Dispersive derivation of the triangle anomaly

Jiří Hořejší*
Nuclear Center, Faculty of Mathematics and Physics, Charles University Prague, V Holešovičkách 2, Praha 8, Czechoslovakia
(Received 5 February 1985)

We present a straightforward generalization of the results of some previous treatments, in which the Adler-Bell-Jackiw anomaly has been recovered with the help of dispersion relations. We consider the absorptive part of the *VVA* triangle diagram with the external momenta *k,p* at vector vertices such that $k^2=p^2\leq 0$ and the fermion mass $m\geq 0$. An integral of the imaginary part of the relevant invariant amplitude is calculated explicitly and shown to produce the desired anomalous contribution to the axial Ward identity. This also enables one to demonstrate the δ-like behavior of such an imaginary part in the limit $k^2\to 0$, $m\to 0$.

The axial anomaly is one of the most intriguing phenomena in quantum field theory. Early observations[1] concerning the peculiar behavior of axial-vector currents were subsequently clarified by Schwinger.[2] The problem was reconsidered later and resolved independently by Adler and by Bell and Jackiw.[3,4] Since then, the anomalous divergence of the axial-vector current or the closely related anomalous Ward identity for the *VVA* triangle diagram have been rederived in many different ways;[5–10] needless to say, our list of relevant literature is far from being complete.

In this Brief Report we will recover the anomalous Ward identity for the *VVA* triangle graph, starting from its absorptive part and using dispersion relations. Some particular calculations along these lines have been performed earlier.[5–7] Here we present a simple generalization of these previous results. The main virtue of such a dispersive approach is that everything can be straightforwardly expressed in terms of *convergent* integrals. In other treatments one deals with divergent quantities which must be given a precise meaning by an explicit regularization.[3,4,8–10] (Of course, one may object that evaluating the triangle graph by means of dispersion relations can be viewed as a unique regularization procedure.)

Let us begin with some definitions and basic formulas. The contribution of the familiar *VVA* triangle diagram (with amputated external legs) is formally given by

$$T_{\alpha\mu\nu}(k,p)=\Gamma_{\alpha\mu\nu}(k,p)+\Gamma_{\alpha\nu\mu}(p,k)\ , \tag{1}$$

$$\Gamma_{\alpha\mu\nu}(k,p)=\int\frac{d^4r}{(2\pi)^4}\mathrm{Tr}\left[\frac{1}{\not r-\not k-m}\gamma_\mu\frac{1}{\not r-m}\gamma_\nu\frac{1}{\not r+\not p-m}\gamma_\alpha\gamma_5\right]\ , \tag{2}$$

where *k,p* are the external momenta outgoing from vector vertices and *m* is the fermion mass. In the sequel we shall also deal with the second-rank pseudotensor $T_{\mu\nu}(k,p)$ which is given by formulas analogous to (1) and (2) with γ_5 replaced by the unit matrix. Our metric and γ-matrix conventions follow that of Bjorken and Drell.[11] The famous result[3,4] for the amplitude (1) and (2) consists in the following: If one imposes the vector Ward identities

$$k^\mu T_{\alpha\mu\nu}(k,p)=p^\nu T_{\alpha\mu\nu}(k,p)=0\ , \tag{3}$$

then the axial Ward identity picks up an anomalous term; i.e.,

$$q^\alpha T_{\alpha\mu\nu}(k,p)=2mT_{\mu\nu}(k,p)+\frac{1}{2\pi^2}\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma\ , \tag{4}$$

where $q=k+p$; the second term on the right-hand side (RHS) of Eq. (4) is just the Adler-Bell-Jackiw anomaly.

For simplicity we shall restrict ourselves to *k,p* such that $k^2=p^2$. The invariant amplitudes (form factors) corresponding to the third-rank Lorentz pseudotensor (1) may be defined as follows (detailed discussion of this point has been given in a preceding paper[12]):

$$\begin{aligned}T_{\alpha\mu\nu}(k,p)=&F_1(q^2;k^2,m^2)\epsilon_{\alpha\mu\nu\rho}(k^\rho-p^\rho)\\&+F_2(q^2;k^2,m^2)(\epsilon_{\alpha\mu\rho\sigma}p_\nu-\epsilon_{\alpha\nu\rho\sigma}k_\mu)k^\rho p^\sigma\\&+F_3(q^2;k^2,m^2)\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma q_\alpha\ .\end{aligned} \tag{5}$$

Note that the notation employed in (5) does not coincide with that of Ref. 12. For $\epsilon_{\mu\nu\rho\sigma}$ we adopt the convention[11] $\epsilon_{0123}=+1$. The second-rank pseudotensor $T_{\mu\nu}(k,p)$ appearing on the RHS of Eq. (4) is described by means of a single form factor *G*, namely,

$$T_{\mu\nu}(k,p)=G(q^2;k^2,m^2)\epsilon_{\mu\nu\rho\sigma}k^\rho p^\sigma\ . \tag{6}$$

