Improved Hamiltonian variational technique for lattice models

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In the context of the Hamiltonian variational approach we propose a systematic method for the improvement of any given trial wave function. The approach has many similarities with the Lanczos scheme and it may be used for quantum-mechanical problems as well as field-theory and statistical-mechanics systems. We apply the method to the Mathieu equation and to the Ising one-dimensional model in a finite lattice. The agreement between our results and the exact ones is excellent in the whole range of parameters. We also briefly discuss the application of these ideas to lattice gauge theories and the similarities with other recently proposed methods.

I. INTRODUCTION

Lattice techniques are the main tool for the treatment of nonperturbative physics in gauge theories.[1] By Monte Carlo simulations many interesting physical magnitudes have been numerically estimated. Nevertheless, it would be very useful to develop analytical methods in order to obtain an intuitive idea of the main features of the different models. Many techniques have been proposed. For example, the mean-field approach with corrections[2] in the Lagrangian framework gives excellent results in the prediction of lattice phase diagrams, although the continuum limit is beyond its capability. Other analytical methods are the variational technique,[3,4] strong-coupling expansion,[5] mean-plaquette,[6] renormalization-group,[7] and finite-lattice approximations.[8] In this paper we study the Hamiltonian variational method. Although this technique gives acceptable results in many systems, perhaps its main drawback is that it is rather difficult to improve the numerical values obtained using a given ansatz. In Ref. 4 we tried to improve the plaquette-independent trial function in a $Z(2)$ gauge model in $2+1$ dimensions by the Bethe-Peierls technique, but the results showed a slow convergence to the accepted values for the model. It is clear that different ideas should be implemented. We propose in this paper a systematic and very simple iterative method for the improvement of any arbitrary ansatz. The application to some well-known models gives excellent results showing a fast convergence to the exact ground-state energy. Although our main purpose is the final implementation of these ideas in lattice gauge theories, the method is suitable for the treatment of quantum-mechanical models as well as field-theory and statistical-mechanics systems.

We remark that our approach is similar in spirit to a recently proposed method[9] which also improves an initial variational ansatz. We have studied the differences between this technique and the approach presented in this paper.

The organization of the paper is as follows. In Sec. II we present the method. In Sec. III we apply it to the Mathieu equation and to the Ising one-dimensional model in a finite lattice. The results are extrapolated to the bulk limit and they are compared with the exact ones. A short discussion is given in Sec. IV.

II. DESCRIPTION OF THE METHOD

The technique that will be described in this section has many similarities with the well-known Lanczos method[10] for the tridiagonalization of matrices. In fact, our idea is inspired by a recent paper[11] where the Lanczos approach was modified in order to extend the range of the strong-coupling expansion.

Suppose that we would like to approximate the ground-state energy of a system described by a Hamiltonian H. In the variational framework we must select a trial wave function ψ_0 which depends on some free parameters $\{\beta\}$ as in the usual Rayleigh-Ritz approach. If we operate with H over ψ_0 the result can always be written as

$$H\psi_0=\epsilon_0\psi_0+b\tilde{\psi}_0 , \tag{1}$$

where $\tilde{\psi}_0$ is a state orthogonal to ψ_0. The constant ϵ_0 $[=(\psi_0,H\psi_0)]$ is the variational energy of ψ_0, i.e., if we minimize ϵ_0 with respect to $\{\beta\}$ we get an upper bound for the energy of the true ground state Φ. The other constant b $[=(\tilde{\psi}_0,H\psi_0)]$ somehow measures the departure of ψ_0 from Φ. The states ψ_0, $\tilde{\psi}_0$ are normalized to 1. Equation (1) may be thought of as the definition of $\tilde{\psi}_0$. Now the idea is obvious. We may improve the initial trial wave function by considering as a new ansatz the linear normalized combination

$$\psi_1=\frac{\psi_0+\alpha_1\tilde{\psi}_0}{(1+\alpha_1{}^2)^{1/2}} , \tag{2}$$

where α_1 is a new variational parameter which may easily be found through minimization of the energy of state Eq. (2),

$$\alpha_1=f-(f^2+1)^{1/2} , \tag{3}$$

where

$$f=\frac{\langle H^3\rangle_0-3\langle H\rangle_0\langle H^2\rangle_0+2\langle H\rangle_0{}^3}{2(\langle H^2\rangle_0-\langle H\rangle_0{}^2)^{3/2}} , \tag{4a}$$

