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New lattice approximation of gauge theories

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A discrete approximation of the SU(2) gauge theory interacting with a matter field in the adjoint representation is proposed. Lattice variables describing the state of the field are gauge invariants of the underlying continuum theory. Topological degrees of freedom corresponding to the monopole strength are discussed.

I. INTRODUCTION

In this paper we construct a lattice field theory, which approximates the continuum theory of an SU(2) gauge field interacting with a matter field in the adjoint representation. This approximation is entirely different from the standard lattice gauge theory proposed by Wilson.[1] Our construction is based on the hydrodynamical description of gauge field models.[2] The essential feature of our theory is that the lattice variables we use are gauge invariants of the underlying continuum theory.

There are two main advantages of this approach.

1. In Wilson's theory the gauge field A_μ and consequently the field strength $F_{\mu\nu}$ are replaced by group-valued lattice variables. Thus, the information about $F_{\mu\nu}$ is "truncated up to $2\pi n$." Here $F_{\mu\nu}$ remains Lie-algebra-valued.

2. Among the lattice variables we construct, there are certain discrete invariants, which describe topologically nontrivial configurations, namely, 't Hooft-Polyakov monopoles[3] appearing in the continuum theory. In this respect our paper is another step toward dealing with topological charges in lattice gauge theories.[4]

Due to the simple structure of the variables we use and due to the fact that they are gauge invariant our model is well adapted to computer analysis.

For pedagogical reasons we demonstrate our method in Sec. II for the case of scalar electrodynamics.

II. SCALAR ELECTRODYNAMICS ON THE LATTICE

Scalar electrodynamics is a theory describing the interaction of the electromagnetic field A_μ with a complex scalar field ϕ. The Lagrangian is given by

$$L=-\tfrac{1}{2}m^2\phi\phi^*+\tfrac{1}{2}D_\mu\phi(D^\mu\phi)^*-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\ , \tag{2.1}$$

where $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $D_\mu\phi=(\partial_\mu+igA_\mu)\phi$. We will formulate this model on a hypercubic lattice Λ in four-dimensional space M, which can be either the Minkowski or the Euclidean space-time. Let us denote by

Z^0—the set of lattice points ,

Z^1—the set of bonds ,

Z^2—the set of plaquettes .

Points will be denoted by $x,y,\dots$ or $x,x+\hat{\mu},\dots$. [The vector $\hat{\mu}$ has the direction of the oriented μth axis and length $\pm\delta$ (lattice spacing), according to whether $\hat{\mu}$ is spacelike or timelike.] Bonds will be denoted by (x,y), $(x,z),\dots$ or $(x,x+\hat{\mu})$, $(x,x+\hat{\nu}),\dots$ and plaquettes by $p=(x,y,z,t)$ or $p=(x,x+\hat{\mu},x+\hat{\mu}+\hat{\nu},x+\hat{\nu})$. A continuous field configuration (A,ϕ) on the whole M gives rise to the lattice variables $Z^0\ni x\to\phi_x\in C^1$, and $Z^1\ni(x,y)\to A_{x,y}\in R^1=u(1)$, defined as $\phi_x\equiv\phi(x)$ and

$$A_{x,x+\hat{\mu}}\equiv\frac{1}{\delta}\int_0^\delta A_{x,x+\hat{\mu}}(\tau)d\tau=-A_{x+\hat{\mu},x}\ , \tag{2.2}$$

where $A_{x,x+\hat{\mu}}(\tau)\equiv A_\mu(x+\tau\hat{\mu}/\delta)$, $\tau\in[0,\delta]$, is the (oriented) restriction of the continuous gauge potential A_μ to the bond $(x,x+\hat{\mu})$.

If we do not want to lose important information about the topological character of (A,ϕ) on the lattice level, we have to assume that the value of the field ϕ on lattice bonds is the covariantly linear interpolation of the values ϕ_x, i.e.,

$$(DD\phi)_{x,x+\hat{\mu}}=0 \tag{2.3a}$$

for every bond $(x,x+\hat{\mu})$, where

$$(DD\phi)_{x,x+\hat{\mu}}(\tau)=[d/d\tau+igA_{x,x+\hat{\mu}}(\tau)]^2\phi(\tau)\ .$$

The easiest way to solve this equation is to pass to such a gauge that $A_{x,x+\hat{\mu}}(\tau)$ is constant along the bond. As a result we obtain the following interpolation formula:

$$\phi(\tau)=\exp(-igA_{x,x+\hat{\mu}}\tau)\left\{[\exp(igA_{x,x+\hat{\mu}}\delta)\phi_{x+\hat{\mu}}-\phi_x]\frac{\tau}{\delta}+\phi_x\right\}\ . \tag{2.3b}$$

The first step of our construction consists therefore in restricting the continuous configuration (A,ϕ) to lattice bonds and imposing the interpolation condition (2.3a). Next we shall characterize this configuration by a full set of invariants. At the beginning we restrict ourselves to the set of generic lattice configurations (configurations for which $\phi\neq 0$ on lattice bonds). Obviously,

$$Z^0 \ni x \to R_x \equiv (\phi_x \phi_x^*)^{1/2} \in R^1_+ \tag{2.4}$$

is a gauge-invariant quantity. The second invariant is given by

$$Z^1 \ni (x,x+\hat{\mu}) \to v_{x,x+\hat{\mu}} \equiv \frac{1}{ig\delta}\int_0^\delta \frac{\phi^*}{|\phi|}\left[D\frac{\phi}{|\phi|}\right]_{x,x+\hat{\mu}}(\tau)d\tau\ , \tag{2.5}$$

where again $D=d/d\tau+igA$.

We observe that

$$v_{x,x+\hat{\mu}} = A_{x,x+\hat{\mu}} + \frac{1}{g\delta}(\Delta\alpha)_{x,x+\hat{\mu}}\ , \tag{2.6}$$

where $(\Delta\alpha)_{x,x+\hat{\mu}}$ is the phase increase of ϕ along the bond $(x,x+\hat{\mu})$. We can always choose a gauge for which $A_{x,x+\hat{\mu}}(\tau)=0$. Then $g\delta v_{x,x+\hat{\mu}}=(\Delta\alpha)_{x,x+\hat{\mu}}$ and ϕ is a linear function on $(x,x+\hat{\mu})$, according to (2.3b). We conclude that

$$-\frac{\pi}{\delta} < g v_{x,x+\hat{\mu}} < \frac{\pi}{\delta}\ . \tag{2.7}$$

Decomposing

$$\phi_x = R_x e^{i\alpha_x},\quad \alpha_x \in]-\pi,\pi]\ , \tag{2.8}$$

we can write down

$$(\Delta\alpha)_{x,x+\hat{\mu}} = \alpha_{x+\hat{\mu}} - \alpha_x + 2\pi n_{x,x+\hat{\mu}}\ , \tag{2.9}$$