The Ward identities (3) and (4) may then be recast consecutively as

$$F_1=k^2F_2 \tag{7}$$

and

$$q^2F_3-2F_1=2mG+\frac{1}{2\pi^2}\ . \tag{8}$$

When the form factors F_j, $j=1,2,3$, or *G* are considered as functions of a complex variable q^2 (at a fixed value of k^2), these should possess a cut along the real axis, extending from $q^2=4m^2$ to infinity.[5–7,13] The corresponding discontinuity of a form factor F_j or *G*, divided by $2i$, will be called its absorptive (imaginary) part and denoted by A_j or *B*, respectively. In order to avoid the cuts with respect to the variable k^2, we shall consider the values $k^2=p^2\leq 0$ in what follows. The functions A_j, $j=1,2,3$, and *B* can be calculated explicitly[5,6,12] [notice that the formula (11.50) in Ref. 7 is incorrect]. It turns out that the unsubtracted dispersion integrals

$$F_j^{(\mathrm{un})}(q^2)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{A_j(t)}{t-q^2}dt,\quad j=1,2,3 \tag{9}$$

and

$$G^{(\mathrm{un})}(q^2)=\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{B(t)}{t-q^2}dt \tag{10}$$

are convergent. This is apparently related[6] to the fact that despite the superficial linear divergence of the integral (2) this is finite when defined with the help of symmetric integration. One may therefore try to define the form factors appearing in (5) and (6) by means of the dispersion relations (9) and (10). The only constraint is the vector Ward identity (7). From (7) and (9) it is easy to see that if we set

$$F_j(q^2;k^2m^2)=F_j^{(\mathrm{un})}(q^2;k^2,m^2),\quad j=1,2,3\ , \tag{11a}$$

$$G(q^2;k^2,m^2)=G^{(\mathrm{un})}(q^2;k^2,m^2)\ , \tag{11b}$$

then (7) is satisfied automatically. Let us stress that this is due to our convenient choice of invariant amplitudes according to (5); there are other options[3,6,12] frequently encountered in current literature, which would necessitate a modification of the definition (11a) through subtractions in the dispersion relations (9) in order to satisfy the requirement (7). Of course, such subtractions have nothing to do with the convergence properties of the integrals (9). Let us also remark that the form factor G may be uniquely calculated directly from the Feynman graph for $T_{\mu\nu}(k,p)$ [cf. Eq. (6)], since the integral obtained from (2) by the replacement $\gamma_5\rightarrow\mathbb{1}$ is perfectly convergent after performing the trace. It can be verified that (10) coincides with the result of such a direct evaluation of G.

From the definitions (9) through (11b) one gets easily for the LHS of the axial Ward identity (8) (taking into account that the absorptive parts obviously satisfy "normal" Ward identities)

$$q^2F_3(q^2;k^2,m^2)-2F_1(q^2;k^2,m^2)=-\frac{1}{\pi}\int_{4m^2}^{\infty}A_3(t;k^2,m^2)dt+2mG(q^2;k^2,m^2)\ . \tag{12}$$

We shall see below that the integral on the RHS of Eq. (12) is convergent. If we want to recover (8), we have to show that

$$\int_{4m^2}^{\infty}A_3(t;k^2,m^2)dt=-\frac{1}{2\pi}\ . \tag{13}$$

The "sum rule" (13) has been verified explicitly by Dolgov and Zakharov[5] for $k^2=0$, $m\neq0$ and by Frishman *et al.*[6] for $k^2<0$, $m=0$. In both cases it is obvious *a priori* (for dimensional reasons) that the integral in (13) is a constant. However, it is not clear how such a statement could be proved on general grounds (using, e.g., some analyticity arguments) in the case when both k^2 and m are nonzero.

Below we shall verify by means of an explicit calculation that Eq. (13) is valid in the case $k^2\leqslant0$, $m\geqslant0$; our result will thus encompass those of the previous treatments.

The function $A_3(q^2;k^2,m^2)$ has been calculated in an earlier paper:[12]

$$A_3(q^2;k^2,m^2)=\frac{1}{2\pi}\frac{2k^2}{q^2}\left[\frac{q^2+2k^2}{(q^2-4k^2)^2}\left(1-\frac{4m^2}{q^2}\right)^{1/2}+\frac{2k^2(q^2-2k^2)}{(q^2)^{1/2}(q^2-4k^2)^{5/2}}\left(\frac{q^2-k^2}{q^2-2k^2}+m^2\frac{q^2-4k^2}{2(k^2)^2}\right)\ln S\right]\ , \tag{14}$$

where

$$S=\frac{q^2-2k^2-[(q^2-4m^2)(q^2-4k^2)]^{1/2}}{q^2-2k^2+[(q^2-4m^2)(q^2-4k^2)]^{1/2}}\ . \tag{15}$$