and

$$\langle H^n\rangle_0=(\psi_0,H^n\psi_0)\ . \tag{4b}$$

It can readily be shown that the state Eq. (2), supplemented by Eq. (3), is the eigenstate of lower energy of the 2×2 matrix

$$\begin{bmatrix}\epsilon_0 & b\\ b & c\end{bmatrix}, \tag{5}$$

where $c=(\tilde{\psi}_0,H\tilde{\psi}_0)$. In general the Hamiltonian may be written as a kinetic energy term plus a potential one, i.e., $H=H_0+xV$, where x is a parameter. If we choose as initial ψ_0 the ground state of H_0, we recover the strong-coupling ($x\ll1$) method of Ref. 11. Nevertheless, if ψ_0 is good enough in the limits $x=0$ and ∞, we prove in Sec. III that our approach gives good results in the whole range of x.

By straightforward algebra and using Eq. (3) the new improved energy can be evaluated,

$$\epsilon_1=\epsilon_0+\alpha_1 b\ . \tag{6}$$

Equation (6) must be minimized (usually numerically) with respect to the parameters $\{\beta\}$ (which, of course, in general, do not coincide with the set $\{\beta\}$ which minimize ϵ_0). The method may easily be iterated, i.e., from H acting over ψ_1 we get a state $\tilde{\psi}_1$ orthogonal to ψ_1. The new improved state is now

$$\psi_2=\frac{\psi_1+\alpha_2\tilde{\psi}_1}{(1+\alpha_2{}^2)^{1/2}}\ . \tag{7}$$

The values of α_2 and ϵ_2 (the energy of ψ_2) are given by Eqs. (3) and (6), respectively, changing $\psi_0,\tilde{\psi}_0$ by $\psi_1,\tilde{\psi}_1$. Using Eq. (2), α_2 and ϵ_2 can be written as a function of $\langle H^n\rangle_0$. Note that in the first step of our iterative method we need to know $n=1,2,3$, in Eq. (4b), and in the second one, $n=1,\ldots,5$. In general two new powers of H must be evaluated in each iteration.

Summarizing, in this section we have proposed a systematic and simple method for the improvement of any trial wave function in Hamiltonian variational techniques. Now we must test the approach in some nontrivial examples.

III. RESULTS

A. Mathieu equation

As a first example of the method described in Sec. II we consider (as in Ref. 11) the Mathieu equation which has a lot of applications in theoretical physics,

$$\left[-\frac{d^2}{d\theta^2}-x\cos\theta\right]\psi(\theta)=E\psi(\theta)\ . \tag{8}$$

It is the Schrödinger equation for the quantum pendulum in a gravitational field, for a particle in a periodic one-dimensional potential, and for the Stark effect of a rigid rotator. This equation also appears in statistical mechanics for the one-dimensional Coulomb gas.[12] In gauge models it has been studied due to its similarities with pure Yang-Mills theories.[13] In lattice gauge theories it also corresponds to the eigenvalue equation for the one-plaquette model[14] (Hamiltonian formulation), where x is proportional to g^{-4} and g is the coupling constant. For this reason we will refer to the region $x\ll1$ ($x\gg1$) in Eq. (8) as the strong (weak) coupling region. The Mathieu equation has also been analyzed by the phase-integral approximation.[15]

We try to reproduce the known results for the ground state of Eq. (8) using a starting trial wave function

$$\psi_0(\theta)=\exp(\tfrac{1}{2}\beta\cos\theta)\ , \tag{9}$$

where β is a variational parameter. It can be shown that this ansatz reproduces the first term of the perturbative expansions in the strong and weak coupling regions [see Eq. (12)]. Recently this type of trial functions [i.e., $\psi_0=\exp(\beta V/2)$] has been widely used[16] in lattice gauge models with acceptable results so they are in general a good starting point for our method. The zeroth-order free energy is

$$\epsilon_0(x,\beta)=\left[\frac{\beta}{4}-x\right]\frac{I_1(\beta)}{I_0(\beta)}\ , \tag{10}$$

where $I_n(\beta)$ are the modified Bessel functions of order n. In Table I we give the results for the first three iterations of our method. The agreement with the exact values[17] is excellent in the whole range of x. The convergence is very fast as is shown in Fig. 1, where we plot the relative error

$$\Delta_i=(\epsilon_{\text{exact}}-\epsilon_i)/\epsilon_{\text{exact}}\ , \tag{11}$$

where ϵ_i is the energy of the ith iteration. Note that the error reduces in one order for the first two iterations. Δ_i has a maximum at $x\sim1$. In Fig. 2 we also compare our results with the strong[17] and weak[18] coupling expansions of the ground state of the Mathieu equation. For completeness we quote here the first terms of these expansions

