of the exponential behavior can be interpreted as a deeper penetration of the Lorentzian paths into the Euclidean area, i.e., the stronger the recession, the higher the velocity $\dot{\phi}_0$ at the bounce point of the Lorentzian paths. However, the magnitude of $\dot{\phi}_0$ is limited as we see from the equation of motion (6): A sufficient expansion can only occur for $2\dot{\phi}_0{}^2<W(\phi_0)$. As for the paths starting near the minimum, we see that they soon contract independently, whether or not a nonvanishing $\dot{\phi}_0$ exists.

The presence of the nonvanishing $\dot{\phi}_0$ opens a new possibility for a type of classical path.

With nonvanishing velocity of the VEV, a universe starting its expansion from a bounce point apart from the local maximum of the potential $W(\phi)$ has the chance to reach the area around the local maximum and thus can become very large. From there, it can either cross the line $\phi=0$, provided that its $\dot{\phi}_0$ exceeded a critical value $\dot{\phi}_c$ at the bounce point, or turn at this line. In both cases the universes expand along the lines $\phi=\pm M$. This situation can be seen in the typical bounce solutions given in Fig. 6.

The classical paths contract again in any case discussed above. This fact, together with the exponential behavior of the wave function near the light cone, can be understood consistently by the interpretation that the class of Lorentzian paths, which will be determined from the wave function by WKB approximation, consists of oscillating universes. To support this picture, the existence of these kinds of paths has to be confirmed. Note that the solutions given in Fig. 6 are not eternally oscillating universes. However, we can get the paths of the eternally oscillating universes by "fine tuning" the initial values as we shall explain in the following.

For further discussion, we define the origin of the parameter time $t=0$ of the path to be at the bounce point of the minimum radius, for convenience.

Generally, a solution which has a bounce at $t=0$ runs from the singularity at $t=-\infty$ to the singularity at $t=+\infty$. Varying the values of $\dot{\phi}_0$ at $t=0$ thereby keeping ϕ_0 fixed, we can find that the path of the solution bounces at its maximum in $t>0$. Then the solution becomes symmetric under time reversion at the maximum

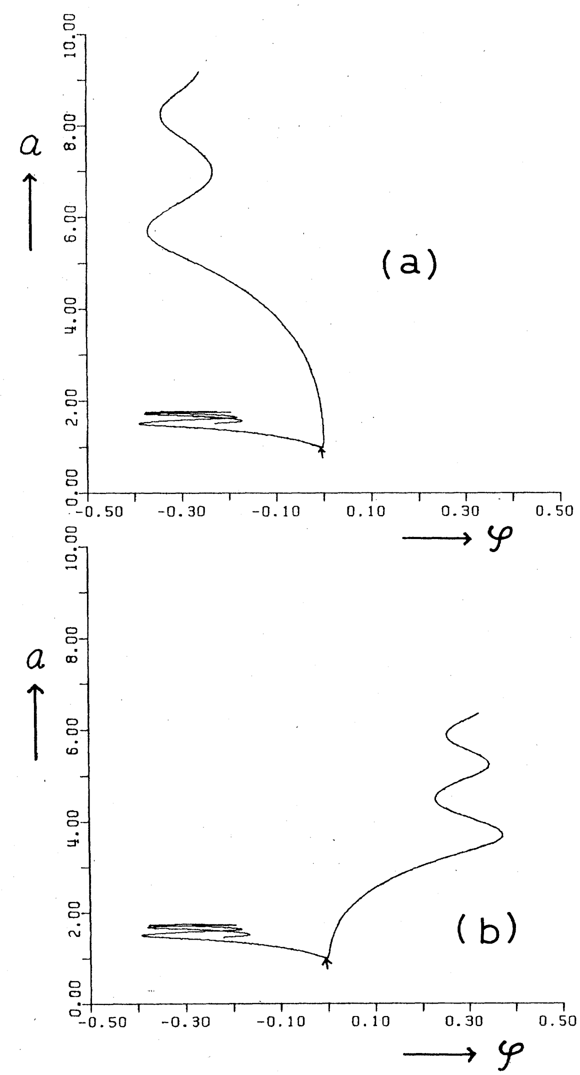

FIG. 6. (a) The bounce solution in (a,ϕ) variables for $M=0.3$ and $\lambda=1$ with the initial values, $\phi_0=-0.005$ and the "velocity" $\dot{\phi}_0=0.024$, i.e., below the critical value at the bounce point. The bounce point is marked by an arrow. (b) The bounce solution in (a,ϕ) variables for $M=0.3$ and $\lambda=1$ with the initial values, $\phi_0=-0.005$ and the "velocity" $\dot{\phi}_0=0.026$, i.e., above the critical value.

