PHYSICAL REVIEW D VOLUME 31, NUMBER 4 15 FEBRUARY 1985

Higher-dimensional extensions of Bianchi type-I cosmologies

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(Received 23 July 1984; revised manuscript received 24 September 1984)

We present some new solutions of the Bianchi type-I space-time in $N=1+d+D$ dimensions. Exact solutions are given both in the vacuum as well in the perfect-fluid case, including a nonvanishing cosmological constant.

I. INTRODUCTION

Higher-dimensional cosmologies are of some interest in view of the modern Kaluza-Klein theories.[1] Such cosmologies have been first investigated by Forgács and Horváth[2] and Chodos and Detweiler.[3] With these models, many interesting results were obtained using the idea that the size of the extra dimensions is small, as required by the physics in the real universe (for further reference, see, e.g., Refs. 4 and 5). However, most of these models are free of matter and therefore cannot describe the behavior under the influence of matter. A first step in the direction of constructing some more realistic models has been recently obtained by Bergamini and Orzalesi.[6] The resulting model is a Bianchi type-I dust solution in $N=4+d$ dimensions. Another solution was given by Sahdev[7] for the radiation-filled Kantowski-Sachs model, which is closely related to the Bianchi type-III model.

In general, in cosmology we are interested in perfect-fluid solutions with equations of state $p=0$ (dust), $p=\epsilon$ (stiff matter), and $p=\epsilon/(N-1)$ (radiation), where p and ϵ are, respectively, the pressure and energy density of matter. In what follows we construct higher-dimensional cosmological Bianchi type-I solutions both in the vacuum case ($\epsilon=0$) as well as in the perfect-fluid case. In addition we consider also models with a nonvanishing cosmological constant Λ.

II. FIELD EQUATIONS AND SOLUTIONS

In choosing a local orthonormal basis σ^μ, we can put the metric of space-time in the form

$$ds^2=\eta_{\mu\nu}\sigma^\mu\sigma^\nu \ , \tag{1}$$

where $\eta_{\mu\nu}=(-1,1,\ldots,1)$ is the Minkowski-metric tensor and $\mu,\nu=0,1,\ldots,N$. We assume $N=1+d+D$. For spatially homogeneous models of Bianchi type-I we have

$$\sigma^0=dt,\ \sigma^i=r_i\omega^i,\ \sigma^n=R_n\omega^n \ \text{(no sum)} \ , \tag{2}$$

where $r_i=r_i(t)$, $R_n=R_n(t)$ are the cosmic scale functions on M^d,M^D, respectively, $i=1,\ldots,d$; $n=d+1,\ldots,d+D$, and $\omega^j=dx^j$ are the time-independent differential forms.

The field equations to be solved are

$$R_{\mu\nu}=[2/(N-2)]\Lambda g_{\mu\nu}+[1/(N-2)](\epsilon-p)g_{\mu\nu} + (\epsilon+p)u_\mu u_\nu \ , \tag{3}$$

where $R_{\mu\nu}$ denotes the Ricci tensor, $g_{\mu\nu}$ the metric tensor on M^N, Λ the N-dimensional cosmological constant, u_μ the velocity N-vector, and ϵ and p are, respectively, the energy density and pressure of the perfect-fluid matter. The N-dimensional gravitational constant $G_N=G_0\Omega_{d+D-3}$ has been absorbed in the definition of ϵ. We consider only nontilted models, i.e., $u_\mu=\delta^0_\mu$.

In the following we are interested in perfect-fluid solutions with equations of state

$$\epsilon=0,\ \epsilon=p,\ p=0,\ p=\epsilon/(N-1) \tag{4}$$

(vacuum, stiff matter, dust, radiation). The perfect-fluid matter obeys the conservation law

$$\dot\epsilon+(\epsilon+p)(dh+DH)=0 \ , \tag{5}$$

where $h=(\ln r)^\cdot$, $H=(\ln R)^\cdot$ are the Hubble parameters, $r=r_d$, $R=R_D$, and $(\ )^\cdot=d/dt$.

Using our new method of reduction of the field equations (3) (see Ref. 8 for details) we obtain the decoupled equations

$$\dot h+h(\ln g)^\cdot=(2\Lambda+\epsilon-p)/(N-2) \ , \tag{6a}$$

$$\dot H+H(\ln g)^\cdot=(2\Lambda+\epsilon-p)/(N-2) \ , \tag{6b}$$

$$\ddot g=g(2\Lambda+\epsilon-p)(N-1)/(N-2) \ , \tag{6c}$$

$$(\ln g)^{\cdot 2}-(dh^2+DH^2)=2(\Lambda+\epsilon) \ , \tag{6d}$$

where $g=r^dR^D$. After specifying the equation of state (4) we obtain from (5) and (6c) a second-order equation for $g=g(t)$, which may be easily solved. The solutions are then completed by solving (6a) and (6b) for $r=r_d$ and $R=R_D$. The constraints on the integration constants are given by Eq. (6d).

