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On the electromagnetic properties of Majorana fermions

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It is proven that *CPT* invariance alone requires the electromagnetic structure of a spin-J Majorana particle ($J=$ half integer) to consist at most in toroid multipole moments and distributions with the multipolarity order $L=1,2,\ldots,2J$; all other usual electric and magnetic multipole characteristics are forbidden. In the spin $J=\frac{3}{2}$ case the covariant form of the electromagnetic vertex is obtained in terms of dipole, quadrupole, and octupole toroid form factors. It is suggested that under certain circumstances Majorana fermions with nonvanishing toroid multipole moments might give rise to Cherenkov or transition radiation.

There is now interest in Majorana particles, which occur commonly in grand unified and supersymmetric theories, as well as in connection with some aspects of neutrino physics. The problem of the possible electromagnetic structure of such truly neutral fermions reveals some interesting but apparently less known and investigated features of these (not yet detected) objects. In the lowest-spin $J=\frac{1}{2}$ case it has been shown[1] that the most general expression for the matrix element of the electromagnetic current of a Majorana fermion is

$$\langle p+k;J=\tfrac{1}{2},J_z^{(f)}|j_\mu(0)|p;J=\tfrac{1}{2},J_x^{(i)}\rangle = iA(k^2)\bar{u}(p+k;J_z^{(f)})[k^2\gamma_\mu-(k\cdot\gamma)k_\mu]\gamma_5\mu(p;J_z^{(i)}) \,, \tag{1}$$

and it has been subsequently realized[2] that this conclusion comes in fact from *CPT* invariance alone. The particular Lorentz structure from Eq. (1), now known as the "Zeldovich's anapole," had been long ago[3] considered in connection with a new kind of electromagnetic interaction (invariant under time reversal but odd under parity) and had been interpreted in terms of a toroid current (i.e., a closed current circulating through a toroidal solenoid). This particular current configuration represents a new dipole characteristic of the system (a "toroid" dipole moment), different from the usual electric or magnetic dipoles. What is special about a spin-$\frac{1}{2}$ Majorana fermion is that for it there is nothing else left except this structure and therefore it can be viewed as the cleanest elementary carrier of the toroid dipole. As seen from Eq. (1), another distinct feature of the anapole (toroid dipole) vertex is that it gives rise to a contact interaction with the external electromagnetic field (i.e., the spin-$\frac{1}{2}$ self-conjugate fermion interacts with the external field only if it overlaps with the source of the latter).

One may ask whether the aforementioned peculiarities of the spin-$\frac{1}{2}$ Majorana particles would hold in the general case of Majorana fermions of arbitrary (half-integral) spin (in the spin-$\frac{3}{2}$ case the matter becomes even more actual if one recalls the central role played by the gravitino in supersymmetry). In the present Rapid Communication we answer this question in the affirmative in the form of a theorem, stated in the first sentence of the abstract. In the proof we have to use as two lemmas some results previously obtained by other authors. Before stating the first lemma, we mention that in a series of papers, summarized in the reviews,[4] it has been shown that in both classical and quantum electrodynamics the toroid dipole moment is only the first element of an independent (third) family of "toroid" multipoles, which, together with the usual electric and magnetic ones, achieve a complete description of a general configuration of charges and currents. What we need here as a first lemma is the following (most general) multipole parametrization[4] of the matrix element of the electromagnetic current $j_\mu(x)=(j_0(x),\mathbf{j}(x))$ taken between one-particle states of mass m, spin J, spin projections $J_z^{(i)}$, $J_z^{(f)}$, and momenta $\mathbf{p}^{(i)}=-\mathbf{k}/2$, $\mathbf{p}^{(f)}=+\mathbf{k}/2$:

$$\left\langle +\frac{\mathbf{k}}{2};J,J_z^{(f)}\middle|j_0(x=0)\middle|-\frac{\mathbf{k}}{2};J,J_z^{(i)}\right\rangle = \sum_{L,M}\frac{[4\pi(2L+1)]^{1/2}}{(2L+1)!!}|\mathbf{k}|^L\frac{C^{J,L,J}_{J_z^{(i)},M,J_z^{(f)}}}{C^{J,L,J}_{J,0,J}}Y_{LM}(\mathbf{n})Q_{L,J}(-\mathbf{k}^2)\,, \tag{2a}$$

