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Finite-temperature deconfinement and chiral-symmetry restoration at strong coupling

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We derive from a d-dimensional lattice gauge theory at finite temperature a $(d-1)$-dimensional effective action in which the dynamical variables are Wilson lines and meson and baryon fields. Analysis of this model shows a first-order deconfinement transition for all values of the bare quark mass, and a second-order chiral transition at a higher temperature for zero bare quark mass. Reasonable values for the two transitions and the hadronic mass spectrum are obtained.

I. INTRODUCTION

Chiral-symmetry breaking is a necessary part of the modern picture of strong-interaction physics. While successful predictions based on chiral-symmetry breaking are almost three decades old, a fundamental understanding of this phenomenon and its derivation from QCD has been elusive. Useful information about chiral symmetry has been obtained, however, using the techniques of lattice gauge theory. Finite-temperature chiral-symmetry restoration is of particular interest at this time, because of its possible relevance to both the physics of heavy-ion collisions and the study of the early universe. This paper discusses the restoration of chiral symmetry at high temperature using strong-coupling methods applied to lattice-gauge-theory models. The loss of confinement at high temperature is also considered here, as the two phenomena are interrelated.

In the remainder of this Introduction, we provide a brief summary of previous work relevant to this one, followed by a discussion of the guiding principles which lead to the model studied here. The second section is a detailed derivation of our model, while the third analyzes the predictions which can be derived from it. A final section summarizes our results.

There are various arguments that suggest that confinement (defined in a variety of ways) necessarily implies chiral-symmetry breaking. This might be thought to imply that chiral-symmetry restoration cannot occur before the deconfinement transition. However, for reasons to be explained shortly, a precise and meaningful general result of this type is difficult to state, much less prove. Nevertheless, it is useful to review what is known.

A simple, intuitive argument that confinement necessarily implies chiral-symmetry breaking has been given by Casher.[1] The basic idea is that, in a semiclassical picture, a spin-independent confining force can change the momentum of a massless quark, but not its spin. Thus, helicity can flip, but this implies chirality is not a good quantum number, so chirality is spontaneously broken.

More formal arguments have been given for the large-N limit, where N is the number of colors. In the zero-temperature case, Coleman and Witten[2] have shown that confinement implies chiral-symmetry breaking; their proof uses the triangle anomaly. For lattice gauge theories at finite temperature, a proof that confinement in the large-N limit implies chiral-symmetry breaking has been given using the Schwinger-Dyson equations.[3]

Chiral-symmetry breaking in the strong-coupling limit of lattice gauge theories at zero temperature has been extensively studied. The earliest results were obtained for the Hamiltonian formalism, exploiting an analogy with quantum Heisenberg antiferromagnets.[4–6] The earliest work in the Lagrangian formalism used graphical techniques,[7] but functional methods have proven very powerful and flexible. The two major approaches have been expansions in $1/N$ (Refs. 8–10) and $1/d$ (Refs. 11,12), where N and d are the number of colors and space-time dimensions, respectively. All of the methods have demonstrated that $\bar{\psi}\psi$ develops a nonvanishing vacuum expectation value. When chiral symmetry is a continuous symmetry, pions are massless Goldstone bosons in the limit of zero bare quark mass. When a small bare quark mass is turned on, the usual PCAC (partial conservation of axial-vector current) result is obtained.

In the following discussion, it will be necessary to have a precise working definition of the chiral and deconfining transitions. What do we mean when we speak of the deconfining transition or the chiral transition? Let us first consider the deconfining transition. For a pure SU(N) gauge theory, without quarks, there is very little doubt that a deconfining phase transition occurs. The order parameter for this transition is the Wilson line, which can be interpreted as the exponential of minus the self-energy of an infinitely massive quark. A global Z_N symmetry requires that the expectation value of the Wilson line be zero, which implies confinement.[13] A rigorous proof that deconfinement occurs in lattice gauge theories (without quarks) at sufficiently high temperature has been constructed,[14] but Monte Carlo simulation[15] provides the best evidence that such a transition might occur in nature, and at a realistic temperature.

It is possible to gain some understanding of the deconfinement transition using a picture developed by Svetitsky and Yaffe,[16] in which the Wilson lines are considered as SU(N)-valued spins. Effective spin models of this type can be derived from lattice gauge theories with some approximations.[17–19] These models can be studied using Monte Carlo simulation or mean field theory, with results for the critical behavior in very good agreement with

Monte Carlo results for the underlying lattice gauge theories.

The inclusion of fermions explicitly breaks the Z_N symmetry of a pure SU(N) gauge theory, so that the Wilson line is generally nonzero. As pointed out by Banks and Ukawa,[20] the effect of fermions on the Wilson lines regarded as spins can be viewed as that of an external magnetic field. In the limit of large fermion mass, this effective external field is proportional to m^{-n_t}, where m is the fermion mass and n_t is the timelike extent of the lattice. For very large values of the mass, it is expected that the deconfining phase transition will still occur: there will be a line of phase transitions, perhaps of the same order as the pure-gauge-theory transition, connecting smoothly with it as the fermion mass goes to infinity. In the case of SU(3), which has a first-order transition, the expectation value of the Wilson line will jump discontinuously from one nonzero value to another. We will refer to all phase transitions on such lines as deconfining transitions, even though the question of confinement is not meaningful once dynamical quarks are included.