where $n_{x,x+\hat{\mu}}$ are integers. Taking the lattice curl of v on $p=(x,x+\hat{\mu},x+\hat{\mu}+\hat{\nu},x+\hat{\nu})$ we get

$$(\text{curl}v)_p \equiv \frac{1}{\delta}(v_{x,x+\hat{\mu}}+v_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} + v_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}}+v_{x+\hat{\nu},x}) = F_p + \frac{2\pi}{g\delta^2}n_p\ , \tag{2.10a}$$

where

$$n_p \equiv n_{x,x+\hat{\mu}} + n_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} + n_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} + n_{x+\hat{\nu},x} \tag{2.10b}$$

is a gauge-invariant quantity, which describes the total phase increase along the boundary of p (the strength of the vortices running through the plaquette) and

$$F_p \equiv \frac{1}{\delta}(A_{x,x+\hat{\mu}} + A_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} + A_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} + A_{x+\hat{\nu},x}) \tag{2.10c}$$

is another invariant, equal to the mean value of the (μ,ν) component of the field strength $F_{\mu\nu}$ on p. Due to (2.10b) we have the identity

$$(\text{div}n)_c = \sum_{p\in\partial c} n_p = 0\ . \tag{2.11}$$

[The total number of vortices through the boundary ∂c of every cube (three-dimensional lattice element) c vanishes.] Thus, we have characterized the configuration (A,ϕ) by the following set of invariants:

$$\begin{aligned}&(a)\ Z^0 \ni x \to R_x \in R^1_+\ ,\\ &(b)\ Z^1 \ni (x,x+\hat{\mu}) \to g v_{x,x+\hat{\mu}} \in \left]-\frac{\pi}{\delta},\frac{\pi}{\delta}\right[\ ,\\ &(c)\ Z^2 \ni p \to n_p \in \mathbb{Z}\ ,\end{aligned} \tag{2.12}$$

which satisfy (2.11).

It is easy to prove that starting from (2.12), one can reconstruct ϕ_x and $A_{x,y}$ up to gauge transformations. Now we allow $g v_{x,x+\hat{\mu}}$ to reach the boundary values $-\pi/\delta$ and π/δ. These situations correspond to the case when a vortex is running through $(x,x+\hat{\mu})$. In this case the value $(\Delta\alpha)_{x,x+\hat{\mu}}$ in (2.9) can be either $+\pi$ or $-\pi$ and correspondingly $n_{x,x+\hat{\mu}}$ may change its value by ± 1. This changes the vortex strength n_p on every plaquette p for which $(x,x+\hat{\mu})$ belongs to its boundary ∂p. Hence, we identify the following configurations:

$$\left[g v_{x,x+\hat{\mu}} = \frac{\pi}{\delta}, n_p\right] = \left[g v_{x,x+\hat{\mu}} = -\frac{\pi}{\delta}, \bar{n}_p\right]\ ,$$

where $\bar{n}_p = n_p+1$ if $(x,x+\hat{\mu})\in\partial p$, $\bar{n}_p=n_p-1$ if $(x+\hat{\mu},x)\in\partial p$, and $\bar{n}_p=n_p$ if $(x,x+\hat{\mu})$ and ∂p are disjoint.

The lattice dynamics of this theory is given by the following action which is the lattice version of (2.1):

$$W = -\frac{\delta^4}{2}m^2\sum_x R_x{}^2 + \frac{\delta^4}{2}\sum_{(x,x+\hat{\mu})}\frac{g^{\mu\mu}}{\delta^2}\left[\left[\frac{R_{x+\hat{\mu}}-R_x}{\delta}\right]^2 + \frac{2}{\delta^2}R_x R_{x+\hat{\mu}}(1-\cos g\delta v_{x,x+\hat{\mu}})\right] - \frac{\delta^4}{4}\sum_p g(p)\left[(\text{curl}v)_p - \frac{2\pi}{g\delta^2}n_p\right]^2\ , \tag{2.13}$$

where for the plaquette $p=(x,x+\hat{\mu},x+\hat{\mu}+\hat{\nu},x+\hat{\nu})$ we put $g(p)=g^{\mu\mu}g^{\nu\nu}/\delta^4=\pm 1$. The second term in (2.13) is a consequence of the interpolation formula (2.3) and the third term follows from (2.10a).

III. GAUGE INVARIANTS FOR THE SU(2) GAUGE FIELD IN THE ADJOINT REPRESENTATION

Now we consider the theory of an SU(2) gauge field A interacting with a three-component matter field ϕ, defined by the Lagrangian

$$L=-\mathscr{V}(|\phi|^2)+\tfrac{1}{2}|D\phi|^2-\tfrac{1}{4}|F|^2 , \tag{3.1a}$$

where $|\ |$ is calculated with the help of the scalar product $h(,)=-\frac{1}{2}K(,)$ (K is the Cartan-Killing form) on su(2) and with the space-time metric $g_{\mu\nu}$. We shall use the orthonormal basis for h: $e_a=-(i/2)\sigma_a$, where (σ_a) are the Pauli matrices. Therefore

$$D_\mu\phi^a=\partial_\mu\phi^a+g\epsilon^a{}_{bc}A^b_\mu\phi^c , \tag{3.1b}$$

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+g\epsilon^a{}_{bc}A^b_\mu A^c_\nu . \tag{3.1c}$$

Moreover, under the identification

$$\mathrm{su}(2)=R^3 \tag{3.2}$$

the adjoint representation Ad of SU(2) is isomorphic to the fundamental representation of SO(3) on R^3. Thus, treating ϕ as an R^3-valued field, we may write down gauge transformations in the following way:

$$\phi'=O^{-1}\phi , \tag{3.3a}$$

$$A'=U^{-1}AU+\frac{1}{g}U^{-1}dU , \tag{3.3b}$$

where $M\ni x\rightarrow U(x)\in \mathrm{SU}(2)$ describes the gauge and $O(x)$ is the orthogonal rotation of R^3 defined by $\mathrm{Ad}\,U(x)$.