$$\left\langle +\frac{\mathbf{k}}{2};J_z,J_z^{(f)}\middle|\mathbf{j}(x=0)\middle|-\frac{\mathbf{k}}{2};J,J_z^{(i)}\right\rangle = \sum_{L,M}\frac{[4\pi(2L+1)(L+1)/L]^{1/2}}{(2L+1)!!}|\mathbf{k}|^L\frac{C^{J,L,J}_{J_z^{(i)},M,J_z^{(f)}}}{C^{J,L,J}_{J,0,J}}\,,$$
$$\times[\mathbf{F}^{(0)}_{LM}(\mathbf{n})M_{L,J}(-\mathbf{k}^2)+|\mathbf{k}|\mathbf{F}^{(+)}_{LM}(\mathbf{n})T_{L,J}(-\mathbf{k}^2)]\,. \tag{2b}$$

$Q_{L,J}(-\mathbf{k}^2)$, $M_{L,J}(-\mathbf{k}^2)$, and $T_{L,J}(-\mathbf{k}^2)$ are, respectively, the charge, magnetic, and toroid multipole (2^L-pole) form factors [which at zero momentum transfer ($\mathbf{k}^2=0$) give the corresponding multipole moments of the considered system]; $Y_{LM}(\mathbf{n})$ are usual spherical functions, while

$$\mathbf{F}^{(0)}_{LM}(\mathbf{n})=\mathbf{Y}_{LLM}(\mathbf{n}),\quad \mathbf{F}^{(+)}_{LM}(\mathbf{n})=i\mathbf{F}^{(0)}_{LM}\times\mathbf{n},\quad \mathbf{n}=\frac{\mathbf{k}}{|\mathbf{k}|}\,;$$

the (spherical basis) components of the vector $\mathbf{Y}_{LLM}$ are

$$\{\mathbf{Y}_{LLM}\}_m=\sum_{M'}C^{L,1,L}_{M',m,M}Y_{LM'}(\mathbf{n})\,;$$

the summation over L and $M=-L,\ldots,+L$ in Eqs. (2a) and (2b) is restricted by the appearing (in view of the Wigner-Eckart theorem) Clebsch-Gordan coefficients [it starts, in fact, with $L=0$ in Eq. (2a) and $L=1$ in Eq. (2b)].

In the case of a Majorana fermion of (half-integral) spin J, TCP invariance restricts the general form of the electromagnetic vertex displayed in Eqs. (2a) and (2b) to a more particular one. To find the resulting selection rules, we need as our second lemma a certain nontrivial phase condition imposed by TCP on single-particle Majorana states which we formulate according to Kayser and Goldhaber[2] as follows: Defining the Majorana particle as a self-conjugate fermion under TCP (rather than under charge conjugation C, which may not always be appropriate) and writing the effect of TCP on the Majorana single-particle state of momentum $\mathbf{p}$, spin J, and spin projection J_z, as

$$TCP|\mathbf{p};J,J_z\rangle=\eta(J_z)|\mathbf{p};J,-J_z\rangle\ , \tag{3}$$

the phase factor $\eta(J_z)(|\eta(J_z)|=1)$ is constrained by TCP invariance alone to satisfy the condition[2]

$$\eta(-J_z)=(-1)^{2J}\eta(J_z)=-\eta(J_z)\ . \tag{4}$$

Since it is known that the electromagnetic interaction is to conserve TCP, $j_\mu(x=0)$ must be TCP-odd also; the second lemma just stated leads then to the following TCP condition on the vertex:

$$\left\langle \frac{\mathbf{k}}{2};J,J_z^{(f)}\middle| j_\mu(0)\middle| -\frac{\mathbf{k}}{2};J,J_z^{(i)}\right\rangle=(-)\eta^*(J_z^{(i)})\eta(J_z^{(f)})\left\langle -\frac{\mathbf{k}}{2};J,-J_z^{(i)}\middle| j_\mu(0)\middle|\frac{\mathbf{k}}{2};J,-J_z^{(f)}\right\rangle\ . \tag{5}$$