point. Therefore, the path comes back to the bounce point at $t=0$ with the velocity $\dot{\phi}=-\dot{\phi}\,|_{t=0}$. Let us call this time t_1. The solution, however, still goes to the singularity on the other end corresponding to $t=\pm\infty$. To make this side finite and bounce at its extremum, we vary the values of ϕ_0 and $\dot{\phi}_0$ at $t=0$, keeping the bounce at $0<t<t_1$. In this way we can construct the Lorentzian path of the eternally bouncing universe. Numerically it is rather hard to find the appropriate parameter, but as long as both paths have an expansion phase this solution will exist. This is certainly true for our examples in Fig. 6.

There are two degrees of freedom to specify completely a classical Lorentzian path, since a Hamiltonian constraint is imposed. In our specification of the initial conditions we have used both degrees of freedom in order to adjust the time reversibility of our solution with respect to its maxima. The consequence is that we pick up an infinite, countable number of Lorentzian paths. These Lorentzian paths can be characterized by a pair of integers which count the number of oscillations in each expansion period.[25] The possibility of a solution describing an eternally oscillating universe without being time reversible with respect to its extrema also remains.[26]

VI. DISCUSSION AND CONCLUSION

In this paper we have investigated a quantum cosmological model of an inflationary universe with a minimally coupled scalar field, the potential of which is a simple double well. We integrated the Wheeler-DeWitt equation with the boundary condition of "no boundary" and found that the resulting wave function forms a nontrivial peak near the potential maximum in a certain parameter range. At the same time, we observed a recession of the exponential behavior in the Euclidean area. The interpretation of this change in the behavior of the wave function led us to a generalized picture of the eternally oscillating universes: The classical universes possess a nonzero "velocity" of the VEV, $\dot{\phi}_0$, at the bounce point $\dot{a}=0$. As we stressed in Sec. V, the nonzero velocity of the VEV implies that the paths of the universes bounce below the line $V=0$ in the Euclidean area.

The new aspects summarized above enlarge the possibilities of the cosmological model:[27] Including the possibility of the nonvanishing velocity $\dot{\phi}_0$, not only the universes with a bounce very near to the maximum, but also those with a bounce apart from the maximum have the chance to experience a long period of inflation compared to the case of vanishing $\dot{\phi}_0$. In fact, provided the velocity $\dot{\phi}_0$ is sufficiently near to the critical velocity at the bounce point, the universes have a rather long period of inflation, while the VEV is climbing up the potential to the maximum.

The meaning of the large peak near the maximum and the change of the behavior of the wave function can be understood consistently by considering the relations between the behavior of the wave function and that of the classical paths as follows.

When we decrease the parameter M, the exponential behavior recedes more, which implies an increase of the velocity of the VEV $\dot{\phi}_0$ of the corresponding classical universe. With the nonvanishing velocity $\dot{\phi}_0$, the universes with bounce points in a wider range around the maximum can reach the area where the peak appears. The numerical analysis confirmed the existence of a large number of classical paths describing these types of oscillating universes. Following our interpretation, we can conclude that our peak is formed where the phases of several individual waves, which correspond to these types of classical paths by the WKB approximation, agree. This circumstance is the well-known constructive interference. Therefore, we can attribute the appearance of the peak to the presence of a bunch of classical Lorentzian paths gathering near the maximum of the potential $W(\phi)$. The formation of the peak is to be understood as a consequence of the constructive interference of the quantum states peaked around these classical paths.

With the above considerations, we can even go a little further: A higher velocity at the bounce point has the effect that the paths gather earlier and closer into the peak. Therefore, the peak is shifted more towards the line $V=0$ and becomes higher with decreasing M. We can see this changing of the behavior of the wave function clearly in Fig. 5. In contrast to this, in the case of large M, the velocity of the VEV $\dot{\phi}_0$ is negligible, and the classical solutions cannot gather enough to form a peak.

