Numerical analysis of inflation

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In this paper the mechanism of a cosmological phase transition is addressed in a new way which avoids weaknesses of previous approaches. The effects of inhomogeneities are included explicitly. A numerical analysis is presented in which for a wide class of models the Universe enters a period of new inflation. The analysis is classical and applies to models in which the scalar field responsible for driving inflation is weakly coupled to other fields. We derive heuristic arguments which determine the boundaries of the region in parameter space for which inflation is realized. The agreement with the numerical results is good. This paper complements a previous analytical analysis.

I. INTRODUCTION

The analysis of cosmological models with a period of exponential expansion of the Universe[1] is generally based on the investigation of the time dependence of the finite-temperature effective potential. Even in models in which $\phi=0$ is a local maximum of the zero-temperature effective potential (Fig. 1), the finite-temperature mass term turns $\phi=0$ into a global minimum at sufficiently high temperatures.[2]

According to the standard view[3] of new inflation, the finite-temperature mass term forces $\phi(\mathbf{x})\simeq 0$ uniformly in space. As the Universe cools down, the temperature-dependent mass term decreases and the scalar field starts to evolve from its initial homogeneous configuration on top of the potential barrier towards the minimum of the potential. The dynamical evolution is described by the classical scalar-field equations of motion. If the curvature of the potential near the origin is small, the scalar field will initially move very slowly. In this "slow rolling" period the stress-energy tensor is dominated by the almost-constant potential-energy term which acts as a cosmological constant and generates exponential expansion of the Universe.

Mazenko, Unruh, and Wald[4] have recently raised serious objections against the standard picture of inflation. They point out that at high temperatures in the early Universe one should expect large fluctuations in the value of the scalar field as a function of space. Hence it is unjustified to use effective-potential methods. Mazenko, Unruh, and Wald argue that as a consequence of large thermal fluctuations spatial domains of $\phi(\mathbf{x})=\pm\sigma$ will form already at the critical temperature T_C. In this case the Universe would never enter an inflationary period.

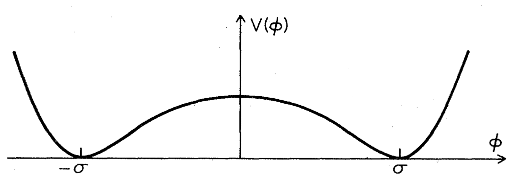

FIG. 1. Form of $V(\phi)$ in models which give inflation.

In a previous publication[5] we presented an analytical analysis of the evolution of a scalar-field configuration $\phi(\mathbf{x},t)$ initialized to be in thermal equilibrium at a given temperature $T_E\geq T_C$. The dynamics is governed by two forces, the expansion of the Universe and the forces due to the nonlinear field-theory potential. The Hubble expansion dampens the amplitude of $\phi(\mathbf{x},t)$ and causes a redshift of wavelengths. Both effects lead to a homogeneous scalar-field configuration localized at a value of ϕ equal to the initial spatial average of $\phi(\mathbf{x})$ and thus, if the latter is small compared to σ, to an inflationary period. The nonlinear forces due to the potential on the other hand favor domain formation. In Ref. 5 we concluded that if the interaction rate Γ of the nonlinear forces is much smaller than the Hubble rate $H(t)$, the Hubble-expansion effects will dominate long enough to ensure a period of inflation of sufficient length to solve the cosmological problems which inflation is meant to solve.

Our previous analysis was nonrigorous. It was based on a perturbative Green's-function method. One aim of this work is to confirm the validity of the analytical method numerically. There is, however, a much more important motivation. In our analytical work we overestimated the domain-forming forces since we only derived bounds on the maximal value of the nonlinear terms in the equations of motion. Hence we were only able to prove the existence of an inflationary period for models with very small coupling constants. In this paper a classical analysis is shown to yield inflation in a much wider class of models. We derive heuristic arguments for the critical values of the coupling constants (values which separate the regions which yield inflation from those which do not).

The conditions on the coupling constants we require in order to demonstrate that the Universe enters an inflationary period are much less restrictive than those which must be imposed in order to get a reasonable answer for the amplitude of energy-density fluctuations.[6]

In the following section we summarize some important previous results. In Sec. III we review our analytical approach and point out in which ways the present analysis is complementary to it. We then describe our numerical method, the particle-physics models we consider, and the physical questions we ask. A special section is devoted to a discussion of initial conditions. In Sec. VI we derive the heuristic conditions on the free parameters of the quantum-field potential under which we expect inflation to be realized. Our numerical results are summarized in Sec. VII and agree quite well with the theoretical predictions. We conclude by discussing some interesting phenomena we observe in some of the numerical runs.

In this paper we use natural units $k_B=\hbar=c=m_{\rm Pl}=1$. The background metric will be that of a spatially flat Friedmann-Robertson-Walker (FRW) universe.

II. PRELIMINARIES

In this section we wish to motivate our investigation in more detail. Experts in the field may want to skip to the next section.

Inflationary-universe models,[1] models in which there is a period of exponential increase in the physical distance between two comoving points, have recently received a lot of attention in the physics literature.[2] They provide a solution of important cosmological problems[1] in the standard big-bang model and allow for a mechanism which for the first time in a causal way generates the primordial energy-density fluctuations required to explain galaxy formation.[6,7]

In inflationary-universe models matter is described in terms of quantum fields, one of which must be a scalar field $\phi(\mathbf{x},t)$ with a nonvanishing vacuum expectation value at zero temperature. The zero-temperature effective potential has double-well shape (see Fig. 1). A simple example is the $\lambda\phi^4$ theory with potential

$$V(\phi)=\lambda(\phi^2-\sigma^2)^2 \ . \tag{1}$$

At high temperatures T the effective potential is approximated by adding a T-dependent mass term to Eq. (1):[8]

$$V_T(\phi)\simeq\lambda(\phi^2-\sigma^2)^2+C\phi^2T^2 \tag{2}$$

with C a constant of the order λ. Thus for temperatures T greater than the critical temperature $T_C=(\lambda/C)^{1/2}\sigma$ the symmetric state $\phi=0$ minimizes the finite-temperature effective potential, whereas for $T<T_C$, $\phi=0$ becomes unstable.

