Does the return flux result in the Aharonov-Bohm scattering amplitude?

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Introducing a new Aharonov-Bohm-type scattering model, the ambiguity of the incoming wave function is avoided and the single-valued wave function is uniquely determined by proper boundary conditions. In our model, however, the return flux plays an important role in deriving the Aharonov-Bohm scattering amplitude.

In recent years, Aharonov-Bohm (AB) scattering has again attracted the attention of physicists.[1-7] The AB scattering is a subtle subject even from the theoretical point of view. Difficulties result from the model, namely, an infinitely long and impenetrable solenoid.

First, the question of the correct choice of the incoming wave function has been presented by several authors.[3,5-7] In the case of strictly two-dimensional scattering, namely, the scattering of an electron by an infinitely long penetrable or impenetrable[7] current-carrying solenoid, the vector potential outside the solenoid can be written as

$$\mathbf{A}=\frac{\Phi}{2\pi r}\hat{\boldsymbol{\theta}} \ , \tag{1}$$

where $\Phi=\alpha\Phi_0$ is the total flux through the solenoid, $\Phi_0=hc/e$ is the quantum unit of flux, and $\hat{\boldsymbol{\theta}}$ is a unit vector in the azimuthal direction. Because this vector potential is long range, the choice of an incoming plane wave is, strictly speaking, incorrect.[6,8] To avoid this difficulty, a switching-on procedure has been recently considered.[6] Second, the AB scattering is a manifest result of the single valuedness of wave functions.[9] In the case of the scattering of an electron by an impenetrable solenoid (a multiply connected space), both the single-valued and the multiple-valued wave functions are allowed.[10-12] However, it has been previously shown that there is no AB scattering where the multiple-valued wave function[5] is used (the AB effect is still existent[11]).

Unlike the authors in Ref. 6, in the present paper we investigate a stationary scattering of an electron by a vector potential confined to a finite region by an additional magnetic field shell.

Because the magnetic field shell cuts off the tail of the long-range vector potential, we can, of course, choose a plane wave as the incident wave. We also will find out that the single-valued wave function naturally results, due to the existence of the magnetic field shell.

Let us suppose there is an infinitely long, impenetrable current-carrying solenoid with radius R_1. Then let us set up a magnetic field shell (as a thought experiment) around a cylindrical surface with radius R_2 ($R_2>R_1$). The direction of the magnetic field shell is opposite to the direction of the magnetic field which is inside the solenoid. For the sake of simplicity, let us ignore the thickness of the magnetic field shell and suppose that the total flux of the magnetic field shell equals the flux Φ that is inside the solenoid.[13]

In a nonsingular gauge the vector potential is

$$\begin{aligned}&\mathbf{A}^{(1)}=\frac{\Phi}{2\pi r}\hat{\boldsymbol{\theta}}, \quad R_2\geqslant r>R_1\\ &\mathbf{A}^{(2)}=0, \quad r>R_2 \ .\end{aligned} \tag{2}$$

Because of the existence of the magnetic field shell, the kinetic angular momentum is not a conserved quantity.[14] The canonical angular momentum is, however, still conserved. Outside the magnetic field shell ($r>R_2$), the wave function should be a free-particle wave function which is, of course, single valued. In order to satisfy the conservation of canonical angular momentum, in the region $R_2>r>R_1$ the wave function should be also single valued. Therefore, the electron wave function can be expanded as

$$\Psi=\sum_{m=-\infty}^{\infty}\phi_m(k,r)e^{im\theta} \ , \tag{3}$$

where ϕ_m is the mth partial wave.