It appears that the deconfining transition can disappear if sufficiently light quarks are present in a gauge-theory model. This can be understood using the spin-model picture; for sufficiently strong external fields, thermodynamic quantities are analytic in the temperature. Monte Carlo results for $n_t=2$ in which fermion effects were simply approximated indicated that sufficiently light quarks might destroy the deconfining transition.[21] However, the latest results,[22,23] using larger lattices ($n_t=4$) and better approximations for the fermion determinant, do show a persistence of the deconfining transition, perhaps even for zero bare quark mass.

Like the deconfining transition proper, the chiral-symmetry-breaking transition is associated with a global symmetry which nature does not really have: in nature, quarks have nonzero bare mass. The order parameter for this transition is the fermion bilinear $\bar{\psi}\psi$, whose expectation value must be zero if chiral symmetry is unbroken. Even though chiral symmetry is lost when fermions have nonzero bare mass, the transition may persist. We will refer to all phase transitions on a line of critical points connected to the chiral-symmetry transition point as chiral transitions.

In order to understand these phase transitions, we can view the phase structure of a gauge theory coupled to fermions at finite temperature in the temperature—quark-mass plane. Something is known about the large-m and small-m limits, but not much about the region of intermediate m. It is easy to see that in nature there may be two, one, or zero phase transitions. If there is but one, it may be a deconfining or chiral transition, or both. In addition, even if neither transition occurs in nature, their effect may show up in rapid fluctuation of thermodynamic quantities if such transitions are nearby in the T-m plane.

Our basic tool for the exploration of the phase structure is a $(d-1)$-dimensional effective action for strong-coupling lattice gauge theories in which the dynamical variables are meson and baryon fields and Wilson lines. The fundamental ingredients used to arrive at this effective action are as follows:

(1) The pure gauge field piece of the action gives rise to an effective interaction between Wilson lines. All other effects of this part of the action are dropped, as in previous work on the zero-temperature strong-coupling limit.

(2) Composite fields are introduced with the quantum numbers of mesons and baryons. We assume that these fields are time-independent, in order to explicitly evaluate certain functional determinants. This assumption has no effect on the zero-temperature consequences of the model, and is certainly reasonable at high temperature.

(3) Given the above, the integrals over the quark fields can be done exactly, as a set of one-dimensional functional determinants. These determinants couple the meson fields to the Wilson lines.

Our simple picture is that mesons are highly localized in space, as a consequence of the strong-coupling limit, but not necessarily in time. When the Wilson lines take on a constant nonzero vacuum expectation value, quark-antiquark pairs are no longer bound together in the timelike direction. As will be seen later, this makes possible chiral-symmetry restoration. On the other hand, timelike quark loops explicitly break the Z_N symmetry of the Wilson lines. Thus the mechanisms for the two transitions are coupled.

Before ending this section, we would like to comment on two interesting issues. The first is that confinement persists in the $(d-1)$-spatial dimensions even at temperatures above the deconfining transition. In the pure gauge theory, the deconfining phase transition is associated with the spontaneous breakdown of a global symmetry of the timelike link variables. Similar symmetries exist if space as well as time is made periodic. However, for large spatial dimensions, these symmetries are not broken. To put it in a slightly different language, timelike Wilson lines having a nonzero expectation value in no way implies that spatial Wilson loops do not have area-law behavior. It is easy to check this in the strong-coupling region.

The other issue is whether the inclusion of timelike quark loops is necessary at all. Because the breakdown of chiral symmetry can be understood using an effective action containing only meson fields,[12] it might be thought that chiral-symmetry restoration at finite temperature could be understood in the same way. The basic idea behind this is that one-loop finite-temperature effects from the functional determinant of the meson propagator can change the form of the effective action; this mechanism has been much discussed for Higgs bosons. There are two reasons why this approach is inadequate, one conceptual and one technical.

As discussed previously, the introduction of fermions explicitly breaks a global Z_N symmetry of the pure gauge theory. The strength of the symmetry breaking depends on the mass of the quarks. However, it is not clear which mass should be used as a parameter, the bare quark mass or the constituent quark mass given by chiral-symmetry breaking. If the latter, then chiral-symmetry restoration will affect the strength of Z_N symmetry breaking. As will be seen later, it is the constituent mass which is relevant. In other words, the effective action for the Wilson line is a function of the fermion condensate $\langle\bar{\psi}\psi\rangle$. This implies that in an effective action in which both

$\langle \bar{\psi}\psi \rangle$ and the Wilson line are independent variables, their behavior will affect one another. This gives an explicit means for the chiral dynamics to be aware of confinement or lack thereof.