According to the standard picture[3] of new inflation, the initial scalar-field configuration minimizes the finite-temperature effective potential and hence is homogeneous and localized at $\phi=0$. As the Universe cools below T_C the configuration becomes unstable and starts moving towards one of the minima of $V(\phi)$ as described by the classical scalar-field equations of motion. If the curvature of $V(\phi)$ at $\phi=0$ is small, then $\phi(\mathbf{x},t)$ will remain close to $\phi=0$ (close on the scale of σ) for a long period during which the equation of state is dominated by the potential energy term which acts as an effective cosmological constant and leads to an exponential expansion of the Universe

$$T_{\mu\nu}\simeq V(0)g_{\mu\nu} \ . \tag{3}$$

The effective potential $V_T(\phi)$ is the free-energy density of a homogeneous scalar-field configuration $\phi(\mathbf{x})=\phi$. At zero temperature, we may alternatively define $V(\phi)$ to be the minimum of the expectation value of the Hamiltonian among all homogeneous states,[9]

$$\begin{aligned} &V(\phi)=\min\langle s\,|\,H\,|\,s\rangle \ , \\ &\langle s\,|\,s\rangle=1 \ , \\ &\langle s\,|\,\phi(\mathbf{x})\,|\,s\rangle=\phi \ . \end{aligned} \tag{4}$$

Mazenko, Wald, and Unruh pointed out[4] that it is unreasonable to restrict attention to homogeneous scalar-field configurations. In the early Universe there will be temperature fluctuations. The static minimum-energy configuration will be inhomogeneous. It will consist of spatial domains in which $\phi(\mathbf{x})=\pm\sigma$. Hence it is unjustified to use effective-potential arguments in the early Universe.

We agree with the objections against using the effective potential to argue for inflation. We also agree that given an initial thermal state at a fixed temperature $T_E\geq T_C$ the final field configuration will consist of spatial domains. However, the domains do not form instantaneously but slowly as a result of the nonlinear force term $V'(\phi)$ in the scalar-field equation of motion. For times shorter than the typical time scale Γ^{-1} of domain formation the Hubble expansion of the Universe is the dominant force effecting the scalar-field configuration. The Hubble damping of the amplitude and the red-shift of wavelengths tend to lead to a homogeneous field configuration localized at a value of ϕ equal to the initial spatial average of $\phi(\mathbf{x})$. The balance of the two effects must be determined by a detailed dynamical analysis.

III. ANALYTICAL RESULTS

In a previous paper[5] we analyzed the evolution of an initial scalar-field configuration analytically and concluded that provided the nonlinear forces $V'(\phi)$ are sufficiently small (compared to H^3) there will be a period of new inflation sufficiently long to solve the cosmological problems. In the case of a Coleman-Weinberg model[10] we showed that new inflation is realized provided

$$g<\frac{\sigma}{m_{\rm Pl}} \ , \tag{5}$$

where g is the gauge coupling constant. For the $\lambda\phi^4$ model of Eq. (1) the condition is

$$\sigma>m_{\rm Pl} \ . \tag{6}$$

Our results were based on determining the dynamical evolution of an initial classical scalar-field configuration in which each Fourier mode is thermally excited at the initial temperature $T_E \geq T_C$. We are thus analyzing typical configurations which contribute to the quantum-field functional integral. We show that each such configuration leads to an equation of state which gives inflation. Since we only consider classical-field configurations which satisfy the field equations we cannot definitely conclude that the result will persist when taking the full quantum average. We hope that the physical understanding gained in this analysis will apply to the complete quantum problem. It must be stressed that the usual approach of coupling the quantum average to gravity cannot be used since it totally neglects the effects of spatial inhomogeneities.

The equation of motion for the scalar field in an expanding FRW universe is

$$\ddot{\phi}+3\frac{\dot{a}(t)}{a(t)}\dot{\phi}(t)-a^{-2}(t)\nabla^2\phi=-V'(\phi) . \tag{7}$$

If we replace the potential by $\frac{1}{12}R^2\phi^2$ (R is the Ricci scalar) we obtain the equation of motion of a free conformally coupled scalar field. The general solution $\phi(\mathbf{x},t)$ is

$$\phi(\mathbf{x},t)=a^{-1}(t)\hat{\phi}(\mathbf{x},\tau) , \tag{8}$$

where $\hat{\phi}(\mathbf{x},\tau)$ is a general solution of the wave equation in flat spacetime and τ is conformal time given by $dt^2 \equiv a^2(t)d\tau^2$. A typical solution in flat spacetime is a standing wave. We thus conclude that the expansion of the Universe produces two important effects. It leads to a Hubble damping of the amplitude of ϕ and it red-shifts wavelengths. Since energy density and pressure of the scalar field are given by

$$\rho=\tfrac{1}{2}\dot{\phi}^2+\tfrac{1}{2}(\nabla\phi)^2a^{-2}(t)+V(\phi) ,$$
$$p=\tfrac{1}{2}\dot{\phi}^2-\tfrac{1}{6}(\nabla\phi)^2a^{-2}(t)-V(\phi) , \tag{9}$$

both effects drive the configuration towards one which is localized at the initial spatial average of $\phi(\mathbf{x})$ and homogeneous. The equation of state becomes $p=-\rho$ and gives rise to inflation.

The force which opposes the free-field contraction and favors domain formation is due to the nonlinear potential $V(\phi)$. The main idea in Ref. 5 is to estimate the maximal effect of this force and hence the minimal time it will take until domains form. As indicated above, we choose the initial scalar-field configuration to be such that each Fourier mode has thermal energy. In a finite volume V we expand the conformal field $\hat{\phi}(\mathbf{x},\tau)$

$$\hat{\phi}(\mathbf{x},\tau)=V^{-1/2}\sum_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}}q_{\mathbf{k}}(\tau) . \tag{10}$$

We use a perturbative Green's-function method to estimate the maximal effect of $V(\phi)$ for each mode $q_{\mathbf{k}}$. We write

$$q_{\mathbf{k}}(\tau)=q_{\mathbf{k}}^{(0)}(\tau)+q_{\mathbf{k}}^{(I)}(\tau) . \tag{11}$$

$q_{\mathbf{k}}^{(0)}(\tau)$ is the result for free evolution, $q_{\mathbf{k}}^{(I)}(\tau)$ is the effect of the potential to lowest order in λ. In order to estimate the source term in the equation of motion for $q_{\mathbf{k}}(\tau)$ it was crucial to assume random initial phases for the $q_{\mathbf{k}}(\tau)$ modes.

Modes with $|\mathbf{k}|=T_E$ have maximal phase space. Those with higher $|\mathbf{k}|$ are suppressed by the thermality condition. We thus claim that the corrections in the scalar-field evolution (compared to the free-field theory dynamics) will be small as long as for the modes with maximal phase space

$$|q_{\mathbf{k}}^{(I)}(\tau)|<q_{\mathbf{k}}^{(0)}(\tau) \tag{12}$$

(on the right-hand side we mean the amplitude of the zeroth-order oscillation). Condition (12) will be satisfied up to a certain time τ_p. Provided $\tau_p>H^{-1}$ there will be a period of new inflation before domains form. For the models we considered this condition yields Eqs. (5) and (6).

The above analysis is nonrigorous. It is based on a perturbative scheme. A further weak point is assuming initial random phases for the Fourier modes. Finally the condition (12) for the existence of an inflationary period is somewhat *ad hoc*. The numerical analysis we present in this paper eliminates the first and last of these weak points. The evolution of a scalar-field configuration is indeed a small perturbation from the free-field evolution. The deviation grows in time and inflation ends soon after it has become of the same order of magnitude as the unperturbed value.