Another remarkable observation we have made is the similarity of the phase of the wave function with that of the de Sitter universe in a certain parameter range, along the line $x=0$. From this, we can expect that the dominant contribution to the WKB approximation of the wave function along the line $x=0$ is given by de Sitter—type universes which are running along this line (see Fig. 7).

Furthermore, in the parameter range where the peak exists, the density of such classical de Sitter—type paths will be high. It would follow that we should be able to see an array of peaks along the line $x=0$. However, we cannot see a clear structure over the whole axis which we could call an array of peaks, although there exists a small number of peaks along this line, for a special parameter, which are definitely bigger than the wavelet structure. In fact, if we see the wave function of smaller M, the phase itself starts to deviate from that of the de Sitter universe. This seems to contradict our picture; nevertheless this behavior can be understood consistently as follows.

The peak near the maximum of the potential $W(\phi)$ is formed by the contributions of classical paths which spread to both sides $\phi=+M$ and $\phi=-M$, i.e., the classical paths run apart from each other with growing distance from the peak. The density of the classical paths near the line $x=0$ is therefore decreasing. Furthermore, we have

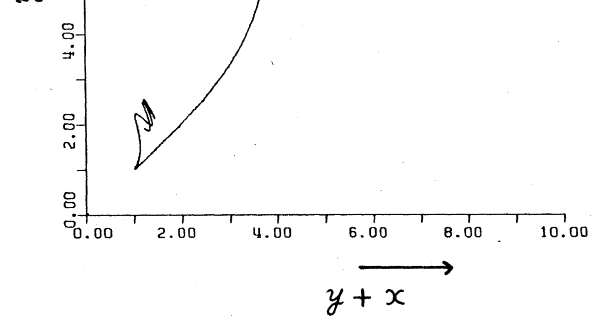

FIG. 7. The bounce solution of Fig. 6(a) in (x,y) variables.

to recall that universes which bounce with a velocity $\dot{\phi}$ larger than the critical $\dot{\phi}_c$ are also present. These classical Lorentzian paths will cross each other and their contributions to the wave function interfere. The more we decrease M, the more interference we get and therefore the expected array of peaks is smeared out and the phase starts to deviate from that of the wave function of the de Sitter universe.

On the other hand, the considerations above suggest the possibility that, in a certain parameter range, many universes start off around the peak and run parallel for a finite distance along the $x=0$. In this parameter range, therefore, we can expect an array of some large peaks along the line $x=0$ which is formed by the universes running parallel along there. Our observations support the discussion given above: If we look carefully at the wave function with the parameter $M=0.3$ in Fig. 4(c), and the corresponding wave function on the line $x=0$ in Fig. 5, we can recognize that the first several peaks along the line $x=0$ are rather high compared to the wave function in other parameter ranges[28] (and also to the case of the de Sitter universe).

We have seen that for a certain parameter the constructive interference, which is responsible for the formation of the peak, is kept along the line $x=0$, visible as an array of peaks. As we already discussed, since the classical paths start to run apart from each other or to cross each other, the array disappears after a finite distance. Nevertheless, this array of peaks can be called a "wave packet."[7] The wave function of our model is a superposition of quantum states of expanding and contracting universes and it is a real function in (x,y) (Ref. 13). Therefore, the array of the peaks which we observe is to be understood as the superposition of the wave packets corresponding to exponentially expanding and contracting universes. It is quite interesting that all these structures could emerge from the wave function obeying the boundary condition of no boundary.

The wavelet structure appearing in that Lorentzian area which includes the peak, for the case of smaller M, is also to be understood as the interference of the contributions of many classical universes.

The complicated oscillatory structure along the lines $\phi=\pm M$ can also be interpreted by the fact that all Lorentzian paths finally expand along there (see Fig. 7). The envelope of the wave function along these lines is an increasing function in a. This is in a good agreement with the fact that the density of the classical paths becomes higher.

In our model, we restricted the number of degrees of freedom of the universe into two, but if we consider a larger or infinite number of degrees of freedom, the interference structure of the wave function may become stronger and the peak may become higher and sharp. In this case, the tendency that the bouncing universes gather near the maximum would increase.

Our model contains two parameters, M and λ. However, since we saw that the Euclidean paths are rather independent of λ we fixed λ ($\lambda=1$) and mainly discussed the effects occurring by varying M alone. The wave function as a whole, of course, changes its behavior when varying λ over a large range. This will be discussed elsewhere. Here, we investigated the change of the properties of the wave function in a range where the behavior of our wave function is relatively "stable" under variation of λ.