The second reason is a technical one. Previous workers on chiral-symmetry breaking at strong coupling have derived effective potentials for $\langle \bar{\psi}\psi \rangle$ which contain a term of the form $\ln\langle \bar{\psi}\psi \rangle$ for zero bare mass. This term never allows the chirally symmetric solution $\langle \bar{\psi}\psi \rangle = 0$ to occur. Our solution to this problem is simple: we include exactly the effects of quark propagation in the timelike direction. This at once cures the problem and couples the quarks to the Wilson lines. The next section is a derivation of our effective action.

II. DERIVATION OF THE EFFECTIVE ACTION

In this section we derive an effective action describing the interaction of Wilson lines and (color-singlet) meson fields in the limit where the coupling constant $g^2 = 2N/\beta$ becomes large. Our starting point is the usual lattice form of the QCD action

$$S = S_G + S_F \,, \tag{2.1}$$

where S_G is given by

$$S_G = \frac{\beta}{2N} \sum_P \mathrm{Tr}[U(P) + U^\dagger(P)] \tag{2.2}$$

and S_F is

$$S_F = -\sum_x \left\{ \frac{1}{2} \sum_\mu \eta_\mu(x)[\bar{\chi}(x)U_\mu(x)\chi(x+\mu) + \bar{\chi}(x+\mu)U_\mu^\dagger(x)\chi(x)] - im\bar{\chi}(x)\chi(x) \right\} . \tag{2.3}$$

We employ Kogut-Susskind fermions, that is, $\chi(x)$ in (2.3) is an N-component Grassmann variable and $\eta_\mu(x) = (-1)^{x_1 + \cdots + x_{\mu-1}}$ takes the place of γ_μ in this formulation. Only one flavor is used which, however, corresponds to $[2^{d/2}]$ flavors in the continuum limit. In the absence of the mass term (m is the bare fermion mass) Eq. (2.3) has a well-known invariance under

$$\begin{aligned} x \text{ odd } & \begin{cases} \chi'(x) = e^{i\alpha}\chi(x), \\ \bar{\chi}'(x) = e^{-i\beta}\bar{\chi}(x) \,, \end{cases} \\ x \text{ even } & \begin{cases} \chi'(x) = e^{i\beta}\chi(x), \\ \bar{\chi}'(x) = e^{-i\alpha}\bar{\chi}(x) \,. \end{cases} \end{aligned} \tag{2.4}$$

Equation (2.4) generates a $U_c(1)\times U(1)$ symmetry group of the massless action. The mass term breaks this symmetry down to the diagonal subgroup $U_\Delta(1)$. Kluberg-Stern *et al.*[11,12] in the framework of a $1/d$ expansion, showed that at infinite coupling the $U(1)\times U(1)$ symmetry is spontaneously broken. Since the pattern of symmetry breaking was identical to that induced by a bare-mass term, the authors interpreted this as a breakdown of chiral symmetry. The associated Goldstone boson is the pion. In this section we will follow their work closely. Their notation is used as well, to facilitate comparison of formulas and results. To introduce finite temperature into our system it is from now on understood that the summation in (2.2) and (2.3) is restricted to points x whose "time" coordinate lies within the finite interval $[0, n_t]$. The physical temperature is as usual related to n_t by $T = (n_t a)^{-1}$ where a is the lattice spacing. From now on we will associate the dth coordinate with the "time" variable. The first step in the derivation of the effective action is to integrate out the spatial links in the partition function

$$Z = \int [dU][d\chi][d\bar{\chi}]e^S \,. \tag{2.5}$$

This can obviously only be done approximately. The integration over the gauge field part of the action can be performed using the methods developed in Ref. 12. At strong coupling this amounts to simply expanding the exponential and then integrating over Haar measure the spatial link variables in the timelike gauge field action. The lowest nonvanishing term will be $O(\beta^{n_t})$. Each integration produces a factor of $1/N$ and we obtain a nearest-neighbor interaction between Wilson lines of the form[18,19]

$$S_{\mathrm{WL}} = J \sum_{\mathrm{n.n.}} (\mathrm{Tr}W_i \mathrm{Tr}W_j^\dagger + \mathrm{H.c.}) \,, \tag{2.6}$$

where $J = (\beta/2N^2)^{n_t}$. The integral over spatial links in the fermionic part of the action is $\prod_{x,i} Z(x,i)$, where

$$Z(x,i) = \int dU_i(x) \exp\{-\tfrac{1}{2}\eta_i(x)[\bar{\chi}(x)U_i(x)\chi(x+i) + \bar{\chi}(x+i)U_i^\dagger(x)\chi(x)]\} \,. \tag{2.7}$$