Since our analytical method was based on establishing upper bounds on the effects of nonlinearities, the bounds in Eqs. (5) and (6) are conservative. The numerical analysis presented below allows us to significantly extend the class of models for which we can show that new inflation is realized.

IV. NUMERICAL APPROACH

We numerically integrate the Klein-Gordon equation (7) for a scalar field $\phi(\mathbf{x},t)$ in a given background FRW metric. The method works for any scalar-field potential $V(\phi)$, but in the evaluation we consider two specific cases. The first is a "Coleman-Weinberg" model, a single scalar-field toy model with potential

$$V(\phi)=\lambda\phi^4\left[\ln\frac{\phi^2}{\sigma^2}-\tfrac{1}{2}\right]+\tfrac{1}{2}\lambda\sigma^4 . \tag{13}$$

This potential can be viewed as the effective dynamical potential for the scalar field in a real Coleman-Weinberg model,[10] a model in which a scalar field with potential $\lambda\phi^4$ (and vanishing mass term) is coupled to gauge fields, and in which the self-couplings of ϕ are suppressed compared to the couplings between ϕ and the gauge fields. This model was used in the original analysis of new inflation.[3]

The second class of models we consider are single scalar-field models with a double-well $\lambda\phi^4$ potential [Eq. (1)]. In many particle physics models the scalar field which drives inflation is very weakly coupled to other fields. The dynamics of the scalar-field configuration is then determined by its interaction with gravity alone, i.e., by Eq. (7). Models of this type naturally arise in supergravity models.[11] Decoupled scalar fields driving inflation have also been proposed in nonsupersymmetric models.[12]

We start the system in the hot radiation-dominated phase at a temperature $T_E \geq T_C$. Since the energy-momentum tensor is dominated by the homogeneous radiation fluid, it is justified to take the metric to be a FRW metric. The scale factor is increasing proportional to $t^{1/2}$. Given an initial scalar-field configuration we follow the time evolution of $T_{\mu\nu}(\phi(\mathbf{x},t))$, the contribution of the scalar field to the equation of state. We determine for which values of the free parameters λ and σ the equation of state becomes inflationary as a consequence of the expansion of the universe, i.e., for which

$$p(\phi(\mathbf{x},t)) \simeq -\rho(\phi(\mathbf{x},t)) \ , \tag{14}$$

where $p(\phi(\mathbf{x},t))$ and $\rho(\phi(\mathbf{x},t))$ are given in Eq. (9).

We choose plane-wave initial conditions of the scalar field:

$$\phi(t,x,y,z) = A \sin kz \ ,$$

$$\dot{\phi}(t,x,y,z) = 0 \ . \tag{15}$$

We also considered the effect of superposing several modes, of adding a small asymmetry (i.e., a small constant) to Eq. (15), and of adding random fluctuations. Choosing plane-wave initial conditions reduces the problem to an effective (1 + 1)-dimensional problem.

In our analytical analysis we chose the initial scalar-field configuration by thermally exciting all Fourier modes. In our opinion the important physical effects can be seen using a single plane-wave initial configuration. We want to verify that as a consequence of the expansion of the Universe the amplitude of the plane wave decreases, and that the coupling to other modes has a negligible effect for small values of the coupling constant. These conclusions will then also apply for a finite superposition of plane waves.

We do not address the question of the length of the inflationary period. Since we show that in models for which the scalar-field equation of state becomes inflationary the scalar-field configuration is essentially homogeneous in space and localized at $\phi=0$, the usual estimates of the length of the inflationary period[3] should be valid.

V. INITIAL CONDITIONS

The first set of initial conditions are "quasithermal" initial conditions. We imagine that the modes of the scalar field have been excited by interactions with other fields. We pick as initial temperature $T_E = \sigma$. Since modes with $k > \sigma$ are exponentially suppressed by the Boltzmann factor and since the phase-space density of modes is proportional to k^3 we pick $k=\sigma$ in Eq. (15). The amplitude A is then determined by demanding

$$\rho_\phi(T_E) = \frac{1}{N}\rho_{\rm rad}(T_E) = \frac{\pi^2}{30}T_E{}^4 \ , \tag{16}$$

$\rho_{\rm rad}$ and ρ_ϕ are the energy densities in radiation and in the scalar field. N is the number of particle species in thermal equilibrium at T_E. On the other hand

$$\rho_\phi(T_E) = \tfrac{1}{2}A^2k^2 \ . \tag{17}$$

Thus

$$A = \left[\frac{2\pi^2}{30}\right]^{1/2} \sigma \simeq \sigma \ . \tag{18}$$

Hence our initial conditions are $A=k=\sigma$ and

$$H(t_0) = \left[\frac{8\pi^3}{90}\right]^{1/2} N^{1/2}\sigma^2 \ . \tag{19}$$

Since a single-mode pure state rather than a mixed state is chosen, we call these quasithermal initial conditions.

The reasons for assuming quasithermal initial conditions are the following. In the standard big-bang model matter is described by a gas in thermal equilibrium for $T < m_{\rm Pl}$. Since the scalar field corresponds to just another particle, it too should be in thermal equilibrium. In particular, at T_E all modes of the scalar field will be thermally excited. More pragmatic reasons are that thermal equilibrium is generally postulated at T_E in models which give new inflation. Mazenko, Unruh, and Wald[4] also assume thermal initial conditions.

Severe criticism can be raised against assuming quasithermal initial conditions for $\phi(\mathbf{x},t)$. For weakly coupled $\lambda\phi^4$ models there will be insufficient time for scalar-field self-interactions to thermalize the configuration. For Coleman-Weinberg—type models (which are strongly coupled to other fields), interactions will thermalize the scalar-field configuration. The coupling to other fields will remain important in the dynamical evolution for $T < T_E$. Taking these into account by simply replacing $V(\phi)$ by $V_{\rm eff}(\phi)$, the one-loop effective potential (13) will be a very crude approximation.

We thus extended our analysis to more general initial conditions. Classical equipartition (although not strictly applicable in this nonlinear system) suggests that at the initial temperature $T_E = \sigma$:

$$\tfrac{1}{2}k^2A^2 = \tfrac{1}{2}\lambda A^4 = \tfrac{1}{2}\sigma^4 \ . \tag{20}$$

This gives $A = \lambda^{-1/4}\sigma$ and $k = \lambda^{1/4}\sigma$, i.e., longer wavelength fluctuations with larger initial amplitude.

The second extension is to include perturbations with wavelength equal to the Hubble radius at $T=\sigma$. We vary the amplitude of these long wavelength perturbations from σ to $\lambda^{-1/4}\sigma$. We also considered initial conditions with nonvanishing spatial average of $\phi(\mathbf{x})$ and with random fluctuations in $\phi(\mathbf{x})$.