Now we want to define the lattice approximation of a continuous configuration (A,ϕ). As in the previous section we will take the restriction of (A,ϕ) to lattice bonds. However, here we will need additional topological information about the behavior of (A,ϕ) on plaquettes. Similarly as in the U(1) case we assume the field ϕ to be covariantly linear and nonvanishing on bonds. This enables us to define the following invariants:

$$Z^0\ni x\rightarrow R_x\equiv|\phi(x)| , \tag{3.4a}$$

$$Z^1\ni(x,x+\hat{\mu})\rightarrow v_{x,x+\hat{\mu}}\equiv\frac{1}{g\delta}\int_0^\delta\left|\left[D\frac{\phi}{|\phi|}\right]_{x,x+\hat{\mu}}(\tau)\right|d\tau . \tag{3.4b}$$

It follows from linear covariance and nonvanishing of ϕ on bonds that

$$0\le g\delta v_{x,x+\hat{\mu}}<\pi . \tag{3.5}$$

The easiest way to see this is to pass to a gauge in which A vanishes along the bond $(x,x+\hat{\mu})$. Then ϕ is linear on $(x,x+\hat{\mu})$. Denoting by $\beta(\tau)$ the angle between $\phi(x)$ and $\phi(x+\tau\hat{\mu}/\delta)$ we have $\beta(0)=0$ and $0<\beta(\delta)<\pi$. On the other hand the integrand in (3.4) is equal to $(d/d\tau)\beta(\tau)$. Thus $g\delta v_{x,x+\hat{\mu}}=\beta(\delta)$. If we allow the field ϕ to vanish on bonds, we have to allow the value π for $g\delta v_{x,x+\hat{\mu}}$.

Now we denote

$$\left[D_\mu\frac{\phi}{|\phi|}\right](x)\equiv\left.\left[D_\mu\frac{\phi}{|\phi|}\right]\left[x+\tau\frac{\hat{\mu}}{\delta}\right]\right|_{\tau=0} \tag{3.6}$$

and observe that the quantities

$$w_{x;\hat{\mu}}\equiv\left[D_\mu\frac{\phi}{|\phi|}\right](x)\left|\left|\left[D_\mu\frac{\phi}{|\phi|}\right](x)\right|\right|^{-1} \tag{3.7}$$

are unit vectors lying in the plane orthogonal to $\phi(x)$. This plane can be identified with the complex plane C^1, but there is no canonical identification. Each two such identifications differ by a multiplicative factor $\zeta\in C^1$, $|\zeta|=1$. Therefore, at each $x\in Z^0$ we have a collection of unit complex numbers $w_{x;\hat{\mu}}\in S^1\subset C^1$, corresponding to the collection of bonds starting at x. This collection can be simultaneously rotated, but the numbers

$$S_{x;\hat{\mu},\hat{\nu}}\equiv w_{x;\hat{\mu}}{}^{-1}w_{x;\hat{\nu}}\in S^1 \tag{3.8}$$

are gauge invariant. They satisfy the identity

$$S_{x;\hat{\mu},\hat{\nu}}S_{x;\hat{\nu},\hat{\rho}}S_{x;\hat{\rho},\hat{\mu}}=1 , \tag{3.9}$$

for any triplet $[(x,x+\hat{\mu}),(x,x+\hat{\nu}),(x,x+\hat{\rho})]$ of bonds.

At the end we notice that if $v_{x,x+\hat{\mu}}=0$ then, of course, formula (3.7) is meaningless. In this case we choose $w_{x;\hat{\mu}}\in S^1$ and $w_{x+\hat{\mu};-\hat{\mu}}\in S^1$ arbitrarily. We will show later that this arbitrariness does not change the Lagrangian.

The invariants constructed in this section are not sufficient to reconstruct the field configuration (A,ϕ) modulo gauge. We will need additional information of topological character.

IV. TOPOLOGICAL INVARIANTS

The easiest way to construct topological invariants is to use a gauge in which

$$\phi^1(\xi)=\phi^2(\xi)=0,\ \ \phi^3(\xi)=|\phi(\xi)| , \tag{4.1}$$

for ξ lying on lattice bonds.

We decompose the restriction of A to lattice bonds into a part belonging to the Lie algebra of the stabilizer $H=\mathrm{U}(1)$ of $e_3\in\mathrm{su}(2)$ and its orthogonal complement. Due to (3.2) the first part is $(0,0,A^3)$ and the second part is $(A^1,A^2,0)$. We denote

$$B_{x,x+\hat{\mu}}(\tau)\equiv A^3_{x,x+\hat{\mu}}(\tau) , \tag{4.2a}$$

$$V_{x,x+\hat{\mu}}(\tau)\equiv A^1_{x,x+\hat{\mu}}(\tau)+iA^2_{x,x+\hat{\mu}}(\tau)\in C^1 . \tag{4.2b}$$

We still have a freedom of $H=\mathrm{U}(1)$ gauge transformations on bonds [preserving condition (4.1)]:

$$B'_{x,x+\hat{\mu}}(\tau)=B_{x,x+\hat{\mu}}(\tau)+\frac{1}{g}\frac{d}{d\tau}\lambda(\tau) , \tag{4.3a}$$

$$V'_{x,x+\hat{\mu}}(\tau)=e^{-i\lambda(\tau)}V_{x,x+\hat{\mu}}(\tau) . \tag{4.3b}$$

In terms of differential geometry the field B is a U(1)-connection form on a bundle, which is in general nontrivial.[5] This nontriviality (due to the existence of monopoles) is manifested by the fact that, unlike in electrodynamics, the quantity

$$(\mathrm{curl}B)_p \equiv \frac{1}{\delta}(B_{x,x+\hat{\mu}}+B_{x+\hat{\mu},x+\hat{\mu}+\hat{\nu}} + B_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}}+B_{x+\hat{\nu},x}) , \tag{4.4a}$$

where

$$B_{x,x+\hat{\mu}} = \frac{1}{\delta}\int_0^{\delta} B_{x,x+\hat{\mu}}(\tau)d\tau , \tag{4.4b}$$

is not invariant. To see this we take a homotopy

$$[0,\delta]\times[0,\delta]\ni(\tau,\eta)\rightarrow U(\tau,\eta)\in \mathrm{SU}(2) , \tag{4.5}$$

such that

$$U(0,\eta)=U(\delta,\eta)=U(\tau,0)=id_{\mathrm{SU}(2)} , \quad U(\tau,\delta)=\exp\left[4\pi n\frac{\tau}{\delta}e_3\right] . \tag{4.6}$$

We perform a gauge transformation on the whole plaquette putting

$$U\left[x+\tau\frac{\hat{\mu}}{\delta}+\eta\frac{\hat{\nu}}{\delta}\right]\equiv U(\tau,\eta) \tag{4.7}$$

and observe that the gauge condition (4.1) is preserved on bonds belonging to this plaquette. However,

$$B'_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} = B_{x+\hat{\mu}+\hat{\nu},x+\hat{\nu}} + \frac{4\pi}{g\delta}n , \tag{4.8}$$

whereas the B's on the other bonds remain unchanged. Therefore,

$$(\mathrm{curl}B)'_p = (\mathrm{curl}B)_p + \frac{4\pi}{g\delta^2}n . \tag{4.9}$$