Introducing the dimensionless variables

$$\beta=\frac{-k^2}{m^2},\quad x=\frac{q^2}{4m^2} \tag{16}$$

we obtain from (14), (15), and (16)

$$\int_{4m^2}^{\infty}A_3(q^2;k^2,m^2)dq^2=-\frac{1}{2\pi}[I_1(\beta)+I_2(\beta)+I_3(\beta)]\ , \tag{17}$$

where

$$I_1(\beta)=\tfrac{1}{2}\beta\int_1^{\infty}\frac{x-\frac{1}{2}\beta}{x(x+\beta)^2}\left(\frac{x-1}{x}\right)^{1/2}dx\ ,$$

$$I_2(\beta)=-\tfrac{1}{4}\beta^2\int_1^{\infty}\frac{x+\frac{1}{4}\beta}{x^{3/2}(x+\beta)^{5/2}}\ln\frac{x+\frac{1}{2}\beta-[(x-1)(x+\beta)]^{1/2}}{x+\frac{1}{2}\beta+[(x-1)(x+\beta)]^{1/2}}\ , \tag{18}$$

$$I_3(\beta)=-\tfrac{1}{2}\int_1^{\infty}\frac{x+\frac{1}{2}\beta}{x^{3/2}(x+\beta)^{3/2}}\ln\frac{x+\frac{1}{2}\beta-[(x-1)(x+\beta)]^{1/2}}{x+\frac{1}{2}\beta+[(x-1)(x+\beta)]^{1/2}}\ .$$

Evaluation of the integrals (18) is elementary and the result is

$$I_1(\beta)=\frac{5}{4}-\frac{2\beta+5}{4\beta}\left(\frac{\beta}{\beta+1}\right)^{1/2}\ln[\beta^{1/2}+(\beta+1)^{1/2}]\ ,$$

$$I_2(\beta)=-\frac{1}{4}+\frac{2\beta^2-3\beta-8}{4\beta^2}\left(\frac{\beta}{\beta+1}\right)^{1/2}\ln[\beta^{1/2}+(\beta+1)^{1/2}]-\frac{2}{\beta}(\beta+1)^{1/2}I_0(\beta)\ , \tag{19}$$

$$I_3(\beta)=\frac{2\beta+2}{\beta^2}\left(\frac{\beta}{\beta+1}\right)^{1/2}\ln[\beta^{1/2}+(\beta+1)^{1/2}]+\frac{2}{\beta}(\beta+1)^{1/2}I_0(\beta)\ ,$$

where

$$I_0(\beta)=\int_{-1}^{1}\frac{dt}{(1+\beta t^2)^{1/2}(\beta t-\beta-2)} .$$

The integral $I_0(\beta)$ can be also easily expressed in terms of elementary functions, but this is not necessary for our purposes. From (19) it is readily seen that

$$I_1(\beta)+I_2(\beta)+I_3(\beta)=1 . \tag{20}$$

Equations (20) and (17) then immediately imply (13) and this is the desired result. Further, from (13) and (14) it easily follows that

$$\lim_{\substack{m\to 0\\ k^2\to 0}} A_3(q^2;k^2,m^2)=-\frac{1}{2\pi}\delta(q^2) . \tag{21}$$

Equation (21) also represents a generalization of the earlier results,[5-7] which in fact established (21) only for particular limiting procedures, namely, $k^2\to 0$ followed by $m\to 0$ and vice versa.

To summarize, Eqs. (13) and (21) [supplemented with the intermediate formula (19)] constitute the main results of the present paper, demonstrating that the earlier calculations[5,6] may be extended in an elementary way.

I am indebted to P. Exner for an enlightening discussion.

---

*Present address: Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141 980 Dubna, U.S.S.R.

[1]H. Fukuda and Y. Miyamoto, Prog. Theor. Phys. **4**, 347 (1949); H. Fukuda *et al.*, *ibid.* **4**, 477 (1949); S. Ozaki, S. Oneda, and S. Sasaki, *ibid.* **4**, 524 (1949); J. Steinberger, Phys. Rev. **76**, 1180 (1949).

[2]J. Schwinger, Phys. Rev. **82**, 664 (1951).

[3]S. Adler, Phys. Rev. **177**, 2426 (1969).

[4]J. Bell and R. Jackiw, Nuovo Cimento **60A**, 47 (1969).

[5]A. D. Dolgov and V. I. Zakharov, Nucl. Phys. **B27**, 525 (1971).

[6]Y. Frishman *et al.*, Nucl. Phys. **B177**, 157 (1981).

[7]K. Huang, *Quarks, Leptons, and Gauge Fields* (World Scientific, Singapore, 1982).

[8]L. Brown, R. Carlitz, and C. Lee, Phys. Rev. D **16**, 417 (1977); R. Jackiw and C. Rebbi, *ibid.* **16**, 1052 (1977).

[9]K. Fujikawa, Phys. Rev. D **21**, 2848 (1980); G. Christos, Z. Phys. C **18**, 156 (1983); G. Siopsis, Caltech. Report No. CALT-68-1187, 1984 (unpublished).

[10]V. N. Gribov, Budapest Report No. KFKI-1981-66, 1981 (unpublished); J. Ambjørn, J. Greensite, and C. Peterson, Nucl. Phys. **B221**, 381 (1983).

[11]J. Bjorken and S. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).

[12]J. Hořejší, Dubna Report No. E2-84-783, 1984 (unpublished).

[13]C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).