Furthermore, we have to keep in mind that in the case where M becomes very small ($M\cong 0.3$) the coupling λ/M^4 is of the order of $\cong 100$, i.e., in the strong coupling region. Therefore, it is not possible to undertake a direct comparison with the perturbative region of the theory.

ACKNOWLEDGMENTS

The authors would like to thank Professor H. Sato and Professor T. Inami for valuable discussions and careful reading of the manuscript. They also thank Professor T. Yoneya and Dr. S. Wada for useful discussions. One of the authors (U.C.) is indebted to Professor H. Sato for his hospitality at the Research Institute for Fundamental Physics (RIFP). The other author (S.W.) would like to acknowledge Professor Z. Maki for his hospitality at RIFP. He would also like to thank the Yukawa Foundation for a grant.

APPENDIX

The numerical calculation of the wave function is performed on a lattice (see Fig. 8) by using a finite-difference scheme.[15] The declination of the wave function in the y direction and x direction can be written in finite-difference form. The distance between the lattice points is denoted by h. The second derivative in the y direction, for example, is expressed by

$$\frac{\partial^2}{\partial y^2}\Psi=\frac{1}{h}\left[\frac{\Psi_N-\Psi_C}{h}-\frac{\Psi_C-\Psi_S}{h}\right]. \tag{A1}$$

We put these finite-difference expressions into the Wheeler-DeWitt equation (11) and get

$$\Psi_N=\Psi_E+\Psi_W-\Psi_S-h^2V(x,y)\tilde{\Psi}_C\ , \tag{A2}$$

where $\tilde{\Psi}_C$ is the effective value of the wave function at the center and given by

$$\tilde{\Psi}_C=\frac{\beta}{2}(\Psi_E+\Psi_W)+(1-\beta)\Psi_C\ , \tag{A3}$$

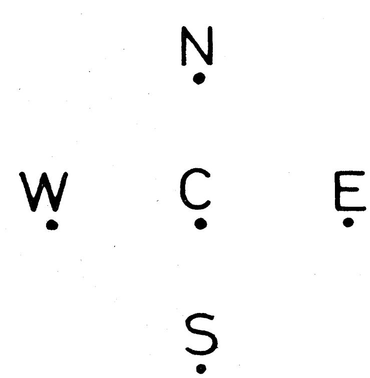

FIG. 8. The convention of the lattice points for the numerical calculation.

where β is taken as 0.5 for the purpose of stability in the numerical calculation. Using these equations, we can express the value of the wave function at a point N on the lattice by its "previous" values at the points E, W, S, and C. Once the values on two arrays along the light cone $y = |x|$ are known, we can calculate the whole wave function on the lattice.

Therefore, we need values on the light cone and values on the lattice points next to the light cone to define the declination of the wave function, numerically. Since the step size is finite, we have to be careful about the fact that the Euclidean area along the light cone is shrinking considerably with the distance from the origin. Actually, we soon reach an area where we "step" into the Lorentzian area already with the first difference step h away from the light cone. There, we have to take into account that a part of the initial points is determined from Euclidean and the others are from the Lorentzian wave function.

For this, using the results in Sec. III, we take the values of the wave function of a de Sitter universe in order to determine the value of the points next to the light cone. As we can see from the WKB approximation, however, near the light cone the wave function is very smooth and the difference between these values and the values on the light cone ($\Psi = 1$) is very small. Furthermore, we observed that the value of the wave function is enough near unity along the $V = 0$ line for large ϕ in the Lorentzian region. This justifies the numerical calculation.

---

[1]A. H. Guth, Phys. Rev. D **23**, 347 (1981); K. Sato, Mon. Not. R. Astron. Soc. **195**, 467 (1981).

[2]A. D. Linde, Rep. Prog. Phys. **47**, 925 (1984), and references therein.

[3]A. Vilenkin, Phys. Lett. **117B**, 25 (1982); Phys. Rev. D **27**, 2848 (1983).

[4]A. Vilenkin, Phys. Rev. D **30**, 509 (1984).

[5]A. D. Linde, Lett. Nuovo Cimento **39**, 401 (1984).

