Proton decay cannot be suppressed kinematically

J. Dittrich and P. Exner*
Nuclear Physics Institute, Czechoslovak Academy of Sciences, 250 68 Řež, Czechoslovakia
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Using a recently derived decay-law formula for relativistic unstable particles, we examine how the decay of a free proton could be affected by the size and shape of the wave packet. For simplicity, the proton is assumed to be spherically symmetric and spinless. The decay probability is found to be lessened comparing to the theoretically expected one; however, the effect is insignificant under realistic experimental conditions. Hence the proton decay is not likely to be suppressed kinematically, contrary to the prediction of a paper by Fleming.

I. INTRODUCTION

The problem of consistent relativistic description of decays has attracted attention for a long time.[1] It is connected with some peculiar difficulties. One of them concerns the fact that one cannot regard the commonly used formula

$$P_\psi(t) = |(\psi, e^{-iHt}\psi)|^2 \tag{1}$$

as an exact expression of the decay law for a free unstable particle.[1,2] This is so because (1) relies on the assumption that the unstable particle is characterized by a one-dimensional subspace $\mathscr{H}_u$ of the state Hilbert space spanned by ψ. However, translational invariance, together with the fact that momentum operators have purely continuous spectra, requires $\mathscr{H}_u$ to be infinite-dimensional.

Making a physically reasonable choice of this subspace, we have been able in Ref. 2 to derive a consistent decay-law formula [cf. (5) below]. It gives a nondecay probability $P_\psi(t)$ which depends not only on the mass distribution of the initial state ψ, but also on its three-momentum spread. Fortunately, this does not complicate the standard treatment of decays, because the deviations of $P_\psi(t)$ from the simple formula related to (1) are very small provided the initial spatial localization of the particle is not extremely sharp. The estimates performed in Ref. 2 show that this is true in practically every decay experiment; hence the decay-law dependence on kinematical characteristics of the initial state is negligible within the limits of experimental errors.

There is a possible exception, however. It concerns the proton decay which is sought vigorously at present in order to confirm (or to refute) this prediction of various grand unified theories.[3] The recent experimental results give a lower bound of[4] 10^{32} yr on the proton lifetime, which seems to be longer than the prediction of the most popular SU(5) model of grand unification (about 10^{31} yr, however, with large uncertainties[3]). Before claiming it as inadequate, one must be sure that the decay is not suppressed by an additional effect. There have been some speculations on this point recently. In most of them, the decay slowdown is regarded as a consequence of the short-time nonexponentiality of the decay law;[5–9] it appears that the effect is negligible unless some highly questionable assumptions are made.[6,10] Miglietta and Rimini[11] predict that the slowdown region is followed by the region of accelerated decay. Again, this region is experimentally hardly accessible. A possible dynamical mechanism for the slowdown of decay was discussed by Goldhaber, Goldman, and Nussinov.[12]

Another suggestion[13] is that the decay might be modified by the spreading of the proton wave packet which is very fast in the time scale given by the theoretically expected lifetime. Such a possibility cannot be excluded *a priori*, and the present paper is devoted to discussion of this problem. Our treatment is based on the above-mentioned decay-law formula because the latter provides a natural framework in which the effects of the size and shape of the initial wave packet on the decay law can be studied. Such an analysis should yield a more reliable result than the rough argument employed in Ref. 13, which, by the way, leads to an effective proton lifetime as large as 10^{36} yr.

As we have mentioned, the appropriate conclusions of Ref. 2 do not apply to the proton. The estimates which worked effectively for the other unstable particles gave a very weak restriction in this case because of the extremely long lifetime.[14] Since the task of finding better estimates is difficult, one must approach the problem directly by calculating $P_\psi(t)$, or rather the decay probability

$$Q_\psi(t) = 1 - P_\psi(t) \ . \tag{2}$$

This will be done in the following sections for a realistic choice of the mass distribution; in order to simplify the calculations, we neglect spin of the proton and assume the wave packet to be spherically symmetric.