In particular, we obtain the following explicit solutions:

Vacuum:

(i) $\Lambda=0$:

$$r_d=\tilde r_d t^{p_d},\quad R_D=\tilde R_D t^{p_D} \ ,$$
$$dp_d+Dp_D=1,\quad dp_d{}^2+Dp_D{}^2=1,\quad g=at \ ; \tag{7a}$$

(ii) $\Lambda\neq0$:

(a) $g=a\sinh(ut)$, $\Lambda>0$:

$$r_d=\tilde r_d[\sinh(ut)]^{1/(N-1)}[\tanh(ut/2)]^{p_d} \ ,$$
$$R_D=\tilde R_D[\sinh(ut)]^{1/(N-1)}[\tanh(ut/2)]^{p_D} \ ,$$
$$dp_d+Dp_D=0,\quad dp_d{}^2+Dp_D{}^2=2\Lambda/u^2 \ ,$$
$$u^2=2\Lambda(N-1)/(N-2) \ ; \tag{7b}$$

(b) $g = a\sin(ut)$, $\Lambda < 0$:

$$r_d = \tilde{r}_d[\sin(ut)]^{1/(N-1)}[\tan(ut/2)]^{p_d} \ ,$$
$$R_D = \tilde{R}_D[\sin(ut)]^{1/(N-1)}[\tan(ut/2)]^{p_D} \ ,$$
$$dp_d + Dp_D = 0, \quad dp_d{}^2 + Dp_D{}^2 = 2\Lambda/u^2 \ ,$$
$$u^2 = -2\Lambda(N-1)/(N-2) \ ; \tag{7c}$$

(c) $g = a\exp(ut)$:

$$r_d = R_D = \tilde{r}_d \exp(vt) \ ,$$
$$u^2 = 2\Lambda(N-1)/(N-2), \quad v^2 = 2\Lambda/(N-1)(N-2) \ . \tag{7d}$$

Dust:

(i) $\Lambda = 0$:

$$g = at(t+t_0) \ ,$$
$$r_d = \tilde{r}_d t^{p_d}(t+t_0)^{2/(N-1)-p_d} \ ,$$
$$R_D = \tilde{R}_D t^{p_D}(t+t_0)^{2/(N-1)-p_D} \ ,$$
$$dp_d + Dp_D = 1, \quad dp_d{}^2 + Dp_D{}^2 = 1, \quad \epsilon = M/g \ ; \tag{8a}$$

(ii) $\Lambda \neq 0$:

(a) $g = a\sinh(ut) + b[\cosh(ut) - 1]$, $\Lambda > 0$:

$$r_d = \tilde{r}_d\{a\sinh(ut) + b[\cosh(ut)-1]\}^{1/(N-1)-p_d} \times [\cosh(ut)-1]^{p_d/u} \ ,$$
$$R_D = \tilde{R}_D\{a\sinh(ut) + b[\cosh(ut)-1]\}^{1/(N-1)-p_D} \times [\cosh(ut)-1]^{p_D/u} \ ,$$
$$dp_d + Dp_D = 0, \quad dp_d{}^2 + Dp_D{}^2 = 2\Lambda/u^2 \ ,$$
$$u^2 = 2\Lambda(N-1)/(N-2), \quad b = M/2\Lambda, \quad \epsilon = M/g \ ; \tag{8b}$$

(b) $g = a\sin(ut) + b[\cos(ut) - 1]$, $\Lambda < 0$:

$$r_d = \tilde{r}_d\{a\sin(ut) + b[\cos(ut)-1]\}^{1/(N-1)-p_d} \times [\cos(ut)-1]^{p_d/u} \ ,$$
$$R_D = \tilde{R}_D\{a\sin(ut) + b[\cos(ut)-1]\}^{1/(N-1)-p_D} \times [\cos(ut)-1]^{p_D/u} \ ,$$
$$dp_d + Dp_D = 0, \quad dp_d{}^2 + Dp_D{}^2 = 2\Lambda/u^2$$
$$u^2 = -2\Lambda(N-1)/(N-2), \quad b = M/2\Lambda, \quad \epsilon = M/g \ ; \tag{8c}$$