According to the expansion (3), the effective potential for the mth partial wave is

$$\begin{aligned}&U_m^{(1)}=\frac{2m\alpha}{r^2}+\frac{\alpha^2}{r^2}, \quad R_2\geqslant r>R_1 \ ,\\ &U_m^{(2)}=0, \quad r>R_2 \ .\end{aligned} \tag{4}$$

The mth partial wave function can be obtained as

$$\begin{aligned}\phi_m^{(1)}&(k,r,R_1,R_2,\alpha)\\ &=A_m[J_{|m+\alpha|}(kr)N_{|m+\alpha|}(kR_1)\\ &\qquad -N_{|m+\alpha|}(kr)J_{|m+\alpha|}(kR_1)] \ ,\\ &\qquad\qquad \text{for } R_2\geqslant r>R_1 \ ,\end{aligned} \tag{5}$$

and

$$\phi_m^{(2)}(k,r,R_1,R_2,\alpha)=B_mJ_m(kr)+C_mN_m(kr) \ , \quad \text{for } r>R_2 \ . \tag{6}$$

Matching the boundary condition at $r=R_2$ (Ref. 14) and considering the wave-function behavior in the asymptotic region, we get

$$\begin{aligned}\Psi \underset{r\to\infty}{\sim} & e^{-ikr\cos\theta}+f(k,R_1,R_2,\alpha,\theta)\frac{e^{ikr}}{\sqrt{r}}\\ &=\sum_{m=-\infty}^{\infty}\left[(-i)^mJ_m(kr)e^{im\theta}+f_m(k,R_1,R_2,\alpha)\frac{e^{ikr}}{\sqrt{r}}e^{im\theta}\right] \ .\end{aligned} \tag{7}$$

The scattering amplitude for the mth partial wave is obtained as

$$f_m(k,R_1,R_2,\alpha)=(-1)^{m+1}\left[\frac{1}{2\pi k}\right]^{1/2}e^{-i\pi/4}\frac{2iD_1}{D_1+D_2} \ , \tag{8}$$

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where

$$D_1 = N_{|m+\alpha|}(kR_1)\Delta_1 - J_{|m+\alpha|}(kR_1)\bar{\Delta}_1 \ , \quad (9)$$

$$D_2 = N_{|m+\alpha|}(kR_1)\Delta_2 - J_{|m+\alpha|}(kR_1)\bar{\Delta}_2 \ , \quad (10)$$

$$\Delta_1 = J_{|m+\alpha|}(kR_2)N_m'(kR_2) - N_m(kR_2)J_{|m+\alpha|}'(kR_2) \ , \quad (9a)$$

$$\bar{\Delta}_1 = N_{|m+\alpha|}(kR_2)N_m'(kR_2) - N_m(kR_2)N_{|m+\alpha|}'(kR_2) \ , \quad (9b)$$

$$\Delta_2 = J_m(kR_2)J_{|m+\alpha|}'(kR_2) - J_{|m+\alpha|}(kR_2)J_m'(kR_2) \ , \quad (10a)$$

$$\bar{\Delta}_2 = J_m(kR_2)N_{|m+\alpha|}'(kR_2) - N_{|m+\alpha|}(kR_2)J_m'(kR_2) \ . \quad (10b)$$

J_m and N_m are the usual Bessel and Neumann functions.

The physical significance of the scattering amplitude (8) is that both the source of the electrons and the detector should be located outside the magnetic field shell. In order to compare our result with the AB scattering amplitude, we investigate the limit of the scattering amplitude (8) for large R_2.

In the limit $kR_2 \to \infty$ (large R_2 and high energy) and using the asymptotic behavior of the Bessel function

$$J_\nu(kr) \underset{kr\to\infty}{\sim} \left[\frac{2}{\pi kr}\right]^{1/2} \cos[kr - (\nu + \tfrac{1}{2})\pi/2] \ , \quad (11)$$

$$N_\nu(kr) \underset{kr\to\infty}{\sim} \left[\frac{2}{\pi kr}\right]^{1/2} \sin[kr - (\nu + \tfrac{1}{2})\pi/2] \ , \quad (12)$$

the scattering amplitude (8) reduces to

$$f_m \underset{kR_2\to\infty}{\sim} f_m^{\mathrm{AB}}(k,\alpha) - (-1)^m \left[\frac{1}{2\pi k}\right]^{1/2} \times e^{-i\pi/4} \frac{2e^{2i\delta_m(\alpha)} J_{|m+\alpha|}(kR_1)}{H_{|m+\alpha|}^{(1)}(kR_1)} \ . \quad (13)$$

This result is just the scattering amplitude by the impenetrable solenoid with radius R_1,[7] where

$$f_m^{\mathrm{AB}}(k,\alpha) = \frac{(-1)^m}{\sqrt{2\pi k}} e^{-i\pi/4}(e^{2i\delta_m(\alpha)} - 1) \quad (14)$$

is the AB scattering amplitude by the line flux, and

$$\delta_m(\alpha) \equiv \frac{\pi}{2}(|m| - |m+\alpha|) = \begin{cases} -\dfrac{\pi\alpha}{2} & \text{for } m \geq 0 \ , \\ \dfrac{\pi\alpha}{2} & \text{for } m < 0 \ ; \end{cases} \quad (15)$$

$H^{(1)}$ is the Hankel function.