The results can be summarized briefly as follows. The calculated decay probability is actually less than the "theoretical" one, but its decrease is insignificant. Up to higher-order terms, we have[15]

$$Q_\psi(t) \approx \Gamma t [1 - s(g)] \ , \tag{3a}$$

where $s(g)$ is a positive quantity depending on the function g which characterizes the three-momentum distribution of the initial state [cf. (4) below]. A rough estimate yields

$$s(g) \lesssim c(\Delta q)^{-2} , \tag{3b}$$

where Δq is the position spread of ψ and the coefficient c is of order 10^{-28} cm^2. Hence the correction might be substantial only for a proton which is localized initially to a volume of subnuclear size, but in this case the arguments leading to (3) are no longer applicable. Since we are not interested in the protons bound in nuclei,[16] the relations (3) show that the decay under consideration cannot be suppressed by some "kinematical fragmentation" as suggested in Ref. 13. This supports the common opinion that there is no other key to the problem than a thorough investigation of the dynamical mechanism that governs the proton decay.

II. THE DECAY LAW

According to Ref. 2, the proton wave function (in p representation) at the initial time $t=0$ is assumed to be of the form

$$\psi(m,\mathbf{p})=f(m)g(\mathbf{p}) . \tag{4a}$$

In fact, g is a two-component function here, but if we neglect spin of the proton, it can be regarded as a scalar one. For technical reasons, one assumes that g is supported by the ball $B_\epsilon=\{\mathbf{p}:\ |\mathbf{p}|<\epsilon\}$ of a radius ϵ. The function ψ belongs to the Hilbert space $\mathscr{H}$ which is the carrier space of the direct integral of (unitary, irreducible) representations of the Poincaré group referring to the isolated system formed by the proton itself and its decay products.[2] The mass distribution $|f(\ )|^2$ will be specified in Sec. IV.

The subspace $\mathscr{H}_u \subset \mathscr{H}$ which corresponds to the proton alone consists of all functions of the form (4a) with $g \in L^2(B_\epsilon)$. In order to write down the decay law explicitly, one needs the projection E_u onto $\mathscr{H}_u$. Its action can be conveniently expressed using a suitable orthonormal basis in $\mathscr{H}_u$; in this way we obtain[14]

$$P_\psi(t)=\sum_{kl\mu}\left|\int_{m_0}^{\infty}dm\,|f(m)|^2\int_0^\epsilon \frac{p^2dp}{2(m^2+p^2)^{1/2}}\bar{h}_k(p)\int_{4\pi}d\Omega_{\mathbf{p}}Y_{l\mu}(\Omega_{\mathbf{p}})g(\mathbf{p})\exp[-it(m^2+p^2)^{1/2}]\right|^2 , \tag{5}$$

where $Y_{l\mu}$ are the spherical functions, $p=|\mathbf{p}|$, and $\{h_k\}$ is a complete system of functions with supports in $[0,\epsilon]$ which fulfill the orthonormality condition

$$\int_{m_0}^{\infty}dm\,|f(m)|^2\int_0^\epsilon \frac{p^2dp}{2(m^2+p^2)^{1/2}}\bar{h}_j(p)h_k(p)=\delta_{jk} . \tag{6}$$

Suppose now that the initial wave function is spherically symmetric, i.e.,

$$\psi(m,\mathbf{p})=f(m)g(p) , \tag{4b}$$

where $g:[0,\epsilon]\rightarrow\mathbb{C}$. This is certainly an *ad hoc* assumption, but it gives us a possibility to evaluate the decay probability from (5). At the same time, our conclusions are not likely to alter qualitatively if the initial state is not rotationally invariant (or if we take the spin into account). For the wave function (4b), one can simplify (5) using orthonormality of the spherical functions: it yields

$$P_\psi(t)=4\pi\sum_{k}\left|\int_{m_0}^{\infty}dm\,|f(m)|^2\int_0^\epsilon \frac{p^2dp}{2(m^2+p^2)^{1/2}}\bar{h}_k(p)g(p)\exp[-it(m^2+p^2)^{1/2}]\right|^2 . \tag{7}$$