This could suggest that the information about the integer part of $(g\delta^2/4\pi)(\mathrm{curl}B)_p$ is completely lost. But this is not the case, because we are able to recover this information from the topological behavior of the matter field ϕ on the plaquette p. Again we restrict ourselves to the generic case when $\phi(\xi)\neq 0$ on plaquettes. (Nongeneric configurations can be discussed in a similar way as it was shown in Sec. II.) The mapping

$$p\ni\xi\rightarrow\Psi_p(\xi)\equiv\frac{\phi(\xi)}{|\phi(\xi)|}\in S^2 \tag{4.10}$$

takes, because of (4.1), the value e_3 on the boundary ∂p of p. We denote by m_p the topological degree of Ψ_p. [$|m_p|$ tells us, how many times S^2 is covered by Ψ_p and sign(m_p) depends on whether Ψ_p preserves the orientation or not.]

One can easily check that after the gauge transformation (4.7) the degree of Ψ_p takes the new value $m'_p=m_p-n$. Therefore,

$$f_p \equiv (\mathrm{curl}B)_p + \frac{4\pi}{g\delta^2}m_p \tag{4.11}$$

is gauge invariant under (4.7). This result obviously extends to any gauge transformation preserving (4.1) on bonds. The quantity f_p can be interpreted as $(\mathrm{curl}B)_p$ in the topologically trivial gauge, where the condition (4.1) is satisfied not only on bonds, but on the whole plaquette (and consequently $m_p=0$). The topologically trivial gauge can be simultaneously performed on all plaquettes belonging to the boundary ∂c of a lattice cube c if and only if the following number vanishes:

$$Q_c=\delta(\mathrm{div}m)_c=\sum_{p\in\partial c} m_p , \tag{4.12}$$

which is usually not the case. The number Q_c assigned to each lattice cube c is the topological degree of the mapping

$$\partial c\ni\xi\rightarrow\Psi(\xi)=\frac{\phi(\xi)}{|\phi(\xi)|}\in S^2 , \tag{4.13}$$

and does not depend on any gauge condition we used earlier. The invariant quantity

$$\frac{4\pi}{g}Q_c=\delta^3(\mathrm{div}f)_c=\frac{4\pi}{g}\sum_{p\in\partial c} m_p \tag{4.14}$$

is interpreted as the total monopole charge contained in c.

Now we will show that $\exp(ig\delta^2 f_p)$ can be calculated from invariants (3.8). For this purpose we write down the condition of covariant linearity of ϕ in the gauge (4.1). We use the following notations [for simplicity we drop the index $(x,x+\hat{\mu})$]:

$$R(\tau)=\phi^3(\tau)=|\phi(\tau)| , \tag{4.15a}$$

$$DV(\tau)=\dot{V}(\tau)+igB(\tau)V(\tau) , \tag{4.15b}$$

where the overdot denotes differentiation along the bond. The $DD\phi=0$ reads

$$(DD\phi)^3=\ddot{R}-g^2|V|^2R=0 , \tag{4.16a}$$

$$(DD\phi)^1+i(DD\phi)^2=-ig(2V\cdot\dot{R}+R\cdot DV)=0 . \tag{4.16b}$$

Decomposing $V=v\cdot\omega$, where $v=|V|, |\omega|=1$, (4.16b) gives

$$[\ln(R^2v)]^{\cdot}=0 , \tag{4.17a}$$

$$D\omega=\dot{\omega}+igB\omega=0 . \tag{4.17b}$$

Therefore, denoting by $\omega_{x;\hat{\mu}}$ the value of ω corresponding to the bond $(x,x+\hat{\mu})$ and calculated at the point x we have

$$-\omega_{x+\hat{\mu};-\hat{\mu}}=\omega_{x;\hat{\mu}}\exp(-ig\delta B_{x,x+\hat{\mu}}) . \tag{4.18}$$

Thus, for a plaquette $p=(x,x+\hat{\mu},x+\hat{\mu}+\hat{\nu},x+\hat{\nu})$ we have

$$\begin{aligned}\exp(ig\delta^2 f_p)&=\exp[ig\delta(B_{x,x+\hat\mu}+B_{x+\hat\mu,x+\hat\mu+\hat\nu}+B_{x+\hat\mu+\hat\nu,x+\hat\nu}+B_{x+\hat\nu,x})]\\&=w_{x;\hat\mu}w_{x+\hat\mu;-\hat\mu}{}^{-1}w_{x+\hat\mu;\hat\nu}w_{x+\hat\mu+\hat\nu;-\hat\nu}{}^{-1}w_{x+\hat\mu+\hat\nu;-\hat\mu}w_{x+\hat\nu;\hat\mu}{}^{-1}w_{x+\hat\nu;-\hat\nu}w_{x;\hat\nu}{}^{-1}\\&=S_{x;\hat\nu,\hat\mu}S_{x+\hat\mu;-\hat\mu,\hat\nu}S_{x+\hat\mu+\hat\nu;-\hat\nu,-\hat\mu}S_{x+\hat\nu;\hat\mu,-\hat\nu}\,.\end{aligned}\tag{4.19}$$

We introduce the following notation: $y=x+\hat\mu$, $z=x+\hat\mu+\hat\nu$, and $t=x+\hat\nu$. Moreover, we denote $S_x=S_{x;\hat\nu,\hat\mu}$, $S_y=S_{x+\hat\mu;-\hat\mu,\hat\nu}$, $S_z=S_{x+\hat\mu+\hat\nu;-\hat\nu,-\hat\mu}$, and $S_t=S_{x+\hat\nu;\hat\mu,-\hat\nu}$. Thus

$$\exp(ig\delta^2 f_p)=S_xS_yS_zS_t\,. \tag{4.20}$$

Now we choose at each $x\in Z^0$ a square root $\chi_{x;\hat\mu,\hat\nu}$ of $S_{x;\hat\mu,\hat\nu}$: $\chi^2_{x;\hat\mu,\hat\nu}=S_{x;\hat\mu,\hat\nu}$, and denote by $\theta_{x;\hat\mu,\hat\nu}\in\,]-\pi,\pi]$ its phase:

$$\chi_{x;\hat\mu,\hat\nu}=\exp(i\theta_{x;\hat\mu,\hat\nu})\,. \tag{4.21}$$

Then (4.20) reads

$$g\delta^2 f_p=2(\theta_x+\theta_y+\theta_z+\theta_t)+4\pi(n_p+\tfrac{1}{2}\epsilon_p)\,, \tag{4.22}$$