Keeping only the leading term in the expansion of the exponential in (2.7) and using

$$\int [dU] U_{ij} U_{kl}^\dagger = \frac{1}{N}\delta_{il}\delta_{jk} \tag{2.8}$$

we obtain the following partition function:

$$Z = \int [dU_d][d\chi][d\bar{\chi}] \exp\left[S_{\mathrm{WL}} - \sum_x \{ \tfrac{1}{2}\eta_d(x)[\bar{\chi}(x)U_d\chi(x+d) + \bar{\chi}(x+d)U_d^\dagger(x)\chi(x)] + 2\bar{m}NM \} - \frac{N}{2(d-1)} \sum_i M(x)M(x+i) \right] . \tag{2.9}$$

Here we have defined

$$M(x)=i\frac{d-1}{2}\frac{1}{N}\sum_a \chi_a(x)\bar{\chi}_a(x),\quad \bar{m}=\frac{m}{[2(d-1)^{1/2}]}\ . \tag{2.10}$$

The reader should note that the steps leading from (2.5) to (2.9) are not rigorously part of a systematic strong-coupling expansion. We do, however, expect that $S_{\rm WL}$ in (2.9) represents the most important contribution of the gauge fields in such an expansion. The dependence on the color-singlet meson field M can now be linearized using a functional Laplace transform

$$e^{A(M)}=\int [d\lambda]e^{A(\lambda)+\lambda M}\ . \tag{2.11}$$

Introducing the operator $V_{x,x'}$ defined as

$$V_{x,x'}=\frac{1}{2(d-1)}\sum_i (\delta_{x',x+i}+\delta_{x',x-i}) \tag{2.12}$$

we obtain for (2.9)

$$Z=\int [dU_d][d\chi][d\bar{\chi}][d\lambda]\exp\left[S_{\rm WL}-\sum_x \tfrac{1}{2}\eta_d(\bar{\chi}U_d\chi+\bar{\chi}U_d^\dagger\chi)-N\left[\sum_{x,x'}\tfrac{1}{2}\lambda(x)V_{xx'}^{-1}\lambda(x')+\sum_x(\lambda+2\bar{m})M(x)\right]\right]\ . \tag{2.13}$$

The crucial step in our analysis is that in the following we will take the field $\lambda(x)$ to depend on the spatial coordinates only, $\lambda(x)=\lambda(\mathbf{x})$. This in effect means that we are dealing with a theory exhibiting confinement in the spacelike direction and at the same time admitting propagation of colored objects in the time direction. This approximation allows us to treat the propagation of timelike quarks exactly. With a t-independent λ we are led to consider the following integral:

$$I_0=\int [d\chi(t)][d\bar{\chi}(t)]\exp\left[-\sum_t\{\tfrac{1}{2}[\bar{\chi}(t)U_0(t)\chi(t+1)+\bar{\chi}(t+1)U_0^\dagger(t)\chi(t)]-i(\epsilon\lambda+m)\bar{\chi}(t)\chi(t)\}\right]\ . \tag{2.14}$$

One of these integrals must be evaluated for each spatial lattice site; the $\mathbf{x}$ coordinates have been suppressed. Equation (2.14) is easily recognized as the partition function for a fermion in one dimension, leading to the determinant of the Dirac operator in the Kogut-Susskind formulation. To evaluate this determinant appropriate boundary conditions must be specified. With the conventions of Kluberg-Stern *et al.*[12] adopted here, the variable $\chi(t)$ satisfies $\chi(t)=\chi(t+n_t)$. Equation (2.14) is most easily calculated by fixing a gauge. We choose a gauge in which U_d satisfies

$$U_d^{ab}=e^{i\phi_a}\delta^{ab}\ . \tag{2.15}$$

In this gauge, we can write

$$\ln I_0=\sum_a\sum_{k_0}\tfrac{1}{2}\ln[\mu^2+\cos^2(k_0-\phi_a)]\ , \tag{2.16}$$

where $\mu=\epsilon\lambda+m$ and k_0 takes on the values $2\pi m/n_t$ with $m=0,\dots,n_t-1$. A shift in k_0 of π puts this in a more familiar form:

$$\ln I_0=\sum_a\sum_{q_0}\tfrac{1}{2}\ln[\sin^2(q_0-\phi_a)+\mu^2]\ , \tag{2.17}$$

where $q_0=\pi(2m+1)/n_t$. The derivative with respect to μ can be evaluated by the contour integral

$$\frac{\partial \ln I_0}{\partial\mu}=\sum_a\frac{n_t\mu}{2\pi i}\int_C\frac{dz}{z}\frac{z^{n_t}}{1+z^{n_t}}\frac{1}{\mu^2-\frac{1}{4}\left[zw_a-\frac{1}{zw_a}\right]^2}\ , \tag{2.18}$$

where $w_a=e^{i\phi_a}$ and C is an appropriately chosen contour. This integral can be evaluated, and I_0 determined. The final result is simple, very much resembling the continuum result:

$$\ln I_0={\rm Tr}\{n_tE+\ln[1+e^{-n_tE}W]+\ln[1+e^{-n_tE}W^\dagger]\}\ . \tag{2.19}$$