The Aharonov-Bohm scattering amplitude (14), which is interpreted as a scattering of an electron by the vector potential,[9] is, however, an extreme case of our scattering amplitude in the limit $R_1 \to 0$. Using our model, the AB scattering amplitude is derived without any ambiguity. The price of doing so, however, is that an additional magnetic field accessible to the electron is introduced. The role of the magnetic field shell is clear. When the incident electron passes through the shell it applies a torque to the electron. Therefore, the kinetic angular momentum of the electron is shifted from $m\hbar$ to $(m-\alpha)\hbar$. Correspondingly, the radial wave function for the mth partial wave changes from $J_m(kr)$ to $J_{|m+\alpha|}(kr)$.[15] This change results in the phase shift δ_m in the asymptotic region. It has been pointed out that the AB scattering as well as the bound-state AB effect is due to the shift of kinetic angular momentum.[16] In our model the shift of kinetic angular momentum is from the magnetic field shell. If we consider the shell as the return flux of a finite but long solenoid as a simplification, the AB scattering amplitude can be understood as a result of return flux, at least in our model.

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[13]The infinite intensity of the magnetic field at $r = R_2$, which is due to the discontinuity of the vector potential **A**, should not puzzle readers who are familiar with the scattering by a rectangular potential. Let us, for example, consider the scattering of an electron by a thin magnetic field layer which is perpendicular to the incoming velocity of the electron. For a sufficiently high incoming velocity of the electron, it is easy to find out in classical mechanics

$$\sin\theta = \frac{e\Phi}{\mu c V_{\mathrm{in}}} \ ,$$

where μ is the mass of electron, Φ is flux per unit length of the layer, and V_{in} is the incoming velocity of the electron. The scattering angle θ between the incoming and outgoing velocities of the electron is dependent on the flux only. The thickness of the magnetic field layer can be zero.

[14]The boundary conditions are

$$\phi_m^{(1)}(r)|_{r=R_2} = \phi_m^{(2)}(r)|_{r=R_2} \ ,$$

$$\left.\frac{d\phi_m^{(1)}(r)}{dr}\right|_{r=R_2} = \left.\frac{d\phi_m^{(2)}(r)}{dr}\right|_{r=R_2} \ ,$$

which guarantee that the density $\rho = \psi^*\psi$ and

$$J_r = \frac{\hbar}{2mi}\left[\psi^*\frac{\partial}{\partial r}\psi - \psi\frac{\partial}{\partial r}\psi^*\right] - \frac{e}{mc}A_r(\mathbf{r})\psi^*\psi$$

are continuous. However,

$$J_\theta = \frac{\hbar}{2mi}\left[\psi^*\frac{1}{r}\frac{\partial}{\partial\theta}\psi - \psi\frac{1}{r}\frac{\partial}{\partial\theta}\psi^*\right] - \frac{e}{mc}A_\theta(\mathbf{r})\psi^*\psi$$

is discontinuous at the boundary $r=R_2$; correspondingly, the kinetic angular momentum is not conserved. It is easy to understand that because of the magnetic field layer which would apply a torque to a particle passing through the boundary, the kinetic angular momentum of the particle will be changed.

[15]In the limit that $R_1 \rightarrow 0$ and R_2 is large, the wave function for the mth partial wave in the scattering region $\propto J_{|m+\alpha|}(kr)$ (see Ref. 7).

[16]See, for example, Lindsay J. Tassie and Murray Peshkin, Ann. Phys. (N.Y.) **16**, 177 (1961): "The direct effect of the inaccessible field is to shift the allowed values of $(r \times mv)_z$."