TABLE I. The energy ϵ_i of the ith iteration compared with the exact result for the ground state of the Mathieu equation at some representative values of x ranging from the strong-coupling region to the weak-coupling regime (we omit the minus sign for all the energies).

x	ϵ_0	ϵ_1	ϵ_2	ϵ_3	ϵ_{exact}
0.2	0.019 616	0.019 660	0.019 661	0.019 662	0.019 662
0.6	0.156 542	0.158 303	0.158 355	0.158 357	0.158 358
1.0	0.372 644	0.378 172	0.378 456	0.378 478	0.378 489
2.0	1.056 478	1.069 194	1.069 984	1.070 079	1.070 130
5.0	3.469 634	3.483 862	3.484 201	3.484 233	3.484 245
10.0	7.819 052	7.828 273	7.828 344	7.828 347	7.828 347

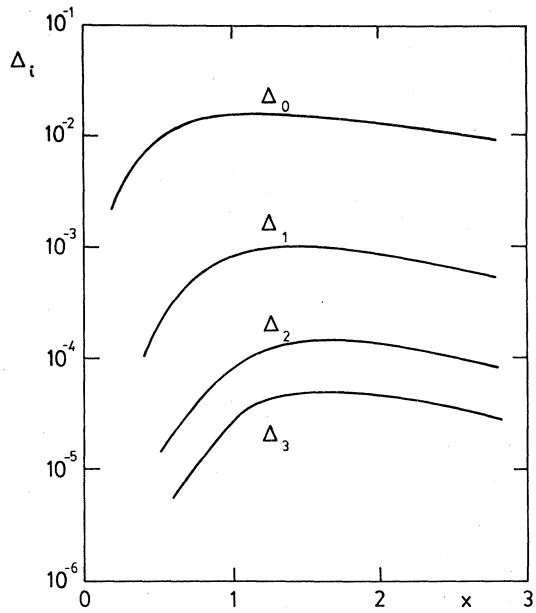

FIG. 1. The relative error $\Delta_i=(\epsilon_{\text{exact}}-\epsilon_i)/\epsilon_{\text{exact}}$ for the Mathieu model. x is in the range of maximum error.

(for the weak-coupling series more terms are given in Ref. 18),

$$\epsilon_{\text{SC}}\approx -0.5x^2+0.218\,750x^4-0.201\,389x^6 +0.232\,907x^8\,, \tag{12a}$$

$$\epsilon_{\text{WC}}\approx -x+\left[\frac{x}{2}\right]^{1/2}-\frac{1}{2^4}-\frac{1}{2^8}\left[\frac{2}{x}\right]^{1/2} -\frac{6}{2^{12}x}-\frac{53}{2^{18}}\left[\frac{2}{x}\right]^{3/2} -\frac{594}{2^{21}x^2}-\frac{7922}{2^{28}}\left[\frac{2}{x}\right]^{5/2}\,. \tag{12b}$$

Note that our variational energy in Fig. 2 (which is indistinguishable from the exact result) interpolates smoothly

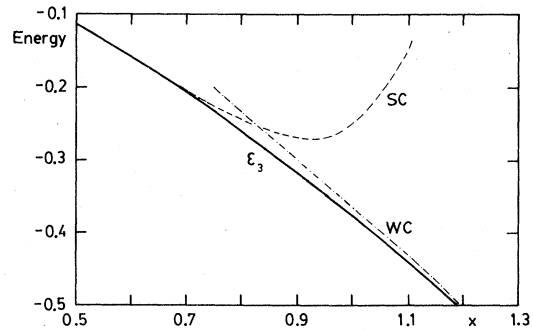

FIG. 2. Plot of the energy ϵ_3 of the third iteration for the Mathieu equation (continuous line) in comparison with the strong coupling (SC) expansion given in Eq. (12a) (dashed line) and the weak coupling (WC) one of Eq. (12b) (dashed-dotted line). The exact result of the model is indistinguishable from ϵ_3.