Since they belong to $\mathscr{H}$, the functions $(m,p)\mapsto f(m)h_k(p)$, $f(m)g(p)$ are square-integrable[2] with respect to $dm\otimes p^2dp/2(m^2+p^2)^{1/2}$. The integrations are then interchangeable by the Fubini theorem and we can write

$$P_\psi(t)=4\pi\sum_k\left|\int_0^\epsilon \bar{h}_k(p)g(p)G(p,t)p^2dp\right|^2 , \tag{8}$$

where

$$G(p,t)=\int_{m_0}^{\infty}\frac{|f(m)|^2}{2(m^2+p^2)^{1/2}}\exp[-it(m^2+p^2)^{1/2}]dm . \tag{9}$$

The orthonormality condition (6) now reads

$$\int_0^\epsilon \bar{h}_j(p)h_k(p)G(p,0)p^2dp=\delta_{jk} . \tag{10}$$

This relation suggests yet another reformulation: the decay law (8) equals

$$P_\psi(t)=4\pi\sum_k|\langle h_k,s_t\rangle_G|^2 , \tag{11a}$$

where

$$s_t(p)=g(p)G(p,t)G(p,0)^{-1}$$

[notice that $G(p,0)>0$ in view of (9)] and $\langle\ ,\ \rangle_G$ is the inner product in $L^2([0,\epsilon],\ G(p,0)p^2dp)$. Since $\{h_k\}$ is an orthonormal basis in this Hilbert space according to (10), the Parseval equality yields

$$P_\psi(t)=4\pi\|s_t\|_G{}^2 , \tag{11b}$$

or

$$P_\psi(t)=4\pi\int_0^\epsilon |g(p)|^2|G(p,t)|^2G(p,0)^{-1}p^2dp , \tag{12}$$

where $G(p,t)$ is given by (9). The formula (12) is essential for our following calculations. Since $P_\psi(0)=1$ by definition, the function g has to fulfill the normalization condition

$$4\pi \int_0^{\epsilon} |g(p)|^2 G(p,0) p^2 dp = 1 . \tag{13}$$

Combining the last two relations with (12), we can express the decay probability

$$Q_\psi(t) = 4\pi \int_0^{\epsilon} |g(p)|^2 \frac{G(p,0)^2 - |G(p,t)|^2}{G(p,0)} p^2 dp . \tag{14}$$

III. SIMPLE PROPERTIES OF $P_\psi(t)$

Notice first that the decay law given by (12) is well defined. As we have mentioned, $G(p,0)$ is positive for each $p \in [0,\epsilon]$ unless $f(m)=0$ for almost all m in $[m_0,\infty)$, but this is impossible since ψ is a unit vector in $\mathscr{H}$. In fact, the mass distribution is normalized conventionally as

$$\int_{m_0}^{\infty} |f(m)|^2 dm = 1 ; \tag{15}$$

it fixes the normalization of g through (13). We have $|G(p,t)| \le G(p,0)$ for all p,t, and therefore $0 \le P_\psi(t) \le P_\psi(0) = 1$ as it must be. Since the integrated functions in (9) and (12) have t-independent majorants, the dominated-convergence theorem implies the functions $G(p,\ )$ and $P_\psi(\ )$ to be continuous. In a similar way, one can prove

$$\lim_{t\to\infty} P_\psi(t) = 0 ; \tag{16}$$

it is enough to change the integration variable to $\lambda = (m^2+p^2)^{1/2}$ in (9) and to apply the Riemann-Lebesgue lemma which shows that $G(p,t) \to 0$ as $t \to \infty$.