(c) $g = a\exp(ut) + c$,

$$r_d = \tilde{r}_d \exp(up_d t)[a\exp(ut) + c]^{1/(N-1)-p_d} \ ,$$
$$R_D = \tilde{R}_D \exp(up_D t)[a\exp(ut) + c]^{1/(N-1)-p_D} \ ,$$
$$dp_d + Dp_D = 1, \quad dp_d{}^2 + Dp_D{}^2 = 1 + 2\Lambda/u^2 \ ,$$
$$u^2 = 2\Lambda(N-1)/(N-2), \quad c = -M/2\Lambda, \quad \epsilon = M/g \ . \tag{8d}$$

Radiation:

$\Lambda = 0$:

$$g^{(N-2)/(N-1)} = L^2\sinh^2(2\eta) \ ,$$
$$dt/d\eta = 4L^{N/(N-2)}[(N-1)/C(N-2)] \times [\sinh(2\eta)]^{N/(N-2)} \ ,$$
$$C^2 = 2M(N-1)/(N-2), \quad L = \text{const} \ ,$$
$$r_d = \tilde{r}_d[\sinh(2\eta)]^{2/(N-2)}[\tanh(\eta/2)]^{p_d} \ ,$$
$$R_D = \tilde{R}_D[\sinh(2\eta)]^{2/(N-2)}[\tanh(\eta/2)]^{p_D} \ ,$$
$$dp_d + Dp_D = 0, \quad dp_d{}^2 + Dp_D{}^2 = 4(N-1)/(N-2) \ ,$$
$$\epsilon = M/g^{N/(N-1)} \ . \tag{9}$$

Stiff matter:

(i) $\Lambda = 0$:

$$r_d = \tilde{r}_d t^{p_d}, \quad R_D = \tilde{R}_D t^{p_D} \ ,$$
$$dp_d + Dp_D = 1, \quad dp_d{}^2 + Dp_D{}^2 = 1 - 2M/u^2 \ ,$$
$$g = ut, \quad \epsilon = M/g^2 \ ; \tag{10a}$$

(ii) $\Lambda \neq 0$:

(a) $g = a\sinh(ut)$, $\Lambda > 0$:

$$r_d = \tilde{r}_d[\sinh(ut)]^{1/(N-1)}[\tanh(ut/2)]^{p_d} \ ,$$
$$R_D = \tilde{R}_D[\sinh(ut)]^{1/(N-1)}[\tanh(ut/2)]^{p_D} \ ,$$
$$dp_d + Dp_D = 0, \quad dp_d{}^2 + Dp_D{}^2 = 2\Lambda/u^2 - 2M/(au)^2 \ ,$$
$$u^2 = 2\Lambda(N-1)/(N-2), \quad \epsilon = M/g^2 \ ; \tag{10b}$$

(b) $g = a\sin(ut)$, $\Lambda < 0$:

$$r_d = \tilde{r}_d[\sin(ut)]^{1/(N-1)}[\tan(ut/2)]^{p_D} \ ,$$
$$R_D = \tilde{r}_d[\sin(ut)]^{1/(N-1)}[\tan(ut/2)]^{p_D} \ ,$$
$$dp_d + Dp_D = 0, \quad dp_d{}^2 + Dp_D{}^2 = 2\Lambda/u^2 - 2M/(au)^2 \ ,$$
$$u^2 = -2\Lambda(N-1)/(N-2), \quad \epsilon = M/g^2 \ , \tag{10c}$$

where $\tilde{r}_d$, $\tilde{R}_D$, p_d, p_D, a, b, t_0, M are constants.

III. CONCLUSION

We have given a complete discussion of the $N = 1 + d + D$-dimensional Bianchi type-I field equations in the vacuum case as well as for various perfect-fluid matter including a nonvanishing cosmological constant. However, it seems to be impossible to integrate the field equations in the radiation case with $\Lambda \neq 0$. In addition, we have to introduce the new time parameter η in the radiation case in order to decouple the corresponding field equations. By setting $D = 0$ we obtain from (7a) the Chodos-Detweiler-Kasner solution.[3] The special case $d = 3$, $D = 1$ has also been discussed by Alvarez and Gavela.[9] The dust solution (8a) in $N = 4 + d$ dimensions has been recently obtained by Bergamini and Orzalesi[6] in a somewhat different form. The method of solving the dust field equations used by these authors is based on the generalization of the four-dimensional approach due to Raychaudhuri.[10] Our higher-dimensional dust solution (8a) is a direct generalization of the solution first given by Schücking and Heckmann.[11,12] (See also Ref. 13.)

We finally remark that the boundary conditions of our solutions (7b), (7c), (8a), (8b), (10a), (10b), and (10c) are $r_d = R_D = 0$ at the initial singularity $t = 0$.

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