TABLE II. Evolution of the variational parameter β_m which minimizes the energy of each iteration. The corresponding value for the third iteration is approximately equal to the second one.

	Iterations		
x	0	1	2
0.6	1.05	1.08	1.09
1.0	1.52	1.59	1.61
2.0	2.29	2.36	2.40
5.0	3.53	3.51	3.54

between both expansions. In Table II we complete the analysis showing the variation of the β parameter which minimizes the energy for some representative points.

From these results it is clear that our method works properly at least in this quantum-mechanical problem.

B. One-dimensional Ising model

As a second example, we perform some exploratory calculations in lattice models. We consider the Ising model in one dimension defined by

$$H=-\sum_i \sigma_i^1 - x\sum_i \sigma_i^3\sigma_{i+1}^3\,, \tag{13}$$

where σ^1, σ^3 are Pauli matrices and the sum is over the sites i of a finite chain of N sites (periodic boundary conditions are assumed). The exact ground-state energy of Eq. (13) is well known,[8]

$$\epsilon_{\text{exact}}=-\sum_{\substack{k=0\\(k\ \text{odd})}}^{2N-1}\left[(1-x)^2+4x\sin^2\left[\frac{\pi k}{2N}\right]\right]^{1/2}\,. \tag{14}$$

It is important to remark that for an infinite-volume system our method presents a problem. In the limit $N=\infty$ it can be proved that α in Eq. (3) is zero. Nevertheless, this fact does not invalidate the approach because we will prove that we can obtain very accurate results by the following trick: apply the method to finite chains and then extrapolate the numbers to the bulk limit using some well-known extrapolating[19] procedures.

As an initial trial function we choose the analog of Eq. (9), i.e.,

$$\psi_0=\exp\left[\tfrac{1}{2}\beta\sum_i \sigma_i^3\sigma_{i+1}^3\right]|0\rangle_{\sigma^1}\,, \tag{15}$$

where $|0\rangle_{\sigma^1}$ represents an eigenvector of σ^1 with eigenvalue $(+1)$ in all the sites of the lattice. This kind of trial state has recently been carefully analyzed.[20] The zeroth-order energy per unit site is

$$\epsilon_0(x,\beta)=-\frac{\cosh^{-2}\beta+x(\tanh\beta+\coth\beta\tanh^N\beta)}{1+\tanh^N\beta}\,. \tag{16}$$

In Table III(a) we can see the results for the ground-state energy as a function of the number of iterations for chains of different lengths at $x=1$, where a phase transition is present for the infinite system (since the method is

TABLE III. (a) The energy per unit site obtained by our iterative method for the ground state of the Ising chain with N sites at $x=1$, compared with the exact value [Eq. (14)] (the common minus sign for all the energies is omitted). The predictions for $N=\infty$ have been obtained by a VBS approximant using the variational results for $N=2, 3, \ldots, 10$ (for $N=2$, the variational energy of the zeroth iteration is just virtually identical to the exact one, so we do not show it in the table). The exact result of the Ising infinite chain in $x=1$ is -1.2734 for the ground and first excited state energies. (b) Same as (a), but for the first excited state. The exact column has been evaluated from Eq. (18).

(a)

	Iterations							
N	0	1	2	3	5	10	20	Exact
4	1.295 74	1.306 56	1.306 56	1.306 56	1.306 56	1.306 56	1.306 56	1.306 56
6	1.262 97	1.286 63	1.287 75	1.287 77	1.287 90	1.287 90	1.287 90	1.287 90
8	1.253 18	1.276 67	1.280 27	1.281 07	1.281 37	1.281 45	1.281 46	1.281 46
10	1.250 76	1.270 85	1.275 60	1.277 21	1.277 99	1.278 37	1.278 49	1.278 49
∞	1.250 21	1.262 74	1.261 10	1.279 29	1.275 26	1.273 53	1.273 24	1.273 24

(b)