In the case of the proton decay, however, one had to wait too long before the limit (16) might become physically interesting. The following fact is much more important: the dependence of $P_\psi(t)$ on kinematical characteristics (i.e., on the shape of g) is suppressed if the three-momentum spread of ψ is small enough. In order to see this, consider the scaling transformation $g \to g_\kappa$, $0 < \kappa \le 1$, where

$$g_\kappa(p) = c(\kappa) g(p/\kappa) . \tag{17a}$$

The normalization factor here is obtained from (13) to be

$$|c(\kappa)| = \kappa^{-3/2} \left[4\pi \int_0^{\epsilon} |g(y)|^2 G(\kappa y,0) y^2 dy \right]^{-1/2} . \tag{17b}$$

Then the decay law $P_{\psi,\kappa}(t)$ referring to the rescaled function g_κ is given by

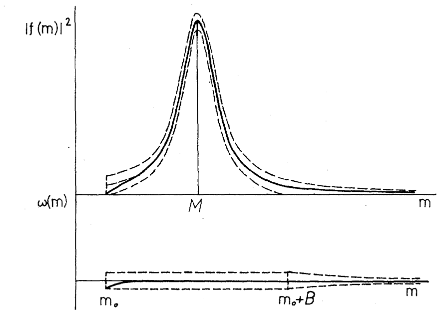

FIG. 1. The mass distribution.

$$P_{\psi,\kappa}(t) = \int_0^{\epsilon} |g(y)|^2 |G(\kappa y,t)|^2 G(\kappa y,0)^{-1} y^2 dy \times \left[\int_0^{\epsilon} |g(y)|^2 G(\kappa y,0) y^2 dy \right]^{-1} . \tag{18}$$

Since the function G is easily seen to be bounded and $g \in L^2(0,\epsilon)$ due to the assumption, one can use the dominated-convergence theorem and perform the limit $\kappa \to 0+$ under the integrals. We obtain

$$\lim_{\kappa\to 0+} G(\kappa y,t) = \int_{m_0}^{\infty} \frac{|f(m)|^2}{2m} e^{-imt} dm$$

so that relation (18) yields

$$\lim_{\kappa\to 0+} P_{\psi,\kappa}(t) = \left| \int_{m_0}^{\infty} \frac{|f(m)|^2}{2m} e^{-imt} dm \right|^2 \left[\int_{m_0}^{\infty} \frac{|f(m)|^2}{2m} dm \right]^{-2} . \tag{19}$$

We see that the limit is independent of g. Moreover, if the mass distribution $|f(\ )|^2$ has a sharp peak around a value M, we can approximate the above expression replacing $(2m)^{-1}$ by $(2M)^{-1}$. It gives

$$\lim_{\kappa\to 0+} P_{\psi,\kappa}(t) \approx \left| \int_{m_0}^{\infty} |f(m)|^2 e^{-imt} dm \right|^2 , \tag{20}$$

i.e., the standard expression related to the formula (1).

The central question is now whether we are near enough to this limit situation in actual experimental arrangements. We will discuss it in the following sections.

IV. CHOICE OF $|f(\ )|^2$

One has to specify first the mass distribution entering into (12) through (9). In any realistic theory, it should be of a more or less Breit-Wigner shape, with the principal contribution resulting from the pole approximation to solution of the full dynamical problem.[1] We choose it in the form

$$|f(m)|^2 = \begin{cases} 0, & m < m_0, \\ N\{(\Gamma/2\pi)[(m-M)^2 + \frac{1}{4}\Gamma^2]^{-1} + \omega(m)\}, & m \ge m_0 , \end{cases} \tag{21}$$

where $M = 938.28$ MeV, further m_0 is the threshold mass and $\Gamma \lesssim 2\times 10^{-60}$ MeV corresponds to the lifetime $T \gtrsim 10^{31}$ yr. The function ω is supposed to obey the following restrictions (Fig. 1):

$$|\omega(m)| \le \begin{cases} (\Gamma/2\pi)[(m-M)^2+\frac{1}{4}\Gamma^2]^{-1}, & m \ge m_0+B, \\ (\Gamma/2\pi)[(m_0-M)^2+\frac{1}{4}\Gamma^2]^{-1}, & |m-M| \le \frac{1}{2}B, \end{cases} \tag{22}$$

where we have denoted $B=2(M-m_0)$. This quantity is of the same order as M, because the lowest open channel in the standard theory[3] is $p \to e^+\pi^0$ which gives $m_0=135.47$ MeV.