where n_p is an integer and $\epsilon_p\in\mathbb{Z}_2$ (i.e., $\epsilon_p=0$ or $\epsilon_p=1$). To any "three-dimensional oriented corner" q, i.e., any triplet $[(x,x+\hat\mu),\ (x,x+\hat\nu),\ (x,x+\hat\rho)]$ of non-co-linear bonds starting at x, we assign the following complex number:

$$\gamma_q\equiv\chi_{x;\hat\mu,\hat\nu}\chi_{x;\hat\nu,\hat\rho}\chi_{x;\hat\rho,\hat\mu}\,, \tag{4.23a}$$

which is either $+1$ or -1 because of (3.9). Moreover, we define

$$\gamma_p\equiv(-1)^{\epsilon_p}\,. \tag{4.23b}$$

One can show that there is such a choice of square roots χ of S that all γ_p and γ_q are equal $+1$. (The proof of this theorem is given in the Appendix.) From now on we always use "good" χ's and θ's. Then

$$g\delta^2 f_p=2(\theta_x+\theta_y+\theta_z+\theta_t)+4\pi n_p \tag{4.24a}$$

and

$$\exp[i(\theta_{x;\hat\mu,\hat\nu}+\theta_{x;\hat\nu,\hat\rho}+\theta_{x;\hat\rho,\hat\mu})]=1=\chi_{x;\hat\mu,\hat\nu}\chi_{x;\hat\nu,\hat\rho}\chi_{x;\hat\rho,\hat\mu}\,. \tag{4.24b}$$

Now we have a complete list of invariants of our theory:

$$\begin{aligned}&\text{(a) } Z^0\ni x\to R_x\in R^1_+\,,\\&\text{(b) } Z^0\ni x\to(\chi_{x;\hat\mu,\hat\nu})_{(\hat\mu,\hat\nu)}\subset S^1\,,\\&\text{(c) } Z^1\ni(x,x+\hat\mu)\to v_{x,x+\hat\mu}\in\left[0,\frac{\pi}{g\delta}\right]\,,\\&\text{(d) } Z^2\ni p\to n_p\in\mathbb{Z}\,.\end{aligned}\tag{4.25}$$

The quantities χ may be viewed as assigned to bonds of a certain lattice Λ^*, which we define in the Appendix (see Fig. 2). It is also worthwhile to notice that for each cube c the number

$$\rho_c=\prod_{p\in\partial c}\gamma_p \tag{4.26}$$

corresponds to the $\mathbb{Z}_2$ monopole considered by Mack, Pietarinen, and Petkova.[6] As it was noticed by these authors, such monopoles do not survive in the continuum limit. In our model they can be simply removed, because of the theorem proved in the Appendix.

V. RECONSTRUCTION THEOREM

In this chapter we show that for a given set of invariants (4.25) fulfilling (4.24b) we are able to reconstruct (up to gauge transformations) the configuration (A,ϕ) on bonds together with the topologically essential behavior of ϕ on plaquettes. This means that the set (4.25) is a complete set of invariants. As a first step we calculate for every p the value f_p, using (4.24a). Next we find for each cube c the monopole number Q_c as the divergence of f according to the formula (4.14). As a consequence we have for each hypercube h

$$\sum_{c\in\partial h}Q_c=0\,. \tag{5.1}$$

Using the Poincaré lemma for integer-valued co-chains on Λ we conclude that we can find a 2-co-chain $Z^2\ni p\to m_p\in\mathbb{Z}$, such that

$$\sum_{p\in\partial c}m_p=Q_c\,. \tag{5.2}$$

Next we take $\bar f_p\equiv f_p-4\pi m_p/g\delta^2$, which satisfies $(\mathrm{div}\bar f)_c=0$. Using again the Poincaré lemma (for real-valued 2-co-chains) we find a 1-co-chain $Z^1\ni(x,x+\hat\mu)\to B_{x,x+\hat\mu}\in R^1$, such that $(\mathrm{curl}B)_p=\bar f_p$. We can now reconstruct (A,ϕ) in the gauge (4.1) on bonds. We choose a point $x_0\in Z^0$ and a collection $\{\omega_{x_0;\hat\mu}\}_{\hat\mu}$, such that for each bond $(x_0,x_0+\hat\mu)$ starting from x_0

$$\omega_{x_0;\hat\nu}=\omega_{x_0;\hat\mu}\chi^2_{x_0;\hat\mu,\hat\nu}\,. \tag{5.3}$$

To find $\omega_{x;\hat\mu}$ at any $x\in Z^0$, we choose an arbitrary path connecting x_0 and x. On each bond of this path we use (4.18). One can easily check that the result does not depend on the choice of the path. Inside bonds we put

$$\omega_\mu\left[x+\tau\frac{\hat\mu}{\delta}\right]=\omega_{x;\hat\mu}e^{-ig\tau B_{x,x+\hat\mu}}\,. \tag{5.4}$$

To reconstruct $R(\xi)=\phi^3(\xi)$ and $v(\xi)=|V(\xi)|$ on bonds we use Eqs. (4.16a) and (4.17a). The latter gives us for each bond the constant $c_{xy}=R^2(\tau)v(\tau)$. Now (4.16a) reads

$$\ddot R=g^2c_{xy}{}^2R^{-3}\,. \tag{5.5}$$

There is a unique solution to this equation fulfilling the boundary conditions $R(x)=R_x$ and $R(y)=R_y$:

$$R^2(\tau)=R_x{}^2(1-\tau/\delta)^2+2[R_x{}^2R_y{}^2-(g\delta c_{xy})^2]^{1/2}\times(1-\tau/\delta)\tau/\delta+R_y{}^2(\tau/\delta)^2\,. \quad (5.6)$$

In our gauge we have $|D_\mu(\phi/|\phi|)|=g|V_\mu|$. Thus (3.4b) gives

$$v_{xy}=\frac{1}{\delta}\int_0^\delta v(\tau)d\tau=\frac{1}{\delta}\int_0^\delta c_{xy}R^{-2}(\tau)d\tau=\frac{1}{g\delta}\arctan\left[\frac{g\delta c_{xy}R_x{}^{-1}R_y{}^{-1}}{[1-(g\delta c_{xy})^2R_x{}^{-2}R_y{}^{-2}]^{1/2}}\right]. \quad (5.7)$$