	Iterations							
N	0	1	2	3	5	10	20	Exact
4	1.194 49	1.207 11	1.207 11	1.207 11	1.207 11	1.207 11	1.207 11	1.207 11
6	1.232 34	1.243 27	1.244 01	1.244 01	1.244 02	1.244 02	1.244 02	1.244 02
8	1.240 81	1.254 71	1.256 20	1.256 58	1.256 76	1.256 83	1.256 83	1.256 83
10	1.245 22	1.260 93	1.261 84	1.262 23	1.262 60	1.262 72	1.262 75	1.262 75
∞	1.254 61	1.260 51	1.260 57	1.253 18	1.273 76	1.273 22	1.273 24	1.273 24

easily programmable all iterations have been obtained through a computer program and the 0th iteration represents the prediction for the initial trial wave function). When x achieves values around $x=1$ the difference between variational and exact energies has a maximum. Nevertheless, note that the results are excellent even in a few iterations (this detail is important for future applications of the technique to more realistic models). When N increases the accuracy decreases as expected from the fact that $\alpha=0$ in the limit $N=\infty$. However, the results show that the influence of this effect is very small for short chains and we can go on with our idea of considering the exact solution of finite chains and then extrapolate them to the bulk limit. The $N=\infty$ values are obtained using the Vanden Broeck-Schwartz approximant with $\alpha_L=-[1-(-1)^L]/2$ (see Ref. 19 for the definition of α_L). We use this approximant because the eigenvalues of the Ising model converge logarithmically to the infinite-chain result in the vicinity of the critical point. This method has been used with success[8] in the $Z(N)$ spin model. Even more the sequence of the column labeled "exact" in Table III(a) converges to the exact bulk result with more than five significant figures indicating that we can use the method with confidence. The extrapolated values for different numbers of iterations are also given in Table III(a). Note the convergence to the exact value $-1.273\,24$ (for $x\neq 1$ this convergence is faster). An unsatisfactory detail is that the exact result is approached from below showing that the extrapolation method does not strictly respect the fact that the variational energy must be an upper bound for the exact energy. Perhaps bigger chains or other extrapolating techniques are needed. Nevertheless the results are highly encouraging.

Using our method the first excited state can also be analyzed.

As an initial trial wave function we chose

$$\psi_0^e=\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\sigma_i^3\psi_n^g\ , \tag{17}$$

where ψ_n^g represents the improved variational ground state obtained using our technique after n iterations. ψ_0^e is an exact eigenvector of H when $x=0$ and for the Ising model in a finite chain using the initial trial state Eq. (15) it is always orthogonal to ψ_n^g. If we consider any trial state or another H, in general we must subtract the projection of the first excited state over the ground state.

The exact energy is

$$\epsilon_{\text{exact}}^e=2(1-x)-\sum_{\substack{k=0\\(k\ \text{even})}}^{2N-1}\left[(1-x)^2+4x\sin^2\left[\frac{\pi k}{2N}\right]\right]^{1/2}\ . \tag{18}$$

For $x>1$ the first term on the right-hand side of Eq. (18) must be eliminated. In Table III(b) we give the results using Eq. (17) as an initial trial function for different numbers of sites and iterations. In this case the jth iteration also means that the n of Eq. (17) is equal to j. As in the ground-state energy the values are very accurate showing that our technique can also deal with excited levels.

In Table IV we show the relative errors of the variation-

TABLE IV. The relative error defined in Eq. (11) of the energy per unit site for the ground ($\Delta\epsilon_g$) and first excited ($\Delta\epsilon_e$) states of the Ising chain at $x=0.5$ and 2.0. The number of sites is 8.

	$x=0.5$		$x=2.0$	
Iterations	$\Delta\epsilon_g$	$\Delta\epsilon_e$	$\Delta\epsilon_g$	$\Delta\epsilon_e$
0	1.07×10^{-3}	7.56×10^{-3}	3.09×10^{-2}	2.53×10^{-2}
1	1.26×10^{-4}	1.85×10^{-3}	1.02×10^{-2}	4.49×10^{-4}
2	1.86×10^{-5}	7.90×10^{-4}	1.07×10^{-3}	4.16×10^{-5}
5	5.29×10^{-7}	2.61×10^{-4}	7.67×10^{-5}	2.36×10^{-6}
10	1.11×10^{-11}	1.09×10^{-4}	3.94×10^{-6}	8.82×10^{-9}

al energies for the ground and first excited states in $x=0.5$ and 2.0 as examples (i.e., a representative value of the strong- and weak-coupling regimes, respectively). The results indicate that the technique gives accurate predictions in the whole range of the x parameter. The convergence is faster in the strong-coupling domain because the state Eq. (15) reproduces the first and second terms of the perturbative series for $x \ll 1$, while in the weak-coupling zone it only gives the leading order (i.e., it is a strong-coupling-type state).