One can use (15) and (21) to find the normalization factor N in (21), or more exactly, to derive the estimate

$$|N-1| \le \frac{4\Gamma}{\pi B}+O(\Gamma^2/B^2) . \tag{23}$$

Since $\Gamma/B \lesssim 10^{-63}$, we set $N=1$ in the following. With this approximation, one can write

$$G(p,t)=G_0(p,t)+G_1(p,t) , \tag{24a}$$

where

$$G_0(p,t)=\frac{\Gamma}{2\pi}\int_{m_0}^{\infty}\frac{\exp[-it(m^2+p^2)^{1/2}]}{(m-M)^2+\frac{1}{4}\Gamma^2}\frac{dm}{2(m^2+p^2)^{1/2}} , \tag{24b}$$

$$G_1(p,t)=\int_{m_0}^{\infty}\frac{\omega(m)}{2(m^2+p^2)^{1/2}}\exp[-it(m^2+p^2)^{1/2}]dm . \tag{24c}$$

V. EVALUATION OF $G(p,t)$

Let us first estimate the right-hand side (RHS) of (24c). The inequalities (22) yield

$$\begin{aligned} |G_1(p,t)| &< \frac{\Gamma}{2\pi}\frac{2}{B^2+\Gamma^2}\int_{m_0}^{m_0+B}\frac{dm}{m}+\frac{\Gamma}{4\pi}\int_{m_0+B}^{\infty}\frac{dm}{m(m-M)^2} \\ &\le \frac{\Gamma}{\pi B^2}\ln\left[1+\frac{B}{m_0}\right]+\frac{\Gamma}{4\pi}\left[\frac{1}{M(M-m_0)}+\frac{1}{M^2}\ln\left[1-\frac{M}{m_0+B}\right]\right] \\ &< \frac{\Gamma}{\pi m_0 B}+\frac{\Gamma}{2\pi M}\left[\frac{1}{B}-\frac{1}{2M+B}\right] < \frac{\Gamma}{\pi B}\left[\frac{1}{m_0}+\frac{1}{2B}\right] . \end{aligned}$$

Since $2B+m_0<4M$, we obtain

$$|G_1(p,t)| < \frac{2\Gamma M}{\pi m_0 B^2} . \tag{25}$$

The RHS of (24b) can be evaluated by contour integration. The integrated function represents restriction to $[m_0,\infty)$ of a function which is analytic with the exception of two simple poles at $M\pm(i/2)\Gamma$ and two cuts on the imaginary axis referring to the factor $(m^2+p^2)^{1/2}$. Thus we have (see Fig. 2)

$$G_0(p,t)=-2\pi i\frac{\Gamma}{2\pi}\left.\frac{\exp[-it(m^2+p^2)^{1/2}]}{2(m^2+p^2)^{1/2}[m-M-(i/2)\Gamma]}\right|_{m=M-(i/2)\Gamma}+G_2(p,t) , \tag{26a}$$

$$G_2(p,t)=-\frac{i\Gamma}{2\pi}\int_0^{\infty}\frac{\exp\{-it[(m_0-iz)^2+p^2]^{1/2}\}}{2[(m_0-iz)^2+p^2]^{1/2}}\frac{dz}{(m_0-iz-M)^2+\frac{1}{4}\Gamma^2} . \tag{26b}$$

Here we have used the fact that the integrated function behaves as $(\Gamma/4\pi)|m|^{-3}$ for large $|m|$, and therefore the integral over $C_2(R)$ vanishes in the limit $R\to\infty$.