Hence $c_{xy}=(g\delta)^{-1}R_xR_y\sin(g\delta v_{xy})$. Finally

$$R^2(\tau)=R_x{}^2\left[1-\frac{\tau}{\delta}\right]^2+2R_xR_y\cos(g\delta v_{xy})\frac{\tau}{\delta}\left[1-\frac{\tau}{\delta}\right]+R_y{}^2\left[\frac{\tau}{\delta}\right]^2 \quad (5.8)$$

and

$$v(\tau)=\frac{1}{g\delta}\frac{\sin(g\delta v_{xy})}{R_xR_y{}^{-1}\left[1-\frac{\tau}{\delta}\right]^2+2\cos(g\delta v_{xy})\frac{\tau}{\delta}\left[1-\frac{\tau}{\delta}\right]+R_x{}^{-1}R_y\left[\frac{\tau}{\delta}\right]^2}\,. \quad (5.9)$$

Thus, we have reconstructed the field (A,ϕ) on bonds. We have also reconstructed the topological degree m_p of the mapping (4.10)

VI. FIELD DYNAMICS

The dynamics of our model will be governed by the difference equations obtained from the lattice approximation of the action defined by the Lagrangian (3.1a). For the scalar term we put

$$W_{\rm I}=-\delta^4\sum_{x\in Z^0}\mathscr{V}(R_x{}^2)\,. \quad (6.1a)$$

The vector term is calculated using the fact that $|D\phi|^2$ is constant on bonds. The result is

$$W_{\rm II}=\frac{\delta^4}{2}\sum_{(x,x+\hat\mu)}\frac{g^{\mu\mu}}{\delta^2}\left[\left[\frac{R_{x+\hat\mu}-R_x}{\delta}\right]^2+\frac{2}{\delta^2}R_xR_{x+\hat\mu}\times[1-\cos(g\delta v_{x,x+\hat\mu})]\right]. \quad (6.1b)$$

To calculate the tensorial part we pass (on each plaquette separately, because simultaneously it is impossible) to the topologically trivial gauge $\phi^3=|\phi|,\phi^1=\phi^2=0$. (Of course, the final result will be expressed in terms of invariants and, therefore, not depending on the gauge chosen inside the plaquette.) Denoting $A_\mu^3=B_\mu$ and $A_\mu^1+iA_\mu^2=V_\mu$ (3.1c) takes in the above gauge the form

$$F_{\mu\nu}^3=\partial_\mu B_\nu-\partial_\nu B_\mu-g\,{\rm Im}(V_\mu V_\nu^*)\,, \quad (6.2a)$$

$$F_{\mu\nu}^1+iF_{\mu\nu}^2=D_\mu V_\nu-D_\nu V_\mu\,, \quad (6.2b)$$

where $D_\mu V_\nu=\partial_\mu V_\nu+igB_\mu V_\nu$. As a lattice version of $\partial_\mu B_\nu-\partial_\nu B_\mu$ on the plaquette p we take its mean value on p, equal to f_p, which we calculate from (4.24a). [We remember that in the topologically trivial gauge $m_p=0$, see (4.11).]

To calculate the lattice version of ${\rm Im}(V_\mu V_\nu^*)$, we replace $|V_\mu|$ on each bond $(x,x+\hat\mu)$ by its mean value $v_{x,x+\hat\mu}$. At each corner x of p we take the value

$$a_{x;\hat\mu,\hat\nu}\equiv-gv_{x,x+\hat\mu}v_{x,x+\hat\nu}{\rm Im}(\chi^2_{x;\hat\nu,\hat\mu})=gv_{x,x+\hat\mu}v_{x,x+\hat\nu}\sin(2\theta_{x;\hat\mu,\hat\nu})\,, \quad (6.3)$$

and average over four corners:

$$(F_{\mu\nu}^3)_p=F_p\equiv f_p+\tfrac{1}{4}(a_{x;\hat\mu,\hat\nu}+a_{x+\hat\mu;\hat\nu,-\hat\mu}+a_{x+\hat\mu+\hat\nu;-\hat\mu,-\hat\nu}+a_{x+\hat\nu;-\hat\nu,\hat\mu})\,. \quad (6.4)$$

To calculate the lattice version of $F_{\mu\nu}^1+iF_{\mu\nu}^2$ we introduce the notation as in Fig. 1: $a_1\equiv\frac{1}{4}a_{x;\hat\mu,\hat\nu}$ and similarly a_2, a_3, and a_4, $B_{12}\equiv B_{x,x+\hat\mu}$, and similarly B_{23}, B_{34}, and B_{41}. The field V will be represented at centers of bonds using the actual phase ω and the mean value v of $|V|$:

$$V_{12}\equiv v_{12}\omega_{x+\hat\mu/2;\hat\mu} \quad (6.5)$$

and similarly V_{23}, V_{34}, and V_{41}. Using (4.18) and (5.3) we have

$$-V_{12}v_{12}{}^{-1}=V_{41}v_{41}{}^{-1}e^{-ig\delta b_1}S_1\,, \quad (6.6)$$

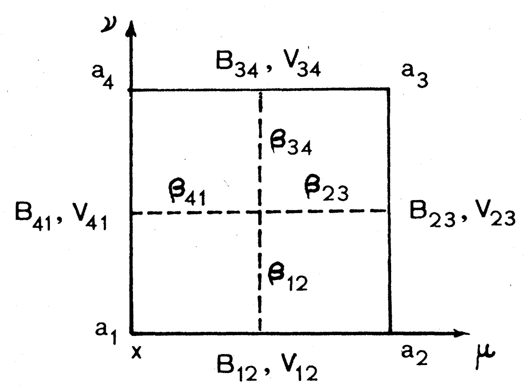

FIG. 1. The fields β interpolate the fields B on the internal bonds of the plaquette.

where

$$b_1=\frac{1}{\delta}\int_0^{\delta/2} B_{x,x+\hat{\mu}}(\tau)d\tau-\frac{1}{\delta}\int_0^{\delta/2} B_{x,x+\hat{\nu}}(\tau)d\tau \qquad (6.7)$$

and $S_1=S_{x;\hat{\nu},\hat{\mu}}$. In the simplest gauge when B is constant on bonds we have $b_1=(B_{41}+B_{12})/2$. Similarly we define b_2, b_3, b_4, and S_2, S_3, and S_4, fulfilling formulas analogous to (6.6). If ω stands for $\omega_{x+\hat{\nu}/2,\hat{\nu}}$, then

$$\begin{aligned} V_{12}&=v_{12}\omega S_1 e^{-ig\delta b_1} ,\\ -V_{23}&=v_{23}\omega S_1 S_2 e^{-ig\delta(b_1+b_2)} ,\\ V_{34}&=v_{34}\omega S_1 S_2 S_3 e^{-ig\delta(b_1+b_2+b_3)} ,\\ -V_{41}&=v_{41}\omega S_1 S_2 S_3 S_4 e^{-ig\delta(b_1+b_2+b_3+b_4)}=v_{41}\omega , \end{aligned} \qquad (6.8)$$