In Fig. 3 we study the sensitivity of our approach to the selection of the variational parameter β. In the first iterations it is clear that we need to estimate with some confidence the value β_m which minimizes the energy but in the subsequent steps good results can be obtained without changing β_m. This fact is very important because the computational effort is appreciably reduced. For example, in Fig. 3 we can see that for $I=5$ or 10 there is a wide domain of β values where the energy is almost equal to the exact one.

Summarizing, the results are satisfactory, showing that the method is also adequate for the treatment of lattice systems.

As was remarked in Sec. I, in Ref. 9 an alternative method to improve a variational ansatz ψ_0 was proposed. It is based on the fact that the state

$$\psi_t=\frac{\exp(-tH/2)\psi_0}{(\psi_0,\exp(-tH)\psi_0)}\ , \tag{19}$$

gives a better approximation than ψ_0 for the ground-state energy for any value of t. In the limit $t\to\infty$ the energy

$$\epsilon(t)=(\psi_t,H\psi_t) \tag{20}$$

converges to the exact value. For the evaluation of Eq. (20) the authors of Ref. 9 expand $\epsilon(t)$ as a power series in t.

The result is

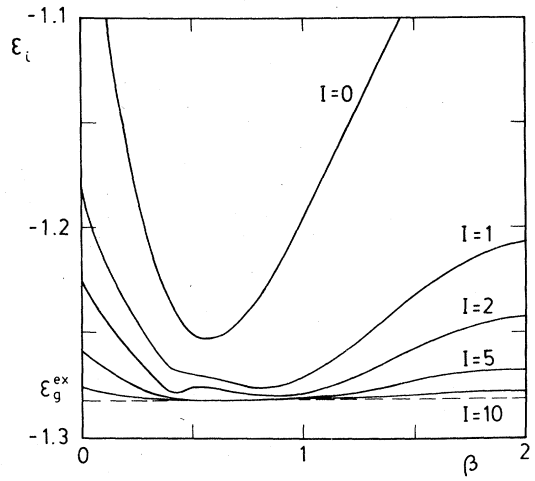

FIG. 3. Plot of the energy ϵ_i of the ith iteration for the Ising chain (with $N=8$ and $x=1$ as examples) as a function of the variational parameter β. The iterations are denoted by I and ϵ_g^{ex} represents the exact ground-state energy which is equal to -1.2815.

$$\epsilon(t)=\sum_{m=0}^{\infty}\frac{(-t)^m}{m!}\langle H^{m+1}\rangle_c\ , \tag{21}$$

where $\langle H^{m+1}\rangle_c$ is defined as

$$\langle H^{m+1}\rangle_c=\langle H^{m+1}\rangle_0-\sum_{p=0}^{m-1}\begin{bmatrix}m\\p\end{bmatrix}\langle H^{p+1}\rangle_c\langle H^{m-p}\rangle_0\ . \tag{22}$$

It can easily be proved that for a lattice model $\langle H^{m+1}\rangle_c$ is proportional to the number of sites N for any value of m so in this case there are no problems with the infinite-volume limit. However the method involves the use of Padé approximants to obtain an estimate of the $t\to\infty$ limit, introducing additional complications. Note the difference with our approach where we obtain very accurate results for a finite lattice and then, using extrapolating techniques, we recover the bulk limit. It is then clear that both approaches are in principle essentially different.

Let us compare the numerical results in the Ising model. From Ref. 9 we know that the difference between the exact result for the ground-state energy and the approximate one [Eq. (21)] is about 4.10^{-3} near $x\sim 0.9$ (and about 10^{-4} in the rest of the parameter's domain). This result was obtained with m up to 6 so it requires the evaluation of $\langle H^7\rangle_0$. In our method three iterations demand the same effort. From Table III(a) we can see that in the $N=\infty$ limit and in $x=1$ the difference between the exact energy and the variational one with three iterations is $\sim 6\times 10^{-3}$ so the numerical precision of both techniques is similar. It would be very useful to look for a combination of both approaches keeping the advantage of each one.

IV. CONCLUSIONS

In this paper we have proposed a systematic method for the improvement of a given trial wave function in the variational approach. The results show that the technique works properly and converges very fast to the correct ground-state energy of the models analyzed. In the Ising model we have also studied the first excited level with good results. In a future publication we hope to apply the method to a gauge model in a finite lattice. Of course our main interest is to apply these ideas to analyze non-Abelian systems in $3+1$ dimensions. Another interesting application lies in the renormalization-group technique in real space,[21] where it is necessary to evaluate the low-lying states of a finite lattice block.