Our intention is to show that $G(p,t)$ is given essentially by the pole term in (26a). Let us estimate the additional term (26b). We have

$$|(m_0-iz-M)^2+\tfrac{1}{4}\Gamma^2| = (z^2+\tfrac{1}{4}B^2)[1+O(\Gamma^2/B^2)] , \tag{27}$$

where the last term can be again neglected. Furthermore,

$$|(m_0-iz)^2+p^2|^2=z^4+2(m_0{}^2-p^2)z^2+(m_0{}^2+p^2)^2 \equiv F(z^2) .$$

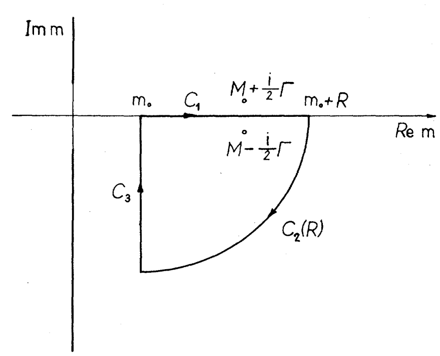

FIG. 2. The integration contour for (24b).

We need to estimate $F(z^2)$ from below. The minimal value $F(y_0)=4p^2m_0{}^2$ is achieved for $y_0=p^2-m_0{}^2$, but we are interested in $F(y)$ for $y=z^2\geq 0$ only. If we therefore restrict our attention to those values of ϵ for which the inequality

$$p\leq\epsilon\leq m_0 \tag{28}$$

holds, then a stronger estimate is possible, namely, $F(y)\geq(m_0{}^2+p^2)^2$, i.e.,

$$|(m_0-iz)^2+p^2|^{1/2}\geq m_0 \ . \tag{29}$$

Finally, $[(m_0-iz)^2+p^2]^{1/2}$ lies in the fourth quadrant for $z\geq 0$ so $|\exp\{-it[(m_0-iz)^2+p^2]^{1/2}\}|\leq 1$ holds for $t\geq 0$. Combining this fact with (26) and (29), we get the inequality

$$|G_2(p,t)|\leq\frac{\Gamma}{4\pi m_0}\int_0^\infty\frac{dz}{z^2+\frac{1}{4}B^2}\ ,$$

which yields the estimate

$$|G_2(p,t)|\leq\frac{\Gamma}{4m_0B}\ . \tag{30}$$

The estimates (25) and (30) together give

$$|G_1(p,t)+G_2(p,t)|<\frac{\Gamma}{m_0B}\left[\frac{2M}{\pi B}+\frac{1}{4}\right]=0.62\frac{\Gamma}{m_0B}\ . \tag{31}$$

We shall show a little later that the modulus of the pole term in (26a) is $\sim(2M)^{-1}$. Taking then the values of M, m_0, and Γ into account, we see that one can neglect the remainder terms and write[17]

$$G(p,t)=\frac{\exp(-it\{[M-(i/2)\Gamma]^2+p^2\}^{1/2})}{2\{[M-(i/2)\Gamma]^2+p^2\}^{1/2}}\ . \tag{32}$$

VI. THE DECAY PROBABILITY

Now one has to insert the calculated $G(p,t)$ into (14). Before doing that, however, it is useful to find a suitable approximative expression to

$$\frac{G(p,0)^2-|G(p,t)|^2}{G(p,0)}=\frac{1-\exp(2t\operatorname{Im}\{[M-(i/2)\Gamma]^2+p^2\}^{1/2})}{2\{[M-(i/2)\Gamma]^2+p^2\}^{1/2}} \tag{33}$$

which would make the integration in (14) easier. Recall that the expression (32) itself is approximative. In view of (31), the corresponding error will not change essentially if we replace the denominator in the RHS of (33) by $2(M^2+p^2)^{1/2}$. Furthermore, we shall assume that

$$\epsilon\ll M\ . \tag{34}$$

In fact, we have already restricted ϵ by (28), where m_0 is at least seven times less than M, but for a reasonable initial localization of the proton, the inequality (34) is fulfilled even much better. In that case, p/M is small too and the square root can be expanded in powers of