because of (4.19) and (4.20). To calculate (6.2b) we will covariantly differentiate V on "internal bonds" connecting centers of bonds (dashed lines in Fig. 1). For this purpose we need the interpolation β of the field B to these four internal bonds, see Fig. 1. We define this interpolation demanding the curvature F^3 to be equal on four internal plaquettes. (The physical meaning of this demand becomes clear, if we analyze the canonical structure of lattice gauge theories.[7]) More precisely, we demand $F_1=F_2=F_3=F_4$, where

$$\begin{aligned} F_1&=\frac{1}{\delta/2}(2b_1-\beta_{12}+\beta_{41})+4a_1\\ &=4(b_1\delta^{-1}+a_1)-2\delta^{-1}(\beta_{12}-\beta_{41}) ,\\ F_2&=4(b_2\delta^{-1}+a_2)-2\delta^{-1}(\beta_{23}-\beta_{12}) ,\\ F_3&=4(b_3\delta^{-1}+a_3)-2\delta^{-1}(\beta_{34}-\beta_{23}) ,\\ F_4&=4(b_4\delta^{-1}+a_4)-2\delta^{-1}(\beta_{41}-\beta_{34}) . \end{aligned} \qquad (6.9)$$

Equations (6.9) are linearly dependent, namely, $(F_1+F_2+F_3+F_4)/4=F_p$. The four quantities β can be therefore calculated up to one arbitrary constant. This arbitrariness reflects the freedom of choosing the U(1) gauge at the center of p. As this constant we take $4\beta=\beta_{12}+\beta_{23}+\beta_{34}+\beta_{41}$. Then we obtain

$$\begin{aligned} \beta_{12}&=\beta+\tfrac{3}{4}(b_1-b_2+\delta(a_1-a_2))\\ &\quad+\tfrac{1}{4}(b_4-b_3+\delta(a_4-a_3)) ,\\ \beta_{23}&=\beta+\tfrac{3}{4}(b_2-b_3+\delta(a_2-a_3))\\ &\quad+\tfrac{1}{4}(b_1-b_4+\delta(a_1-a_4)) ,\\ \beta_{34}&=\beta+\tfrac{3}{4}(b_3-b_4+\delta(a_3-a_4))\\ &\quad+\tfrac{1}{4}(b_2-b_1+\delta(a_2-a_1)) ,\\ \beta_{41}&=\beta+\tfrac{3}{4}(b_4-b_1+\delta(a_4-a_1))\\ &\quad+\tfrac{1}{4}(b_3-b_2+\delta(a_3-a_2)) . \end{aligned} \qquad (6.10)$$

As a lattice approximation of $D_\mu V_\nu$ we take

$$\begin{aligned} (D_\mu V_\nu)_p&=\frac{1}{\delta}(V_{23}e^{ig\delta\beta_{23}/2}-V_{14}e^{ig\delta\beta_{41}/2}) ,\\ (D_\nu V_\mu)_p&=\frac{1}{\delta}(V_{43}e^{ig\delta\beta_{34}/2}-V_{12}e^{ig\delta\beta_{12}/2}) , \end{aligned} \qquad (6.11)$$

and consequently

$$(F^1_{\mu\nu}+iF^2_{\mu\nu})_p=\frac{1}{\delta}(V_{12}e^{ig\delta\beta_{12}/2}+V_{23}e^{ig\delta\beta_{23}/2}+V_{34}e^{ig\delta\beta_{34}/2}+V_{41}e^{ig\delta\beta_{41}/2}) . \qquad (6.12)$$

Using (6.10), (6.8), and (6.4) we obtain after some lengthy calculation

$$(F^1_{\mu\nu}+iF^2_{\mu\nu})_p=\zeta_p G_p , \qquad (6.13)$$

where

$$\begin{aligned} G_p=\frac{1}{\delta}\Bigg[& v_{12}\exp\left\{-i\left[\frac{3}{4}\left[\theta_2-\theta_1+g\frac{\delta^2}{2}(a_2-a_1)\right]+\frac{1}{4}\left[\theta_3-\theta_4+g\frac{\delta^2}{2}(a_3-a_4)\right]\right]\right\}\\ &-(-1)^{n_p}v_{23}\exp\left\{-i\left[\frac{3}{4}\left[\theta_3-\theta_2+g\frac{\delta^2}{2}(a_3-a_2)\right]+\frac{1}{4}\left[\theta_4-\theta_1+g\frac{\delta^2}{2}(a_4-a_1)\right]\right]\right\}\\ &+v_{34}\exp\left\{-i\left[\frac{3}{4}\left[\theta_4-\theta_3+g\frac{\delta^2}{2}(a_4-a_3)\right]+\frac{1}{4}\left[\theta_1-\theta_2+g\frac{\delta^2}{2}(a_1-a_2)\right]\right]\right\}\\ &-(-1)^{n_p}v_{41}\exp\left\{-i\left[\frac{3}{4}\left[\theta_1-\theta_4+g\frac{\delta^2}{2}(a_1-a_4)\right]+\frac{1}{4}\left[\theta_2-\theta_3+g\frac{\delta^2}{2}(a_2-a_3)\right]\right]\right\}\Bigg] \end{aligned} \qquad (6.14)$$

and $\zeta_p\in S^1$ is a phase factor,

$$\zeta_p=(-1)^{n_p}\omega\, e^{ig(\delta/2)\beta}\exp\left\{-ig\frac{\delta}{2}\left[\tfrac{3}{4}(b_1-b_4)+\tfrac{1}{4}(b_2-b_3)\right]\right\}\exp\{-i[\tfrac{3}{4}(\theta_4-\theta_1)+\tfrac{1}{4}(\theta_3-\theta_2)]\} . \qquad (6.15)$$

This factor corresponds to the arbitrariness in choosing the gauge. We remember that if one of the numbers $v_{x,x+\hat{\mu}}$ vanishes, the corresponding phases $w_{x;\hat{\mu}}$ and $w_{x+\hat{\mu};-\hat{\mu}}$ can be chosen arbitrarily. If, e.g., $v_{12}=0$ we have the arbitrariness $\bar{\theta}_1=\theta_1+\alpha$ and $\bar{\theta}_2=\theta_2-\alpha$. It is easy to check that $|G_p|$ does not depend on the choice of α, which proves the consistency of our approach.