VI. THEORETICAL PREDICTIONS

On the basis of qualitative arguments we first derive a criterion which allows us to determine for which values of the free parameters λ and σ of the scalar-field potential to expect new inflation. Three forces enter into the equation of motion of the scalar field, the tension force $T = k^2a^{-2}(t)\phi$, the Hubble damping term $H(t)\dot{\phi}(t) = D$, and the nonlinear force F_N, which is $4\lambda\phi^3\ln(\phi^2/\sigma^2)$ for the Coleman-Weinberg model and $4\lambda\phi(\phi^2-\sigma^2)$ for the $\lambda\phi^4$ model.

The tension force induces time oscillations of a standing wave. For a free conformally coupled scalar field the solution of the Klein-Gordon equation (7) is

$$\phi(t,\mathbf{x})=a^{-1}(t)\hat{\phi}(\tau(t),\mathbf{x}) , \tag{21}$$

where $\tau(t)$ is conformal time and $\hat{\phi}$ denotes a standing wave in flat spacetime. The Hubble damping force is made up of a contribution from $a^{-1}(t)$ and another from $\hat{\phi}$.

$$D=-H^2\phi(t,\mathbf{x})+Ka^{-2}(t)\cos(kt-\mathbf{k}\cdot\mathbf{x}) \tag{22}$$

for $\phi(t,\mathbf{x})=a^{-1}(t)\sin(kt-\mathbf{k}\cdot\mathbf{x})$. Only the first term is a damping force in the sense that it at every instant in time damps the magnitude of ϕ. We call it $\hat{D}$. $\hat{D}$ is the force which drives ϕ towards the initial spatial average of ϕ, i.e., $\phi=0$. The nonlinear force F_N is the cause of domain formation. It always points toward the closer of the minima of $V(\phi)$ at $\phi=\pm\sigma$. If $|\hat{D}|>F_N|$ at all points in space at the initial time, then a large amplitude initial-field configuration will be driven toward the homogeneous configuration $\phi(\mathbf{x})=0$. In this case new inflation will be realized. For the Coleman-Weinberg model the criterion is

$$H^2\phi>4\lambda\phi^3\left|\ln\frac{\phi^2}{\sigma^2}\right| . \tag{23}$$

For quasithermal initial conditions the maximum of the ratio between the right-hand side and left-hand side of (23) occurs for some value $\phi\cong\sigma$. At this point the logarithmic factor will be smaller than 1. Hence (23) yields

$$\lambda<\left[\frac{H}{\sigma}\right]^2 C=\frac{8\pi^3}{90}N\sigma^2 C \tag{24}$$

with a constant $C\cong 1$. For a $\lambda\phi^4$ model there is no logarithmic suppression factor on the right-hand side of (23). Hence Eq. (24) will be true, but with a smaller constant C.

Two crucial predictions from Eq. (24) are that $\lambda_{\max}$, the maximal value of the coupling constant for which inflation is realized, scales as σ^2 and as N. We can understand both scalings easily. For fixed λ, the average curvature of the potential and hence the domain forming forces, increase as σ decreases. Increasing N means increasing the initial Hubble constant and the damping force. Our numerical analysis verifies both predictions.

If the initial amplitude at t_0 is larger than σ, the above argument must be slightly modified. Initially both $\hat{D}$ and N will point in the same direction (for $\phi>\sigma$). The amplitude of the scalar field will be damped according to Eq. (2). Once it becomes of the order σ at time t, the above analysis applies unchanged. But at that time the Hubble parameter is smaller

$$\frac{H(t)}{H(t_0)}=\left[\frac{\sigma}{A}\right]^2 . \tag{25}$$

Hence (24) gets replaced by

$$\lambda<\frac{8\pi^3}{90}CN\sigma^2\left[\frac{\sigma}{A}\right]^4 . \tag{26}$$

The numerical results show that $\lambda_{\max}$ decreases as A increases.

An intuitive picture for inflation follows by rewriting the Klein-Gordon equation as

$$\ddot{\phi}+3H\dot{\phi}=-V'(\phi)-k^2a^{-2}\phi=-V'_{\mathrm{eff},k}(\phi,t) , \tag{27}$$

by evaluating all terms of Eq. (7) for a plane wave with number k. For $k^2>4\lambda\sigma^2$ the effective potential for mode k has a unique minimum at $\phi=0$ (in the case of a $\lambda\phi^4$ theory). Figure 2 represents the phenomenon graphically.

The above intuitive picture also gives a criterion for the effect of a superimposed long-wavelength mode with wave number k_2 and amplitude A_2. Provided $k_2{}^2>4\lambda\sigma^2$ (again for a $\lambda\phi^4$ model) the tension will be strong enough to set an oscillation of this second mode in motion. If the modes are weakly coupled the amplitude of both modes will independently decrease. There are no obstructions to inflation. The only effect of $A_2\neq 0$ is to change the amplitude in Eq. (26).

The intuitive picture also immediately yields a criterion for the time at which the nonlinear forces begin to dominate. As soon as the slope of $V_{\mathrm{eff},k}(\phi,t)$ for $\phi>0$ becomes negative the decrease in the amplitude of oscillation ceases and nonlinear effects take over. In the Coleman-Weinberg model the condition is

$$k^2a^{-2}(t)\phi(t)=4\lambda\phi^3(t)\ln\frac{\phi^2}{\sigma^2} . \tag{28}$$

For initial amplitude A and scale factor

$$a(t)=\exp(tH) , \tag{29}$$

Eq. (28) yields

$$\Delta t\cong\lambda^{-1}\left[\frac{k}{A}\right]^2 H^{-1} \tag{30}$$

for the length of the period during which the amplitude of oscillation decreases. A in this context is the amplitude at the time the Universe becomes inflationary. In work in progress we will attempt to verify this formula numerically.

A nonvanishing initial spatial average $\bar{\phi}$ of $\phi(\mathbf{x},t)$ will evolve according to the classical equation of motion for the $k=0$ mode. If the initial values of $\bar{\phi}$ and $(d/dt)\bar{\phi}$ are small compared to σ and σ^2, respectively, and if the curvature of the potential at $\bar{\phi}(t_0)$ is small compared to H^3, then $\bar{\phi}$ will evolve slowly (on a time scale H^{-1}) toward the closest minimum of the potential. The $k\neq 0$ modes will be damped as described above and the scalar-field configuration will become homogeneous and localized at $\bar{\phi}(t)$. In the de Sitter phase Eq. (7) can be solved explicitly

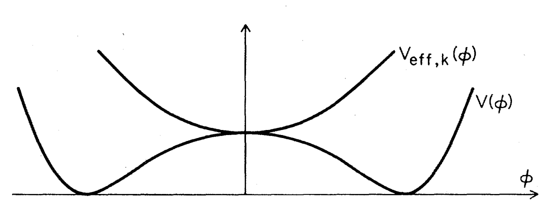

FIG. 2. Potential $V(\phi)$ and effective potential $V_{\mathrm{eff},k}(\phi)$ for a k mode with $k^2>4\lambda\sigma^2$.