$$z=\frac{p^2}{M^2}-i\frac{\Gamma}{M}-\frac{\Gamma^2}{4M^2}$$

so that we obtain

$$2t\operatorname{Im}\{[M-(i/2)\Gamma]^2+p^2\}^{1/2}=-\Gamma t\left[1-\frac{p^2}{2M^2}+\frac{3p^4}{8M^4}+\cdots\right],$$

neglecting again the terms which contain Γ/M. The numerator in the RHS of (33) is then given by

$$\Gamma t\left[1-\frac{p^2}{2M^2}+\frac{3p^4}{8M^4}+\cdots\right]-\tfrac{1}{2}(\Gamma t)^2\left[1-\frac{p^2}{M^2}+\frac{p^4}{M^4}+\cdots\right]+\cdots\ . \tag{35}$$

The second term can be neglected since Γt is very small.[15] If we neglect also the higher powers of p^2/M^2, then

$$\tfrac{1}{2}\{[M-(i/2)\Gamma]^2+p^2\}^{-1/2}\approx\frac{1}{2M}\left[1-\frac{p^2}{2M^2}\right]$$

so the relations (33) and (35) yield

$$\frac{G(p,0)^2-|G(p,t)|^2}{G(p,0)}\approx\frac{\Gamma t}{2M}\left[1-\frac{p^2}{M^2}\right], \tag{36}$$

up to higher-order terms.

Substituting now from (36) to (14), we get

$$Q_\psi(t)\approx\frac{2\pi\Gamma t}{M}\int_0^\epsilon|g(p)|^2\left[1-\frac{p^2}{M^2}\right]p^2dp\ . \tag{37}$$

Furthermore, the normalization condition (13) can be rewritten in the same way as

$$\frac{2\pi}{M}\int_0^\epsilon|g(p)|^2\left[1-\frac{p^2}{2M^2}\right]p^2dp\approx 1\ , \tag{38}$$

again up to the higher-order terms. Combining the last two relations, we can express finally the decay probability in the form

$$Q_\psi(t)\approx\Gamma t\left[1-\frac{\pi}{M^3}\int_0^\epsilon|g(p)|^2p^4dp\right], \tag{39}$$

i.e., we obtain the formula (3a) with $s(g)$ given explicitly.[18]

VII. CONCLUSIONS

The central question now concerns the magnitude of the slowdown coefficient $s(g)$. Let us first derive an upper bound for it. Suppose that the following inequality holds,

$$s(g)\equiv\frac{\pi}{M^3}\int_0^\epsilon|g(p)|^2p^4dp\leq K\ , \tag{40a}$$

then one may multiply its RHS by the expression appearing on the left-hand side of (38), thus obtaining

$$\frac{\pi}{M}\int_0^{\epsilon}|g(p)|^2\left[2K-(1+K)\frac{p^2}{M^2}\right]p^2dp\geq 0\ .$$

This inequality is certainly fulfilled if one requires the term in the large parentheses to be non-negative for $p=\epsilon$. This yields a sufficient condition, namely,

$$K\geq\frac{\epsilon^2}{M^2}\left[2-\frac{\epsilon^2}{M^2}\right]^{-1},$$

under which (40a) is valid. In view of the assumption (34), we get the sought bound,

$$s(g)\lesssim\frac{\epsilon^2}{2M^2}\ll 1\ . \tag{40b}$$

Of course, the value of $s(g)$ is determined by the shape of g, and not only by the three-momentum cutoff ϵ. If, for instance, g has a sharp peak of a width $\eta\ll\epsilon$ and its tail decays rapidly enough in the sense that

$$\int_{\eta}^{\epsilon}|g(p)|^2p^{2k}dp\ll\int_0^{\eta}|g(p)|^2p^{2k}dp,\quad k=1,2\ ,$$

then one can estimate $s(g)$ rather by $\eta^2/2M^2$. On this loose level, therefore, the value of $s(g)$ is determined by the three-momentum spread Δp of ψ,

$$s(g)\lesssim\frac{(\Delta p)^2}{2M^2c^2}\ , \tag{41a}$$

where we have returned to the standard system of units. Continuing this heuristic argument, we express Δp from the uncertainty relation[19] obtaining in this way[20]

$$s(g)\lesssim\frac{3\hbar^2}{8M^2c^2}(\Delta q)^{-2}\ . \tag{41b}$$

Hence we have arrived at the relation (3b); substituting the numerical values we find $c=1.6\times10^{-28}$ cm^2.