Finally we put

$$W_{\rm III}=-\tfrac{1}{4}\delta^4\sum_p(F_p{}^2+|G_p|^2) \tag{6.16}$$

and

$$W=W_{\rm I}+W_{\rm II}+W_{\rm III}\ . \tag{6.17}$$

The above formula for the action W is a good lattice approximation of the action (3.1a). In the case of the classical theory we will derive the difference equations of our theory by varying W with respect to variables (4.25). The equations are elliptic or hyperbolic according to whether we take Euclidean or Minkowski space-time M. [It can be proved, e.g., that in the latter case the Cauchy problem on the lattice is well posed and is causal, i.e., the values of variables (4.25) in a certain region ϑ of the lattice depend only on that part of Cauchy data which lies in the past "light cone" of ϑ. These results will be published elsewhere.] In the quantum case W can be defined for Feynman path integrals. There will be no "ghosts" in such a theory since all the constraints have been removed and the whole theory is formulated in terms of gauge invariants. The integration over the set (4.25a)–(4.25c) of continuous variables will be performed with respect to the measure which we inherit from the continuous theory. We thus have to integrate from 0 to ∞ over $R_x{}^2dR_x$ and from 0 to $\pi/g\delta$ over $dv_{x,x+\mu}$. The integration over χ's is more complicated due to the constraint (4.24b). The constraint defines a plane in the space of variables θ and Dirac δ on this plane has to be used. The choice of independent parametrization of the plane is not straightforward. A possible way to overcome this difficulty is to integrate over unconstrained variables (3.7). Here, all physically meaningful functions have to be invariant with respect to the simultaneous rotation of all w's at each point $x\in Z^0$ independently [it is the case of the Lagrangian (6.17)].

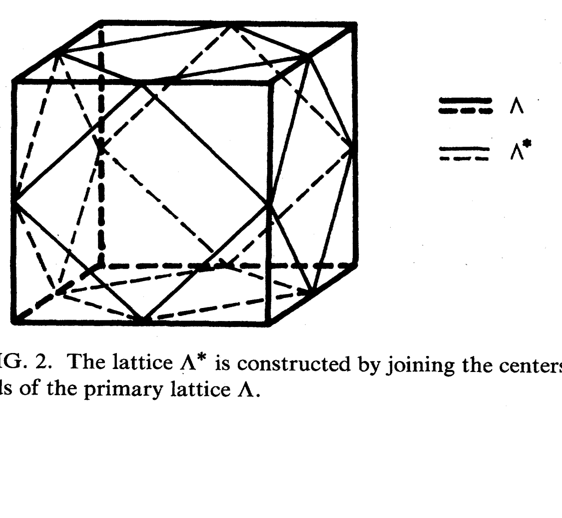

FIG. 2. The lattice Λ^* is constructed by joining the centers of bonds of the primary lattice Λ.

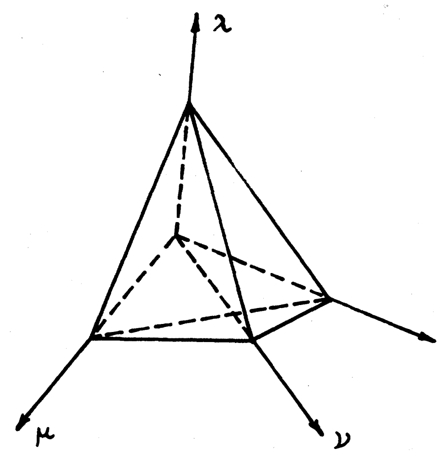

FIG. 3. Typical three-dimensional element of Λ^* with triangular faces.

The integration with respect to the discrete degrees of freedom (4.25d) is simply the summation.

APPENDIX

Theorem. There exists a choice of square roots χ, such that all γ_p and γ_q defined by (4.23) are equal to $+1$.

Proof. Consider a new lattice Λ^* composed of centers of bonds of Λ. Bonds of Λ^* are defined as lines connecting centers of bonds of Λ, which are nearest neighbors (see Fig. 2).

Then we have two types of plaquettes of Λ^*: rectangular ones, corresponding to plaquettes of Λ, and triangular ones, corresponding to "three dimensional oriented corners" of Λ. Define a $\mathbf{Z}_2$-valued 2-co-chain $\gamma\equiv\{\gamma_q,\gamma_p\}_{q,p}$ on Λ^* by putting γ_p [defined by (4.23b)] on rectangular plaquettes and γ_q [defined by (4.23a)] on triangular plaquettes. We prove that the coboundary of γ vanishes, i.e.,

$$(\partial\gamma)_V=\prod_{p\in\partial V}\gamma_p\prod_{q\in\partial V}\gamma_q=1\ ,$$

for each three-dimensional element V of Λ^*. There are two types of such elements: (a) "cubes with cut corners" (see Fig. 2) and (b) elements, which have only triangular faces (Fig. 3). Let V be a "cut cube" corresponding to a cube c of Λ. For any rectangular plaquette of ∂V corresponding to a plaquette $p\in\partial c$ we have

$$\tfrac{1}{2}g\delta^2 f_p-(\theta_1+\theta_2+\theta_3+\theta_4)-2\pi n_p=\pi\epsilon_p\ ,$$

or

$$\gamma_p=e^{i\pi\epsilon_p}=e^{ig\delta^2 f_p/2}(\chi_1\chi_2\chi_3\chi_4)^{-1}\ .$$

Therefore

$$(\partial\gamma)_V=\prod_{p\in\partial c}e^{ig\delta^2 f_p/2}\prod_{i=1,\ldots,24}\chi_i{}^{-1}\prod_{i=1,\ldots,24}\chi_i\ ,$$

where the index i numbers all the 24 corners belonging to the 6 faces of c. The third factor in $(\partial\gamma)_V$ comes from triangular plaquettes. Thus,

$$(\partial\gamma)_V=\exp\left[\frac{i}{2}g\delta^3(\text{div}f)_c\right]=\exp(2\pi i Q_c)=1 \ .$$

Now, let V be an element which has only triangular faces: But for triangular faces γ_q was already defined by the ∂ operator [formula (4.23a) means that $\gamma_q=(\partial\chi)_q$]. Therefore, $(\partial\gamma)_V=1$ is a trivial consequence of $\partial^2=0$. This proves that γ is closed. Now we have from Poincaré lemma that there exists a $\mathbb{Z}_2$-valued 1-co-chain $\{\lambda_i\}$, such that $\gamma=\partial\lambda$ (i corresponds to bonds of Λ^*, i.e., to corners of Λ and $\lambda_i=\pm 1$). We use λ_i to "improve" the choice of square roots χ_i:

$$\bar{\chi}_i=\chi_i\lambda_i \ .$$

Now it is obvious that the co-chain γ for $\bar{\chi}$ vanishes.

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