Here we have introduced the quantity E given by

$$\sinh E=m+\epsilon\lambda\ . \tag{2.20}$$

The gauge-dependent formula

$$W^{ab}=e^{in_t\phi_a}\delta^{ab} \tag{2.21}$$

has been used to recover a gauge-invariant result. Putting all this together, we finally arrive at the following effective action:

$$\begin{aligned}S_{\rm eff}=&\sum_{\rm n.n.}J[{\rm Tr}W(\mathbf{x}){\rm Tr}W^\dagger(\mathbf{y})+{\rm H.c.}]\\&-\tfrac{1}{2}n_tN\sum_{\mathbf{x},\mathbf{y}}\lambda(\mathbf{x})V_{\mathbf{x},\mathbf{y}}^{-1}\lambda(\mathbf{y})\\&+\sum_{\mathbf{x}}{\rm Tr}\{n_tE+\ln[1+e^{-n_tE}W]\\&+\ln[1+e^{-n_tE}W^\dagger]\}\ .\end{aligned} \tag{2.22}$$

Following Kluberg-Stern *et al.*, it is straightforward to evaluate the $O(1/(d-1))$ corrections to the effective action. These corrections have two sources: the first is terms in the original action proportional to $(d-1)^{-1}$, while the second is the functional determinant from the integration over λ. The $(d-1)^{-1}$ term in the functional determinant is just the lowest-order term in a kind of hopping-parameter expansion. It is convenient to define

$$G=\frac{1}{V_4}\langle \mathrm{Tr}\ln[2\cosh n_t E+U+U^\dagger]\rangle_0 . \qquad (2.23)$$

In the work of Kluberg-Stern *et al.*, G is equal to $\ln\lambda$. The first correction is given by

$$S_q=\frac{V_3}{8(d-1)(N-1)}\left[n_t^2 N\left[\frac{dG}{d\lambda}\right]^2+n_t\frac{d^2G}{d\lambda^2}\right] \qquad (2.24)$$

while the second is

$$S_m=\frac{V_3}{8(d-1)}\left[\frac{d^2G}{d\lambda^2}\right]^2 .$$

These effects will turn out to move the chiral transition towards smaller temperatures.

III. ANALYSIS OF THE EFFECTIVE ACTION

In the previous section, the effective action was derived from the strong-coupling limit of lattice QCD. In this section, the effective action will be used to obtain various interesting results, including the phase diagram in the T-m plane.

A. Chiral symmetry made manifest

The vacuum expectation value of λ can be obtained from the λ-dependent part of the effective action for constant λ. This is given by the potential V_λ defined by

$$V_\lambda=\frac{-S_\lambda}{V_4}=+\frac{N}{2}\lambda^2-\mathrm{Tr}\left[E+\frac{1}{n_t}\ln[1+e^{-n_tE}W]+\frac{1}{n_t}\ln[1+e^{-n_tE}W^\dagger]\right] . \qquad (3.1)$$

A useful alternative form for V_λ is

$$V_\lambda=+\frac{N}{2}\lambda^2-\frac{1}{n_t}\mathrm{Tr}\ln[2\cosh n_tE+(W+W^\dagger)] . \qquad (3.2)$$

For a given, fixed W, λ is determined by

$$\frac{\partial V_\lambda}{\partial\lambda}=N\lambda-\left[\frac{\epsilon}{[1+(\epsilon\lambda+m)^2]^{1/2}}\right]\times\mathrm{Tr}\left[1-\frac{e^{-n_tE}W}{1+e^{-n_tE}W}-\frac{e^{-n_tE}W^\dagger}{1+e^{-n_tE}W^\dagger}\right]=0 . \qquad (3.3)$$

It is easy to check that when $m=0$, $\lambda=0$ is always a solution of Eq. (3.3) (the term in parentheses vanishes). Thus chiral-symmetry restoration is possible.

B. The zero-temperature limit

It is of considerable importance to examine the zero-temperature limit of the model. This allows for a comparison with previous work; it also gives a determination of meson and baryon masses.

In the zero-temperature limit V_λ is given by

$$V_\lambda=-\tfrac{1}{2}\lambda^2+|E| \qquad (3.4)$$

and λ is determined from

$$\frac{\partial V_\lambda}{\partial\lambda}=-\lambda+\frac{\epsilon\,\mathrm{sign}(\epsilon\lambda+m)}{[1+(\epsilon\lambda+m)^2]^{1/2}}=0 . \qquad (3.5)$$

[This formula is given incorrectly in Eq. (16) of our previous work.]