Inserting now Eqs. (2a) and (2b) into Eq. (5), taking into account the phase condition Eq. (4) and performing some simple manipulations, among which the hardest are the use of the parity property of $Y_{LM}(\mathbf{n})$ and of the elementary relation

$$C^{J,L,J}_{-J_z^{(f)},M,-J_z^{(i)}}=(-1)^{L+M}C^{J,L,J}_{J_z^{(i)},M,J_z^{(f)}}\ ,$$

one finally proves the theorem expressed in the abstract: For a Majorana fermion of spin J only the toroid form factors $T_{L,J}(-\mathbf{k}^2)$ [with $L=1,2,\ldots,2J$, i.e., all of those appearing in the right-hand side of Eq. (2b)] survive in general; all the other electric and magnetic multipole form factors $Q_{L,J}(-\mathbf{k}^2)$ and $M_{L,J}(-\mathbf{k}^2)$ in Eqs. (2a) and (2b) are ruled out exclusively on TCP-invariance grounds. That the number of the remaining (toroid) form factors makes up the correct required number of independent transitions in the most general parametrization of the electromagnetic vertex of a spin-J Majorana particle, one may convince himself also by an easy count (in the nonrelativistic approximation) of the independent transitions in the crossed vertex $\langle 0|j_\mu(0)|M,M\rangle$, where $|M,M\rangle$ represents a pair of identical Majorana particles of spin $J=(2l-1)/2$ $(l=1,2,\ldots)$. Antisymmetry at the permutation of the two fermions requires $(-1)^{L+S+1}$ to be -1 ($\mathbf{L},\mathbf{S}$ are the orbital angular momentum and the total spin in the MM system). S may be $0,1,2,\ldots,2l-1$ and, for each of these values of S and L, must be such that its coupling with this particular S gives the total angular momentum l carried by the conserved current $j_\mu(x)$. For S given, only states with $L=S-1,S,S+1$ can do that, but for these only $L=S$ will render $(-1)^{L+S+1}$ negative. Therefore, only $|M,M\rangle$ states with $L=S=1,2,\ldots,2l-1$ are allowed and the number of independent form factors is confirmed to be $2l-1=2J$.

For Majorana fermions of spin $J=\frac{3}{2}$ it is more convenient practically to have at hand a covariant form of the electromagnetic vertex rather than the expression

$$\left\langle +\frac{\mathbf{k}}{2};\frac{3}{2},J_z^{(f)}\middle|\mathbf{j}(x=0)\middle|-\frac{\mathbf{k}}{2};\frac{3}{2},J_z^{(i)}\right\rangle=\sum_{\substack{L=1,2,3\\ M=-L,\ldots,+L}}\frac{[4\pi(2L+1)(L+1)/L]^{1/2}}{(2L+1)!!}\frac{C^{3/2,L,3/2}_{J_z^{(i)},M,J_z^{(f)}}}{C^{3/2,L,3/2}_{3/2,0,3/2}}|\mathbf{k}|^{L+1}\mathbf{F}^{(+)}_{LM}(\mathbf{n})T_{L,3/2}(-\mathbf{k}^2)\ ,$$
$$\left\langle +\frac{\mathbf{k}}{2};\frac{3}{2},J_z^{(f)}\middle| j_0(x=0)\middle|-\frac{\mathbf{k}}{2};\frac{3}{2},J_z^{(i)}\right\rangle=0\ , \tag{6}$$

obtained as a particular case from the above considerations. It is

$$\begin{aligned}&\langle p+k;J=\tfrac{3}{2},J_z^{(f)}|j_\mu(x=0)|p;J=\tfrac{3}{2},J_z^{(i)}\rangle\\ &\quad=i\bar{u}_\rho(p+k;J_z^{(f)})\{A(k^2)[k^2\gamma_\mu-(k\cdot\gamma)k_\mu]\gamma_5 g^{\rho\sigma}+[B(k^2)+iC(k^2)\gamma_5][k_\mu k^\rho k^\sigma-\tfrac{1}{2}k^2(g_\mu^\rho k^\sigma+g_\mu^\sigma k^\rho)]\}u_\sigma(p;J_z^{(i)})\ .\end{aligned} \tag{7}$$