[6]Ideas of the quantum creation of the universe are discussed by E. P. Tryon [Nature **246**, 396 (1973)]; R. Brout, F. Englert, and E. Gunzig [Ann. Phys. (NY) **115**, 78 (1978)]; Atkatz and H. Pagels [Phys. Rev. D **25**, 2065 (1982)]; and J. R. Gott [Nature **295**, 304 (1982)].

[7]J. A. Wheeler, in *Battelle Rencontrés*, edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[8]B. S. DeWitt, Phys. Rev. **160**, 1113 (1967).

[9]C. W. Misner, Phys. Rev. **186**, 1319 (1969); in *Magic Without Magic: John Archibald Wheeler, a Collection of Essays in Honor of his 60th Birthday*, edited by J. R. Klauder (Freeman, San Francisco, 1972).

[10]For a review see, e.g., M. Ryan, *Hamiltonian Cosmology* (Springer, New York, 1972); see also M. A. H. MacCallum, in *Quantum Gravity, An Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Oxford University Press, New York 1975), and references therein.

[11]D. J. Kaup and A. P. Vitello, Phys. Rev. D **9**, 1648 (1974); W. F. Blyth and C. J. Isham, *ibid.* **11**, 768 (1975); C. J. Isham and J. E. Nelson, *ibid.* **15**, 3226 (1974).

[12]S. W. Hawking, in *Astrophysical Cosmology*, proceedings of the Study Week on Cosmology and Fundamental Sciences, 1982, edited by H. A. Brück, G. V. Coyne, and M. S. Longair (Pontificia Academia Scientarium, Vatican City, 1982).

[13]J. B. Hartle and S. W. Hawking, Phys. Rev. D **28**, 2960 (1983).

[14]S. W. Hawking, in *Relativity, Groups, and Topology II, Les Houches Lectures, 1983*, edited by B. S. DeWitt and R. Stora (North-Holland, Amsterdam, 1984), p. 333.

[15]I. G. Moss and W. A. Wright, Phys. Rev. D **29**, 1067 (1984).

[16]S. W. Hawking, Nucl. Phys. **B239**, 257 (1984).

[17]S. W. Hawking and Z. C. Wu, Phys. Lett. **151B**, 15 (1985).

[18]A. D. Linde, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 149 (1983); Phys. Lett. **129B**, 177 (1983).

[19]G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2752 (1977).

[20]See, e.g., R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation*, edited by L. Witten (Wiley, New York, 1962), p. 227.

[21]In the numerical calculation, we choose the lapse function as $N = 1$.

[22]S. Coleman, Phys. Rev. D **15**, 2929 (1977); C. Callan and S. Coleman, *ibid.* **16**, 1762 (1977); S. Coleman, in *The Whys of Subnuclear Physics*, proceedings of the International School of Subnuclear Physics, Erice, 1977, edited by A. Zichichi (Plenum, New York, 1979).

[23]In the case of the de Sitter universe, the wave function shows rapid oscillations with a slowly varying amplitude. This wave function can be considered as a standing wave consisting of a superposition of quantum states describing expanding and contracting universes. It is shown that the envelope of the probability distribution $a^p |\Psi(a)|^2$, i.e., the square of the slowly varying amplitude, is proportional to the distribution of the three-spheres in a in the de Sitter universe (Ref. 13). Thus in this context the envelope of the square of the wave function is to be understood as a square of the slowly varying amplitude. In other words, if we think of the semiclassical approximation, the values of this envelope are given by the square of the prefactor. The factor a^p in the probability distribution is demanded by choosing the measure of the minisuperspace in such a way that the differential operator of the Wheeler-DeWitt equation becomes Hermitian. In our case the probability distribution is simply given by $|\Psi(x,y)|^2$ in (x,y) variables.

[24]S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).

[25]The possibility of this type of solution is suggested by Hawking in a different context (Ref. 14) and discussed by Page (Ref. 26). However, in our case this type of solution is inevitable in order to understand the behavior of the wave function.

[26]D. N. Page, Class. Quantum Grav. **1**, 417 (1984).

[27]At the bounce point of a classical path the corresponding classical universe reaches its minimal size and from there goes into a new expansion phase. From this point of view we can construct a cosmological model in which the bounce point represents the beginning of the evolution of a universe.

[28]For a parameter value $M = 0.32$, all peaks within the whole size of the diagram ($a \leq 6$) on the line $x = 0$ are definitely higher than the neighboring wavelet structure. The phase for this parameter is comparable to that of the de Sitter universe.