If the effect of timelike quark loops is not included, V_λ becomes

$$\bar V_\lambda=-\tfrac{1}{2}\lambda^2+\ln[\bar\epsilon\lambda+m] , \qquad (3.6)$$

where $\bar\epsilon=\sqrt{d}/2$. The value of λ follows from

$$-\lambda+\frac{\bar\epsilon}{\bar\epsilon\lambda+m}=0 . \qquad (3.7)$$

The two equations for λ are very similar, but Eq. (3.5) allows $\lambda=0$ as a solution, and Eq. (3.7) does not. In the large-d limit, the solutions are essentially the same. In our case λ is given by

$$\lambda=\frac{(4\epsilon^2+m^2)^{1/2}-m}{2\epsilon} . \qquad (3.8)$$

In the case $m=0$, λ is given by

$$\lambda^2=\frac{(1+4\epsilon^4)^{1/2}-1}{2\epsilon^2} \qquad (3.9)$$

which reduces to $\lambda^2=1$ in the large-d limit. In this way we see that the two approaches agree in the large-d limit, as they should.

C. Large quark mass

It is also interesting to consider the limit of large bare quark mass m. In this case λ becomes small and E is given approximately by

$$E=\ln(2m) . \qquad (3.10)$$

The fermionic determinant can be expanded in powers of m^{-1}. This results in an effective action for Wilson loops of the form

$$S_W=\sum_{\langle ij\rangle}J[\mathrm{Tr}W_i\mathrm{Tr}W_j^\dagger+\mathrm{H.c.}]+\sum\left[\frac{1}{2m}\right]^{n_t}\mathrm{Tr}[W_i+W_i^\dagger] , \qquad (3.11)$$

where i and j are spatial indices. The sum over $\langle ij\rangle$ denotes a sum over nearest-neighbor pairs.

This effective action has been obtained before by strong coupling[17,18] and Migdal-Kadanoff[19] methods. It is known to reproduce rather well both the qualitative and quantitative features of the deconfinement transition.

D. Mean field theory

One tool that has proven useful in the study of the Wilson-line effective action S_W discussed above is mean field theory. This method has given quite reasonable results, consistent with Monte Carlo simulations of the effective spin model and the underlying gauge field theory. Thus we do not hesitate to apply it to our model, in order to carry out the integration over Wilson line variables.

Mean field theory has been studied extensively in the last few years, and a comprehensive review has been given recently,[24] so we will discuss only our particular application. The original problem is the evaluation of a functional integral of the form

$$Z=\int [d\lambda][dB][d\bar{B}][dW]e^{S[\lambda,B,\bar{B},W]} . \tag{3.12}$$

Mean field theory replaces this by

$$Z_{\rm MFT}=Z_0\int [d\lambda][dB][dW] \times\exp\{\langle S[\lambda,B,\bar{B},W]\rangle_0-S_0[W]\} . \tag{3.13}$$

In the above equation $\langle \cdots \rangle_0$ denotes the expectation value with respect to a trial action S_0; Z_0 is defined to be

$$Z_0\equiv \int [dW]\exp S_0[W] . \tag{3.14}$$

We choose S_0 to have the simple form

$$S_0=\sum_i \frac{x}{2}{\rm Tr}[U(i)+U^\dagger(i)] . \tag{3.15}$$

As a first approximation, the study of the deconfinement and chiral transitions reduces to finding solutions of

$$\frac{\partial A}{\partial x}=0 , \tag{3.16a}$$

$$\frac{\partial A}{\partial \lambda(i)}=0 , \tag{3.16b}$$

where A is given by

$$\begin{aligned}A={}&\ln Z_0-\tfrac{1}{2}n_t\sum_{i,j}\lambda(i)V^{-1}(i,j)\lambda(j)\\&+\Big\langle J\sum_{\langle ij\rangle}[{\rm Tr}W(i){\rm Tr}W^\dagger(j)+{\rm H.c.}]\\&+\sum_i {\rm Tr}\{n_tE+\ln[1-e^{-n_tE(i)}W(i)]\\&+\ln[1-e^{-n_tE(i)}W^\dagger(i)]\}\Big\rangle_0 .\end{aligned} \tag{3.17}$$

Restriction of $\lambda(i)$ to constant values gives two nonlinear equations in λ and x which must be solved simultaneously.

Unfortunately, there is no simple form for the expectation value of the logarithm of the fermion determinant. However, this quantity has a simple expansion in inverse powers of $2\cosh n_tE$. From this expansion, it is easy to determine the expectation value for small x. Because x does turn out to be small, and $[2\cosh n_tE]^{-1}<\frac{1}{2}$, this is an adequate representation of the expectation value of the determinant. Details are given in the Appendix.

We define the connected generating functional associated with Z_0 by

$$W_0=\frac{1}{V_3}\ln Z_0 . \tag{3.18}$$

It has an expansion for small x given by[25]

$$W\simeq\tfrac{1}{4}x^2+\tfrac{1}{24}x^3-\tfrac{1}{384}x^5-\tfrac{1}{1536}x^6 . \tag{3.19}$$

We can now write down a useful formula for A:

$$\begin{aligned}\frac{A}{V_3}={}&-\tfrac{3}{2}n_t\lambda^2+W_0-xW_0'+2(d-1)J(W_0')\\&+3\ln[2\cosh(n_tE)]+\frac{W_0'}{\cosh(n_tE)}\\&-\frac{(2N-x)}{\cosh^2(n_tE)}+\frac{W_0'}{4\cosh^3(n_tE)}\\&+\frac{(2-x^2)}{24\cosh^3(n_tE)} .\end{aligned} \tag{3.20}$$

Equations (3.16a) and (3.16b) can now be solved numerically.