Upon completion of this work we received a report[22] where an improved variational method for lattice systems is also proposed. The technique is very similar to ours. In both cases the Lanczos approach is used to express the Hamiltonian matrix in a very convenient base. The authors of Ref. 22 studied carefully the O(3) model in $1+1$ dimensions and gauge systems. They also work on a finite lattice (eight sites), but without using extrapolation techniques to recover the infinite-volume limit. In the gauge model they consider a plaquette-independent ansatz[16] as an initial trial function and in the spin model a

link-independent one, which is a generalization to an O(3) theory of our state equation (15).

Perhaps the main difference between the approach of Ref. 22 and the iterative method presented in this paper is that we need to diagonalize a 2×2 matrix in each step while in the approach of Ref. 22 it is necessary to find the eigenvalues of a bigger one but only once. As was recently remarked[11] in the context of the strong-coupling expansion, the first alternative converges faster than the usual Lanczos technique and it demands less numerical effort.

Apart from these minor details and the presentation of the methods both alternatives seem to be very similar.

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[1]J. Kogut, Rev. Mod. Phys. **55**, 775 (1983).
[2]J. M. Drouffe and J. B. Zuber, Phys. Rep. **102**, 1 (1983); E. Dagotto, Phys. Rev. D **30**, 1276 (1984).
[3]S. Drell, H. Quinn, B. Svetitsky, and M. Weinstein, Phys. Rev. D **19**, 619 (1979); D. Boyanovsky, R. Deza, and L. Masperi, *ibid.* **22**, 3034 (1980); D. Horn and M. Weinstein, *ibid.* **25**, 3331 (1982); N. Hari Dass, A. Patkós, and F. Deák, Nucl. Phys. **B205 [FS5]**, 414 (1982).
[4]E. Dagotto and A. Moreo, Phys. Rev. D **29**, 300 (1984).
[5]J. Kogut, R. Pearson, and J. Shigemitsu, Phys. Rev. Lett. **43**, 484 (1979).
[6]G. Batrouni, Nucl. Phys. **B208**, 12 (1982); E. Dagotto and A. Moreo, Phys. Rev. D **30**, 1271 (1984).
[7]K. Bitar, S. Gottlieb, and C. Zachos, Phys. Rev. D **26**, 2853 (1982).
[8]C. Hamer and M. Barber, J. Phys. A **14**, 241 (1981); **14**, 2009 (1981).
[9]D. Horn and M. Weinstein, Phys. Rev. D **30**, 1256 (1984).
[10]C. Lanczos, J. Res. Nat. Bur. Stand. **45**, 255 (1950); H. Roomany and H. Wyld, Phys. Rev. D **21**, 3341 (1980).
[11]J. Alberty, J. Greensite, and A. Patkós, Phys. Lett. **138B**, 405 (1984).
[12]A. Lenard, J. Math. Phys. **2**, 682 (1961).
[13]H. Neuberger, Phys. Rev. D **17**, 498 (1978).
[14]D. Robson and D. Webber, Z. Phys. C **7**, 53 (1980); S. Wadia, Phys. Lett. **93B**, 403 (1980).
[15]E. Floyd, J. Math. Phys. **17**, 880 (1976); N. Fröman, J. Phys. A **12**, 2355 (1979).
[16]E. Dagotto and A. Moreo, Phys. Rev. D **29**, 2350 (1984).
[17]*Tables Relating to Mathieu Functions,* 2nd ed., National Bureau of Standards (U.S. Government Printing Office, Washington, D.C., 1967).
[18]M. Stone and J. Reeve, Phys. Rev. D **18**, 4746 (1978).
[19]J. M. Vanden Broeck and L. Schwartz, SIAM J. Math. Anal. **10**, 658 (1979); M. Barber and C. Hamer, J. Aust. Math. Soc. B **23**, 229 (1982).
[20]E. Dagotto, L. Masperi and A. Moreo, Bariloche report (unpublished).
[21]S. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D **16**, 1769 (1977).
[22]A. Duncan and R. Roskies, Proceedings of the Argonne National Laboratory Workshop, 1984 (unpublished); Phys. Rev. D **31**, 364 (1985).