It is clear that a more careful analysis is needed if one wants to get some precise statements about the dependence of $s(g)$ on the shape of the wave function. We are convinced, however, that the results of such an effort cannot alter the principal physical conclusion drawn from the above considerations, namely, that the suppression of decay of a free proton due to kinematical effects is unlikely under realistic experimental conditions. We expect also this conclusion to be preserved if one relaxes the technical assumptions made above, i.e., the spherical symmetry and neglect of the spin.

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*Present address: Laboratory of Theoretical Physics, JINR, 141 980 Dubna, U.S.S.R.

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[15]Since $\Gamma t\lesssim 10^{-21}$ during the age of our Universe, the linear approximation to the decay law is pretty good. Recall that the role of small-time nonexponentialities is negligible, as pointed out, e.g., in Ref. 6.

[16]Recall that the decay-law formula (5) which is the backbone of our considerations has been derived for a free unstable particle. Hence our conclusions apply to real experiments only in the cases when the protons involved can be regarded as free in a reasonable approximation; comments on this point are given in Ref. 14. Nuclear effects are considered by L. A. Fernández, R. F. Alvárez-Estrada, and J. L. Sánchez-Gómez, Phys. Rev. D **27**, 2656 (1983).

[17]The estimates have been performed for the threshold mass $m_0\approx 135$ MeV which is given by the standard SU(5) model (Ref. 3, p. 321). If a more complicated Lagrangian is considered, then other decay channels may be open, e.g., $p\rightarrow e^+\nu^c\nu^c$ (Ref. 3, p. 327). However, the increase of M/m_0 to 2×10^3 makes little difference: the RHS of (32) can be again identified with $G(p,t)$ for all practical purposes, a possible error being now of order $\lesssim 10^{-60}$. In fact the space $\mathscr{H}$ should contain the states corresponding not only to the "first" decay, but to all consecutive decays of the decay products as well. This also requires $m_0=m_e$.

[18]The formula (39) can be interpreted as a result of momentum spread and of the relativistic dilatation of lifetime for moving particle. Let τ_0 be the theoretical lifetime at rest. The lifetime of the moving particle with momentum p and mass M is

$$\tau(p)=\tau_0(1-v^2)^{-1/2}=\frac{\tau_0}{M}(p^2+M^2)^{1/2}\ .$$

The mean value over the momentum spread

$$\bar{\tau}=4\pi\int_0^{\epsilon}\tau(p)|g(p)|^2G(p,0)p^2dp$$
$$\approx\tau_0\left[1+\frac{\pi}{M^3}\int_0^{\epsilon}|g(p)|^2p^4dp\right]$$

corresponds just to Eq. (39) in our approximation.

[19]Strictly speaking, the uncertainty relation just says that the dispersion Δp is bounded from below by the appropriate multiple of $(\Delta q)^{-1}$, and hence it is not immediately applicable

here. However, the conventional wisdom which replaces Δp by $\hbar(\Delta q)^{-1}$ in such situations has some point. As a simple example, consider a spinless particle in one-dimensional nonrelativistic quantum mechanics whose wave function in the p representation is of the form $\hat{\phi}(p)=\epsilon^{-1/2}\chi_{[-\epsilon/2,\epsilon/2]}(p)$. In the x representation, $\phi(x)$ has a slowly decaying tail so the dispersion $(\Delta q)_\phi=\infty$. Nevertheless, ϕ forms a distinct peak whose width (measured, e.g., from the distance of the neighboring zeros) is $\sim\hbar/\epsilon$.

[20]In the sense of Ref. 19, Δq here and in (3b) is not necessarily the dispersion, but simply a suitable quantity characterizing the size of the proton wave packet.