E. A qualitative analysis of chiral-symmetry restoration

It happens that for zero bare quark mass, the expectation value of the Wilson lines is approximately constant over a wide range of temperatures. This suggests replacing W by a real constant v, and it is instructive to do so. The potential V_λ is given by

$$V_\lambda=\frac{N}{2}\lambda^2-\frac{N}{n_t}\ln[2\cosh(n_tE)+2v] . \tag{3.21}$$

In order for chiral-symmetry restoration to take place when $m=0$, $\lambda=0$ must be a minimum of V_λ. Expanding V_λ to $O(\lambda^2)$, we obtain

$$V_\lambda=\frac{N}{2}\lambda^2-\frac{Nn_t(d-1)}{4(1+v)}\lambda^2+\cdots . \tag{3.22}$$

The coefficient of λ^2 is zero when

$$T=\frac{1}{n_ta}=\frac{(d-1)}{2(1+v)a} . \tag{3.23}$$

If the transition is second order, then T is the transition temperature of this simplified model. On the other hand, if the transition is first order, T is a bound on the transition temperature: $T_c\geq T$. Distinguishing between a first- and second-order transition requires knowledge of the global behavior of V_λ.

The interpretation we give Eq. (3.23) is that the larger the expectation value of the Wilson line, the lower the temperature at which chiral symmetry can be restored. It turns out that this procedure will give an accurate evaluation of the chiral-transition temperature.

F. Hadron masses

The masses of hadrons can be evaluated using a variant of the procedure of Kluberg-Stern *et al.* To lowest order in $(d-1)$, the inverse meson propagator is

$$G^{-1}(k)=\frac{1}{(d-1)}\sum_{i=1}^{d-1}\cos k_i+\frac{\partial^2E}{\partial\lambda^2} . \tag{3.24}$$

Because we have assumed λ is time-independent we cannot take k_d Euclidean: it is fixed at 0. We can take any of the other momenta Euclidean, however. There are

TABLE I. Comparison of various physical quantities as obtained from the model with either experimental or Monte Carlo results. All quantities are in MeV.

Effective model	Physical value/Monte Carlo
$m_\pi=140$ (input)	140
$m_\rho=780$ (input)	780
$m_\delta=1008$	980
$m_{A_1}=1157$	1100
$\sigma^{1/2}=400$ (input)	≈ 400
$T_D=184$	≈ 200
$T_{\rm ch}=386$	≈ 200

$d-1$ solutions of $G^{-2}(k)=0$ which correspond to different masses. They are given by

$$\cosh M_p=(d-1)\left[\frac{\partial^2 E}{\partial\lambda^2}-1\right]+2p+1\ , \qquad (3.25)$$

where p runs from 0 to $d-2$. In the small-m limit, the m_p are determined by

$$(aM_0)^2=2[2(d-1)^{1/2}]ma\ ,$$

$$aM_p=2\ln[\sqrt{p}+\sqrt{p+1}]+ma\frac{d-1}{[2p(p+1)^{1/2}]}\ , \qquad (3.26)$$

$$p=1,\ldots,d-2\ .$$

Following Ref. 12, we use the mass of the π and the ρ as input. This fixes the bare quark mass m and the lattice spacing a, as well as the mass of other meson states. The mass of the baryon is given by

$$aM_B=\frac{N}{2}\ln[2(d-1)]+\frac{Nma}{[2(d-1)^{1/2}]}\ . \qquad (3.27)$$

The result of this procedure is shown in Table I, where they are compared with the true masses. As can be seen, the results are surprisingly good. The actual values are very close to the results of Ref. 12. It might seem inconsistent to give a value for A_1, which is associated with $p=3$, but this should be acceptable in the large-d limit.

G. The deconfinement transition

Our model has three free parameters: the lattice spacing a, the bare quark mass, and the Wilson line coupling J. This last parameter can easily be related to β in the strong-coupling limit. QCD in the real world has only two free parameters: the bare quark mass (which may be a matrix) and Λ. For a lattice gauge theory and m near the continuum limit, physical observables depend on β and a only through a scale-setting parameter Λ, in a way given by the renormalization group. In the case considered here, we are very far from the continuum limit, so J must be set independently of a.

In order to obtain a value for J, we use the fact that Wilson line correlation functions fall off with spatial distance with a mass given by $n_t\sigma$. In the strong-coupling limit, this implies

$$J=\exp(-n_t\sigma a^2)\ . \qquad (3.28)$$

Thus our model uses three physical quantities as input: m_π, m_ρ, and σ.