Equation (7) may be obtained by the same procedure which has been used by Kayser and Goldhaber[2] to get Eq. (1), i.e., using the TCP requirement [Eq. (5)] and the relation (similar to the corresponding one in the spin-$\frac{1}{2}$ case)

$$u_\rho(p;-J_z)=(-1)^{J_z-1/2}\gamma_1\gamma_3\bar{u}_\rho^T(p;J_z)\ .$$

As far as the interpretation of the three form factors $A(k^2)$, $B(k^2)$, and $C(k^2)$ is concerned, it is easy to show that $A(0)$, $B(0)$, and $C(0)$ express, respectively, the dipole, quadrupole, and octupole toroid moments of the considered spin-$\frac{3}{2}$ Majorana particle. All one has to do is to evaluate the corresponding expectation values of the multipole toroid moments[5]

$$T_{L,M}(t)=\left[\frac{4\pi}{2L+1}\right]^{1/2}\frac{i}{2(L+1)(2L+3)}\times\int\mathbf{j}(\mathbf{x},t)\cdot\nabla\times\left[r^{L+2}(-i\mathbf{r}\times\nabla)Y^*_{LM}\left[\frac{\mathbf{r}}{r}\right]\right]d^3x\ , \tag{8}$$

in the cases of interest $L=1,2,3$ by taking for the appearing matrix element

$$\left\langle +\frac{\mathbf{k}}{2};J=\tfrac{3}{2},J_z=+\tfrac{3}{2}\middle|\mathbf{j}(\mathbf{x},t)\middle|-\frac{\mathbf{k}}{2};J=\tfrac{3}{2},J_z=+\tfrac{3}{2}\right\rangle$$

the form displayed in Eq. (7). After some straightforward calculations one finds

$$\langle \mathbf{0};\tfrac{3}{2},+\tfrac{3}{2}|T_{L,M=0}|\mathbf{0};\tfrac{3}{2},+\tfrac{3}{2}\rangle = (2\pi)^3\delta^3(\mathbf{k}=0)\begin{cases} A(k^2=0), & L=1, \\ -B(k^2=0), & L=2, \\ \dfrac{3}{2m}C(k^2=0), & L=3. \end{cases} \tag{9}$$

Having seen from the previous discussion that Majorana particles may still be left, in general, with a certain electromagnetic structure, identified before as consisting of toroid moments and distributions, the question arises of whether such neutral fermions, on account of this type of structure, might give rise to Cherenkov or transition radiation. Ginzburg and Tsytovich[5] have recently analyzed the Cherenkov and transition radiation of a moving elementary toroid dipole (constant in time in its rest frame; if there is time dependence, there will be some radiation anyway), stating at one point (the end of Sec. 6 of their paper) that "we are not aware of any real problems in which the . . . radiation of relativistic toroid dipole moments would be of any importance." We point out that if Majorana particles are really there, the above remark may lose much of its sadness. Indeed, relying on the classical-electrodynamics analysis of Ref. 5, we are led to the suggestion that Majorana particles (with nonvanishing toroid structure) might, in principle, give rise to Cherenkov and transition radiation, albeit small, by the toroid character of the source, and by the actual values of the toroid moments themselves (determined by small parity- or time-reversal-violating interactions), and small again because the toroid current, as noted in Ref. 5, would emit this kind of radiation mainly, inasmuch as it is filled up by the medium through which it travels, thus making the effect conditional on both the type of media and the actual spatial extension of the traveling Majorana particle. Concerning this latter point, it seems that for more or less normal media much would depend upon the Majorana particles in question being very extended objects, which should not be *a priori* excluded. For extremely dense (nuclear) matter, the requirement of large extension might be less prohibiting and even much less for such a medium as the vacuum itself in the presence of strong electromagnetic fields.

We end with the remark that while Majorana fermions are allowed to possess only toroid multipole moments and distributions as *intrinsic* electromagnetic characteristics, in general there is nothing to prevent them from having electric, magnetic, or toroid[6] polarizabilities as distinct characteristics describing their behavior in external electric (and magnetic) fields or currents. In other words, they may get *induced* electric, magnetic, or toroid moments when external fields are present, irrespective of whether or not parity or time-reversal invariance are good symmetries, because in this case a new direction, that of the external field or current, is available, thus making inoperative (for the *induced* moments) the arguments leading to selection rules.

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