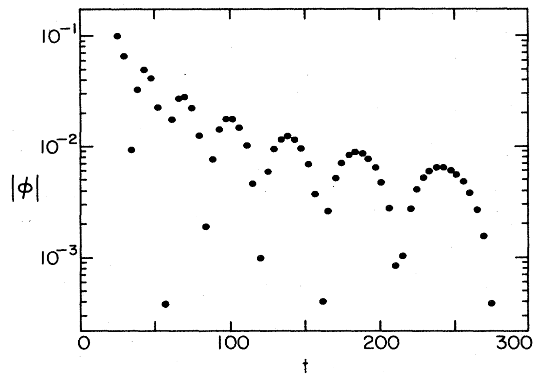

FIG. 3. Value of the scalar field as a function of time at the position of the maximum of the initial standing wave ($\lambda\phi^4$ model, $\sigma=10^{-1}$, $\lambda=10^{-2}$, $N=1, A=k=\sigma$).

given the approximation $(d^2/dt^2)\bar{\phi}=0$. The solution is[6]

$$\bar{\phi}(t)=\left[\frac{3H}{2\lambda}\right]^{1/2}\frac{1}{(t^*+t_0-t)^{1/2}}\,, \qquad t^*=\frac{3H}{2\lambda\bar{\phi}^2(t_0)}\,, \tag{31}$$

t^* gives an order of magnitude estimate for the length of the slow rolling period. It is an open question whether initial values $\bar{\phi}(t_0)$ and $(d/dt)\bar{\phi}(t_0)$ which give a sufficiently long inflationary period are natural in a cosmological context.

VII. NUMERICAL RESULTS

First we give a brief description of the program. We numerically integrate the Klein-Gordon equation (7) as a system of coupled first-order differential equations for ϕ and $\dot{\phi}$ in a radiation-dominated FRW universe. The number of points on the spatial grid is a free parameter. We generally use 100 points for single-mode runs and 200 points for runs with more than one plane wave. Periodic boundary conditions hold. The spatial grid corresponds

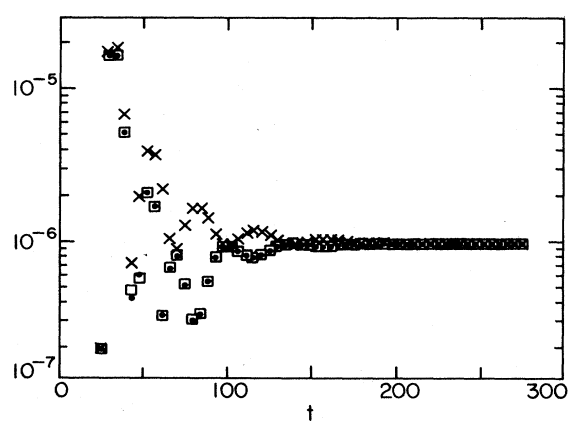

FIG. 4. Equation of state of the scalar field at the position of the maximum of the initial standing wave as a function of time for the same run as in Fig. 3. The horizontal axis is time. Crosses denote $|\rho(t)|$, squares $|p_x(t)|$, and points $|p_z(t)|$.

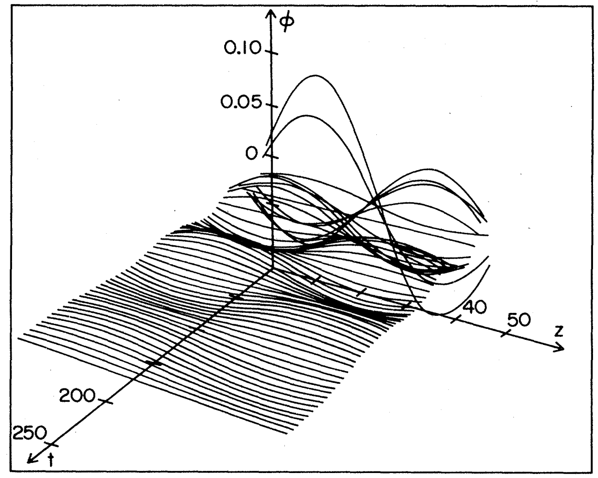

FIG. 5. A three-dimensional plot showing the value of the scalar field (vertical axis) as a function of space (right axis) and time (left axis) for the run considered in Fig. 3.

to n wavelengths. To maximize the resolution we choose $n=1$. We checked the stability of our analysis against changing n.

The important physical free parameters in the program

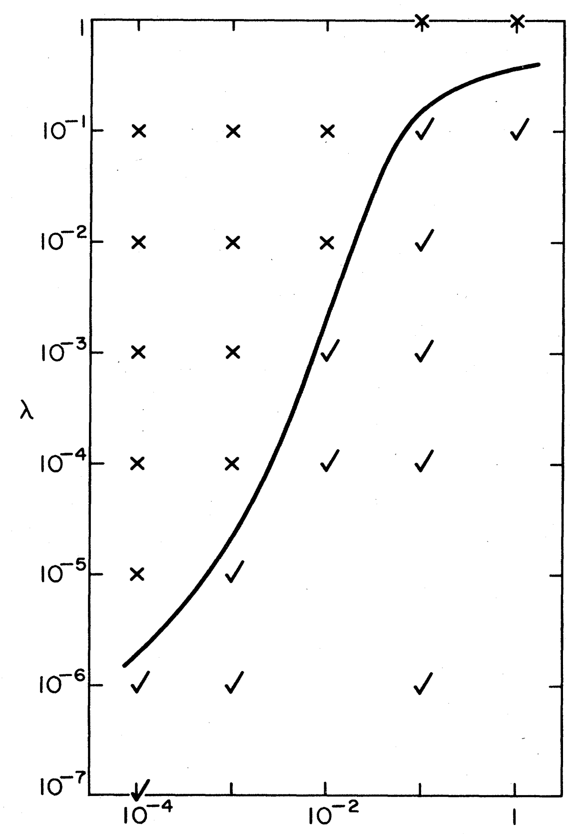

FIG. 6. λ_{max} as a function of σ for plane-wave initial conditions $A=k=\sigma$, $N=1$ in the Coleman-Weinberg model. Checks mark runs which yield an inflationary equation of state, $\times$'s mark runs which fail to give inflation.

TABLE I. Theoretical predictions and numerical results for λ_{max}. Initial conditions in the numerical runs are $N=1$, $k=A=\sigma$.