Once J is known, the deconfinement temperature of the pure gauge theory can be determined in physical units, and compared with the results of Monte Carlo simulations. Taking $\sigma=(400\text{ MeV})^2$, we find $T_d=175$ MeV for the pure gauge theory. This result is in quite reasonable agreement with Monte Carlo results.

As mentioned previously, there is good reason to expect that the deconfining transition persists for large, but finite bare quark mass. We have checked this, and find that a deconfining transition occurs for all quark masses greater than $0.6a^{-1}=264$ MeV. This result was obtained for various values of the bare quark mass by explicitly finding two locally stable solutions for a certain range of temperature. The solution with higher action is stable, and the other metastable; the actions are equal at the transition point.

The order parameter for these transitions is the Wilson line: it jumps discontinuously at a transition. As the bare quark mass is decreased, the size of the jump in the expectation value of the Wilson line decreases, until the jump and the transition disappear at the critical value of the bare mass. The chiral order parameter λ, on the other hand, changes very little as the transition is crossed. The transition temperature is also insensitive to the value of the bare mass. As m is decreased from infinity to the critical value, the critical temperature decreases from 175 to 173 MeV.

H. The chiral transition

For a bare quark mass of zero, $\lambda=0$ is always a solution of Eq. (3.16). This solution can be compared with other solutions, in particular the solution that smoothly connects to the stable, zero-temperature chiral-symmetry-breaking solution. As it happens, this latter solution is always globally stable. As the temperature is increased λ decreases smoothly until it reaches zero at a temperature of 487 MeV. The transition is thus second order, and occurs only for zero bare quark mass.

The value of $v=\langle {\rm Tr}W\rangle/N$ is very stable at these high temperatures with a value of about 0.4. Applying Eq. (3.23) leads to a transition temperature of 471 MeV, in quite reasonable agreement with the more accurate numerical result.

The inclusion of higher-order terms in the action, as derived in II does not change the order of the transition, but does lower the transition temperature significantly, to 386 MeV. However, it is very interesting to note that the difference in action between the $\lambda\neq 0$ and $\lambda=0$ solutions is in fact very small. This suggests that higher-order corrections might easily change the order of the transition; it also explains why we originally thought the transition to be first order.[26]

IV. CONCLUSIONS

Our conclusions are fairly simple. We have seen that a strong-coupling model can reproduce both deconfinement and chiral-symmetry restoration, including reasonable physical values for the critical parameters. Unfortunately, the results are not in qualitative or quantitative agreement with the most recent Monte Carlo results. We have found the following:

(1) The deconfinement transition occurs for a bare quark mass greater than 264 MeV (four flavors), at an almost constant temperature of about 174 MeV. This is not in complete agreement with the latest Monte Carlo results,[22,23] which may indicate that the deconfinement transition occurs for very small values of the quark mass. On the other hand, our results do provide an explanation of this phenomenon: it is the constituent quark mass which determines the strength of Z_N symmetry breaking.

(2) Chiral symmetry is restored via a second-order transition at a higher temperature than the deconfining transitions. This is again at variance with the latest Monte Carlo results, which indicate that the chiral transition is first order and very close to the deconfinement transition. The two transitions may in fact occur together. This discrepancy probably reflects a need to couple the dynamics of the two transitions even more closely in our model.

In summary, we have displayed a relatively simple model which gives reasonable results for the hadron mass spectrum and the deconfinement and chiral-symmetry-restoration transitions.

APPENDIX

In this appendix we derive an approximate formula for the expectation value of the fermion determinant in the case $N=3$. The object to be evaluated can be put in the form

$$D=N\ln[2\cosh n_t E] + \left\langle \mathrm{Tr}\ln\left[1+\frac{1}{2\cosh(n_t E)}(W+W^\dagger)\right]\right\rangle_0 .$$

The logarithm can be expanded, and the following useful identities applied:

$$\mathrm{Tr}U^2=\chi_6-\chi_{\bar{3}} ,$$

$$\mathrm{Tr}U^3=\chi_{10}-\chi_8+1$$

when the χ_α are SU(3) characters. This combines with the expansion

$$\exp\left[\frac{x}{2}(\chi_3+\chi_{\bar{3}})\right] \simeq 1+\frac{x}{2}(\chi_3+\chi_{\bar{3}})+\frac{x^2}{8}(2\chi_1+\chi_3+\chi_{\bar{3}}+\chi_6+\chi_{\bar{6}}+2\chi_8) + O(x^2)$$

to give a formula for D:

$$D=N\ln[2\cosh(n_t E)]+\frac{W_0'}{\cosh n_t E}-\frac{(6-x)}{8\cosh^2 n_t E} + \frac{W_0'}{\cosh^3 n_t E}+\frac{(2-x^2)}{24\cosh^3 n_t E} ,$$

where we have used the more accurate formula

$$\langle \tfrac{1}{2}\mathrm{Tr}(W+W^\dagger)\rangle = W_0'(x)$$

when applicable.

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