σ	Theory [Eq. (24)]	λ_{max} Numerics Coleman-Weinberg model	Numerics $\lambda\phi^4$
10^{-4}	$\sim 10^{-8}$	10^{-6}	...
10^{-3}	$\sim 10^{-6}$	10^{-5}	10^{-7}
10^{-2}	$\sim 10^{-4}$	10^{-3}	10^{-4}
10^{-1}	$\sim 10^{-2}$	10^{-1}	10^{-2}
1	~ 1	10^{-1}	10^{-2}

are N, the number of particles in thermal equilibrium at the starting temperature T_E, λ and σ, the two parameters which determine the shape of the potential, and A and k, which characterize the amplitude and initial wave number of the scalar-field configuration. We follow the evolution of the system for a fixed number of initial Hubble-expansion times. The program stores the value ϕ of the scalar field, the energy density $\rho(\phi)$, the pressure $p_z(\phi)$ in the direction of the standing wave, and the pressure $p_x(\phi)$ perpendicular to the standing wave, as well as the energy density of background radiation, all at a fixed number of time values. In particular we can plot the (absolute value of the) amplitude of ϕ at a given point z as a function of time. We plot the contribution of the scalar field to the equation of state as a function of time for the same point z, and finally we produce three-dimensional (3D) graphs showing the value of the scalar field along the entire spatial grid as a function of time.

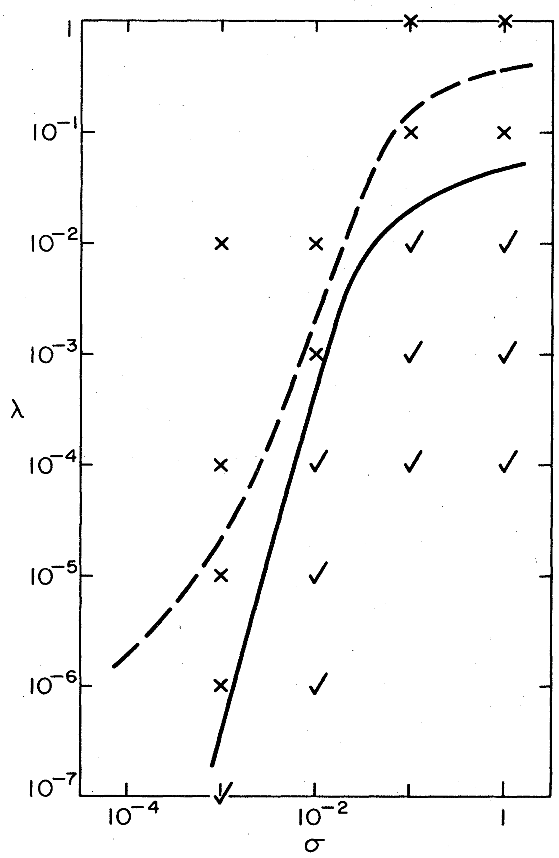

FIG. 7. λ_{max} as a function of σ for the $\lambda\phi^4$ model (same initial conditions as in Fig. 6). The dashed line indicates $\lambda_{max}(\sigma)$ from Fig. 6.

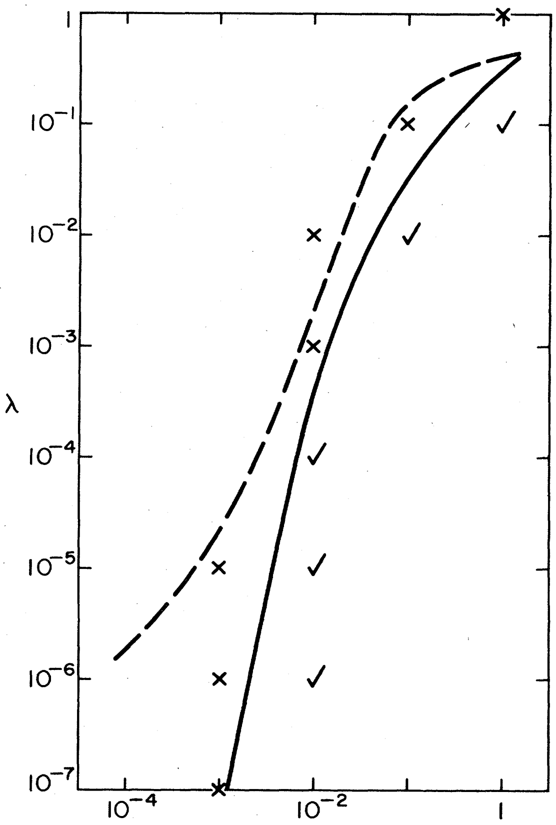

FIG. 8. λ_{max} as a function of σ in the Coleman-Weinberg model for large-amplitude initial conditions ($A=\lambda^{-1/4}\sigma$, $k=\lambda^{1/4}\sigma$).

Figure 3 shows the (absolute value of the) amplitude of the scalar field at the maximum of the initial standing wave for a run with a $\lambda\phi^4$ potential. The run uses $N=1$,

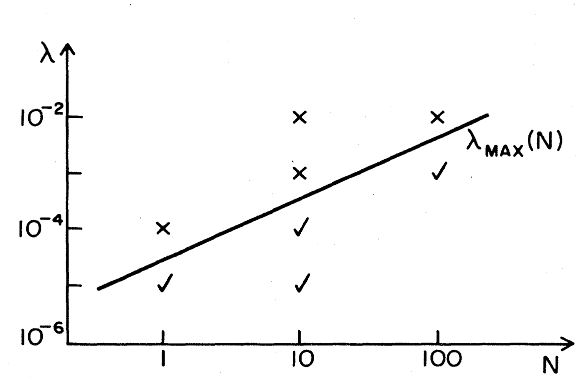

FIG. 9. N dependence of λ_{max} (Coleman-Weinberg model, $\sigma=10^{-2}$, Hubble radius scale fluctuation with amplitude $A_2=\sigma$ included).

TABLE II. Theoretical predictions and numerical results for $\lambda_{\max}$ for large-amplitude initial conditions (Coleman-Weinberg model, $N=1$, $A=\lambda^{-1/4}$, $k=\lambda^{1/4}\sigma$).

	$\lambda_{\max}$	
σ	Theory [Eq. (26)]	Numerics
10^{-3}	$\sim 10^{-9}$	10^{-8}
10^{-2}	$\sim 10^{-6}$	10^{-4}
10^{-1}	$\sim 10^{-3}$	10^{-2}
1	~ 1	10^{-1}

$\sigma=10^{-1}$, $\lambda=10^{-2}$, $A=\sigma$, and $k=\sigma$. We follow the evolution for 10 initial Hubble times. The envelope of the curve in Fig. 3 is decreasing as predicted by the Hubble damping formula. At the end of the run the amplitude is less than 10% of its original amplitude, i.e., the configuration is localized at the top of the potential barrier to better than 10% of the distance to the minima. Correspondingly the equation of state of the scalar field rapidly approaches a de Sitter equation $p_x=p_z=-\rho$. This is shown by the coincidence of the three curves at late times in Fig. 4. The crosses give the value of $|\rho|$, the squares correspond to $|p_x|$, and the points to $|p_z|$. Initially the tension and kinetic terms dominate the equation of state, giving $p\neq-\rho$. As the scalar field settles down near $\phi=0$ the potential term $V(\phi)$ begins to dominate, yielding $p=-\rho$. The 3D plot in Fig. 5 (again for the same run) depicts the time evolution of the scalar-field configuration along the spatial grid. The initial large amplitude wave is damped by the Hubble expansion. The sinusoidal shape of the wave is maintained since the nonlinear forces are too weak to produce visible mode-mixing effects.

The results of our numerical runs for the Coleman-Weinberg model for quasithermal initial conditions $A=k=\sigma$ are summarized in Fig. 6. All runs take $N=1$. We explore the parameter space region $10^{-4}\leq\sigma\leq 1$ and determine $\lambda_{\max}$, the maximal value of λ for which an inflationary period is realized. In Table I we compare the theoretical predictions for $\lambda_{\max}$ with our numerical results. The agreement confirms our heuristic picture of inflation. In particular we verify the σ^2 dependence of $\lambda_{\max}$ predicted in Eq. (24).

The analogous results in the case of a $\lambda\phi^4$ model for quasithermal initial conditions are summarized in Fig. 7.

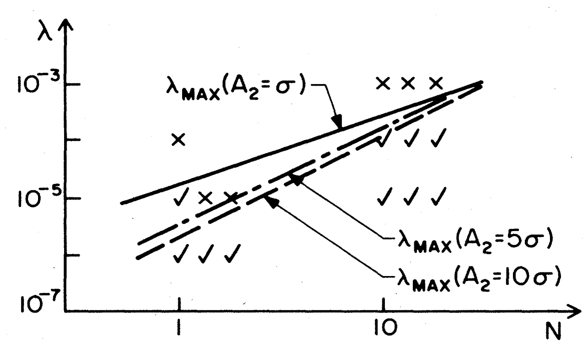

FIG. 10. Dependence of $\lambda_{\max}$ on A_2 (Coleman-Weinberg model, $\sigma=10^{-2}$). The first column gives the results for $A_2=\sigma$, the second for $A_2=5\sigma$, and the third for $A_2=10\sigma$.

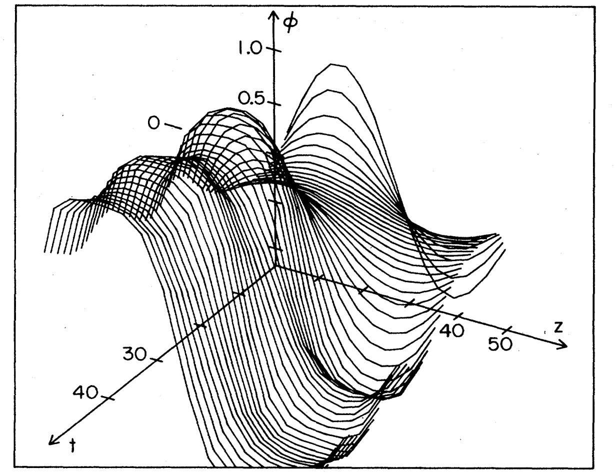

FIG. 11. Three-dimensional plot of the scalar-field evolution as a function of time in a $\lambda\phi^4$ model with $\sigma=1$, $\lambda=10^{-2}$, $N=1$, $A=k=\sigma$. The axes are as in Fig. 5.

The runs are for $N=1$ and explore the range $10^{-3}\leq\sigma\leq 1$. In Table I we compare the theoretical predictions for $\lambda_{\max}$ with the numerical results. The heuristic picture is again confirmed. Except for $\sigma\simeq 1$ the σ^2 dependence of $\lambda_{\max}$ in Eq. (24) is verified. As predicted, the difference between the Coleman-Weinberg model and the $\lambda\phi^4$ model manifests itself in the difference in the constants C of Eq. (25). In the Coleman-Weinberg case $\lambda_{\max}$ is enhanced by a factor of the order 10 due to the extra logarithmic suppression of nonlinear forces.

In the case of the Coleman-Weinberg model we investigated the effect of different initial conditions. We first considered large-amplitude plane-wave initial conditions $A=\lambda^{-1/4}\sigma$, $k=\lambda^{1/4}\sigma$, $N=1$. As predicted by Eq. (26), $\lambda_{\max}$ decreases as A increases. The numerical results are summarized in Fig. 8 and theoretical predictions and numerical results for $\lambda_{\max}$ are compared in Table II. In these runs a conformal mass term has been included in the potential.

Next we considered, again in the case of the Coleman-Weinberg model and again with a conformal mass term in

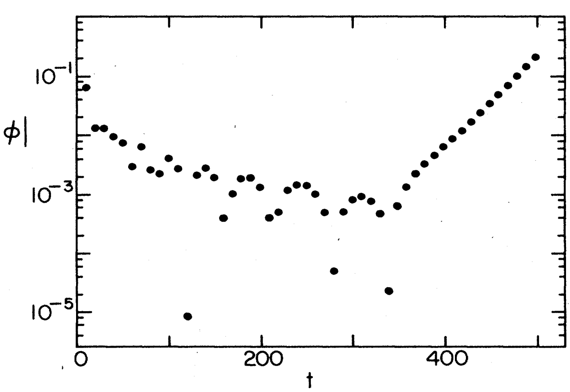

FIG. 12. Value of the scalar field at the point in space corresponding to the maximum of the standing wave as a function of time in a run with a long period of inflation. ($\lambda\phi^4$ model, $\sigma=1$, $\lambda=10^{-4}$, $A=k=\sigma$, $N=1$.)

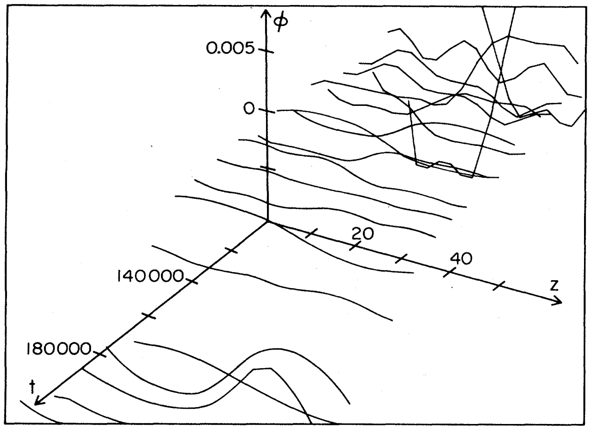

FIG. 13. Three-dimensional plot of the scalar field in a run with large initial amplitude (Coleman-Weinberg, $\sigma=10^{-2}$, $\lambda=10^{-4}$, $N=1$, $k=\lambda^{1/4}$, $A=\lambda^{-1/4}\sigma$).

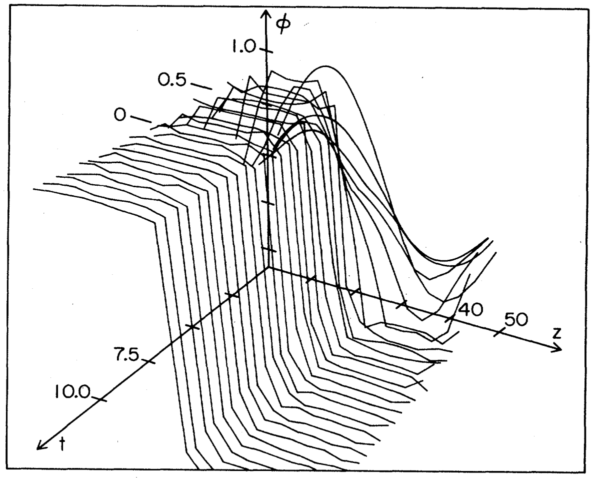

FIG. 14. Immediate domain formation in a model with large coupling constant (Coleman-Weinberg, $\sigma=1$, $\lambda=1$).

the potential, the effect of adding a long wavelength fluctuation. The initial scalar-field configuration is given by the wave numbers $k_1=\lambda^{1/4}\sigma$ and $k_2=H$ and the amplitudes $A_1=\lambda^{-1/4}\sigma$ and A_2 of the two plane waves. We consider $\sigma=10^{-2}$ and analyze how $\lambda_{\max}$ depends on A_2 and N. Figure 9 shows that $\lambda_{\max}$ increases linearly with N as predicted from Eq. (24). In Fig. 10 we plot $\lambda_{\max}$ as a function of A_2 for $A_2=\sigma$, 5σ, and 10σ. There is no obstruction to inflation and, as predicted in Sec. VI, $\lambda_{\max}$ only depends weakly on A_2 since a change in A_2 causes only a small change in the total amplitude.

A preliminary investigation of the effect of adding random initial velocities was performed. We first considered initial conditions with $\phi(\mathbf{x})=0$ and $\dot{\phi}(\mathbf{x})$ chosen randomly in the range $[-\sigma^2,\sigma^2]$ independently at each point in space. The range was determined by demanding that the kinetic energy not exceed the thermal energy of one field at $T=\sigma$. For a $\lambda\phi^4$ potential with $\sigma=1$ and $\lambda=10^{-4}$ the equation of state remained inflationary for a time period greater than 800 initial Hubble-expansion times.

The next set of initial conditions consisted of a plane-wave mode $\phi(\mathbf{x})=\sigma\sin\sigma x$ with random velocities $\dot{\phi}(\mathbf{x})$ chosen in the same manner as above. In the case of the $\lambda\phi^4$ potential with $\sigma=1$ we found inflation for $\lambda=10^{-3}$ and $\lambda=10^{-4}$ but not for $\lambda=10^{-2}$. A more systematic analysis is left for a future publication.

VIII. SOME EXAMPLES

Figures 3–5 show the dynamical evolution of typical initial conditions for which new inflation is realized. In this section we would like to mention some of the other interesting features we observe in the numerical runs.

Figure 11 plots the evolution of the scalar field for initial conditions which show a phase transition. The parameters are $N=1$, $\sigma=1$, $\lambda=10^{-2}$. The run is for $k=A=\sigma$ and for a $\lambda\phi^4$ potential. The total time interval is 200 initial Hubble times. The amplitude of the standing wave decreases until $A(t)\simeq 0.1A$, at which point the equation of state of the scalar field is $p\cong-\rho$. But the nonlinear forces immediately begin to dominate, drive the scalar field towards the minimum of the potential, and cause oscillations about the minimum.

For $\sigma=1$, $\lambda=10^{-4}$ we observe a long period of inflation which terminates near the end of the turn, after 2000 initial Hubble times with a slow rolling of the scalar field, uniformly in space, toward $\phi=\sigma$ (Fig. 12). The run is for a $\lambda\phi^4$ potential and takes $A=k=\sigma$.

If we consider an initial scalar-field configuration with a large amplitude, i.e., $k=\lambda^{1/4}\sigma$, $A=\lambda^{-1/4}\sigma$, we see that the initial plane wave breaks up into its higher harmonics. This reflects the fact that for $|\phi|>\sigma$ the nonlinear terms in the potential are dominant. Figure 13 shows the evolution for a Coleman-Weinberg model for $\sigma=10^{-2}$, $\lambda=10^{-4}$, and $N=1$. The total time interval is $100H^{-1}(t_0)$. Initially the amplitude of the scalar field decreases uniformly in space which demonstrates that the individual modes basically evolve independently. Towards the end of the run the phase transition takes place.

For very large coupling constants, the nonlinear forces dominate *ab initio* which is reflected in immediate domain formation. An example is shown in Fig. 14. It is for a Coleman-Weinberg potential, $\sigma=\lambda=1$, $k=A=\sigma$, and $N=1$. The time interval is $50H^{-1}(t_0)$.

IX. CONCLUSIONS

The evolution of a classical scalar-field configuration in an expanding Friedmann-Roberson-Walker universe with scale factor $a(t)\sim t^{1/2}$ is analyzed numerically. The goal is to determine if the evolution leads to a configuration necessary to produce new inflation, namely localized near $\phi=0$ and homogeneous in space.

We conclude that new inflation can be realized in a large class of models in which a weakly coupled scalar field generates inflation. Theoretical considerations lead to the prediction that $\lambda_{\max}$, the maximal value of the coupling constant for which inflation occurs, should scale as $\sigma^{-2}N$, N being the number of particle species in the initial thermal bath. The numerical results confirm this prediction both for a Coleman-Weinberg potential and a $\lambda\phi^4$ potential.

The basic mechanism which produces a scalar-field configuration which is homogeneous and localized at $\phi=0$ is Hubble damping. The numerical runs show that nonlinear effects and mode-mode couplings produce a much smaller effect over the time period of interest than Hubble damping.

The analysis was performed with various sets of initial conditions. We considered plane-wave initial conditions with amplitude A and wave number k equal to σ (justified by thermal equilibrium arguments). We also considered large amplitude waves, i.e., $A=\lambda^{-1/4}\sigma$, $k=\lambda^{1/4}\sigma$, as well as the effect of additional Hubble-scale long-wavelength fluctuations. Finally we considered random initial velocities.

This analysis complements and strengthens our previous analytical analysis.[5] Mazenko[13] has just completed a numerical analysis of the evolution of an N-vector model coupled to Einstein's equations. In the $N\rightarrow\infty$ limit an inflationary phase is observed. It is however a manifestation of chaotic inflation[14] rather than new inflation.

In a recent report[15] Guth and Pi present a careful analysis of the evolution of the Higgs field in a time-dependent quadratic potential and reach similar conclusions to our results. In work in progress we are studying the evolution of the scalar field for more realistic initial conditions with a 3D code. This will enable us to analyze the mechanism of the phase transition, spinodal decomposition.[16]

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