PHYSICAL REVIEW D VOLUME 31, NUMBER 4 15 FEBRUARY 1985

Galilean-invariant gauge theory

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(Received 15 October 1984)

It is generally characteristic of a field theory with a zero-mass particle that it does not possess a nontrivial Galilean limit. Since all the well-known gauge theories require (at least in the free-field limit) such massless excitations, there are no known examples at this time of Galilean-invariant gauge field theories. However, by making use of a recently formulated gauge theory in two spatial dimensions in which there is no elementary photon, it is shown that there does exist a Galilean theory which incorporates the gauge principle. The general N-particle state for this theory is constructed and subsequently used to obtain the corresponding N-particle Schrödinger equation. In the case of two particles the scattering process is considered explicitly, it is being shown that for all partial waves one obtains the same nonzero phase shift.

I. INTRODUCTION

The Galilean-invariant formulation of wave equations and field theories has been found in recent years to be a useful tool for the study of problems which are exceedingly difficult when addressed within the full formalism of special relativity. In particular, this approach has been used by Levy-Leblond[1] to show that the well-known result for the g factor of an electron is not a consequence of Lorentz invariance, numerous claims to the contrary notwithstanding. The extension of the work of Levy-Leblond from the spin-$\frac{1}{2}$ case to general spin[2,3] led to the proof of the long-standing conjecture of Belinfante that the g factor is quite generally the reciprocal of the spin value. This result, which was first proved[3] in the Galilean case for so-called minimal theories, was subsequently shown to apply in Minkowski space as well.[4] The latter result was readily obtained from the Lorentz-invariant wave equations for arbitrary-spin particles, whose formulation was first achieved in Ref. 4 following the solution of the more tractable Galilean problem.

It would therefore seem to be prudent in the light of these advances to have at one's disposal a Galilean-invariant model for the study of the problems one encounters in modern quantum field theory. Although Galilean field theories do exist and have been studied, their applicability has been limited because they have yet to be extended beyond the level of the simple Lee model and its variants. The reason for this limitation is the fact that there is very little interest currently in any model which does not possess a local gauge invariance. This circumstance, together with the Bargmann superselection rule on the mass, has thus far prevented a successful formulation of a Galilean-invariant theory which could have any significant impact on modern field theory. To spell this out in detail one recognizes that a Galilean theory allows a particular vertex only if the sum of the masses which enter equals the sum of the masses which leave the interaction point. Since the basic interaction in a gauge theory is Yukawa-like, one can only expect a model to have a nontrivial Galilean limit if the massless gauge particle of the relativistic theory can be consistently described in that same limit. One does not at present know how to do this, nor is it clear from a physical point of view that it is reasonable to attempt the incorporation of massless particles into a Galilean (or "nonrelativistic") framework.

One recent development[5] does, however, offer an opportunity to fill this void by accommodating the Bargmann superselection rule in a relatively harmless manner. The model to which reference is being made here is in a world of two spatial dimensions and is described in its Poincaré-invariant version by the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\phi^{\mu}\epsilon_{\mu\nu\alpha}\partial^{\alpha}\phi^{\nu}+g\phi^{\mu}j_{\mu}+\mathscr{L}' , \qquad (1)$$

where $\mathscr{L}'$ consists of terms not containing the gauge field ϕ^{μ}. The Lagrangian (1) is invariant (up to a divergence) under

$$\phi^{\mu}\rightarrow\phi^{\mu}+\partial^{\mu}\Lambda ,$$

thereby allowing one to select, for example, the radiation gauge

$$\nabla\cdot\boldsymbol{\phi}=0 \qquad (2)$$

or the axial gauge

$$\mathbf{n}\cdot\boldsymbol{\phi}=0 , \qquad (3)$$

where $\mathbf{n}\cdot\mathbf{n}=1$. In the former case one finds that the equations implied by (1) yield for the ϕ^{μ} fields the results

$$\phi_i(x)=-g\epsilon_{ij}\partial_j\int d^2x'\mathscr{D}(x-x')j^0(x') \qquad (4)$$

and

$$\phi^0(x)=g\int d^2x'\mathbf{j}(x')\times\nabla'\mathscr{D}(x-x') , \qquad (5)$$

where $\mathscr{D}(x)$ is defined by

$$-\nabla^2\mathscr{D}(x)=\delta(\mathbf{x})$$

and ϵ_{ij} is the Levi-Civita tensor in two spatial dimensions. The relevant observation here is that $\phi^{\mu}(x)$ is merely a function of the current operator components and consequently does not have an elementary particle associated with it. Because of this, each of the terms in the Lagrangian will conform to the Bargmann superselection rule in the Galilean limit and the goal of constructing a Galilean-covariant gauge theory can be realized. Although the structure of the ϕ^μ fields is substantially different in the axial gauge, identical conclusions are reached in that case concerning the existence of a Galilean gauge theory.

In the following section the model is presented and its invariances discussed. Section III considers formal questions related to the definition of bilinear operators and shows that the technique of point splitting is in general necessary in order to preserve gauge invariance. In Sec. IV the general N-particle state is constructed and the Schrödinger equation derived for the corresponding wave function. Two-body scattering is examined in V with strikingly simple (yet somewhat bizarre) results being obtained. The relationship between N-body wave functions in different gauges is examined in Sec. VI and some concluding remarks offered in Sec. VII.

II. GAUGE INVARIANCE IN A GALILEAN FIELD THEORY

In order to avoid nonessential complications as much as possible, what will be considered here is a mass-m spinless particle described by the field $\psi(x)$. The gauge field ϕ^μ of the Minkowski-space theory becomes in the Galilean limit the two-vector $\phi_i(x)$ and the scalar $\phi(x)$. The proposed Lagrangian is

$$\mathscr{L}=i\psi^\dagger\left[\frac{\partial}{\partial t}+ig\phi\right]\psi$$
$$+\frac{1}{2m}[\boldsymbol{\psi}^\dagger\cdot\boldsymbol{\psi}+\boldsymbol{\psi}^\dagger\cdot(\nabla-ig\boldsymbol{\phi})\psi-\psi^\dagger(\nabla-ig\boldsymbol{\phi})\cdot\boldsymbol{\psi}]$$
$$-\frac{1}{2}\phi\nabla\times\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\phi}\times\nabla\phi-\frac{1}{2}\boldsymbol{\phi}\times\frac{\partial}{\partial t}\boldsymbol{\phi} \qquad (6)$$

which implies the equations of motion

$$-\nabla\times\boldsymbol{\phi}=g\rho ,$$
$$\epsilon_{ij}\left[\frac{\partial}{\partial t}\phi_j+\nabla_j\phi\right]=gj_i ,$$
$$i\frac{\partial}{\partial t}\psi=g\phi\psi+\frac{1}{2m}(\nabla-ig\boldsymbol{\phi})\cdot\boldsymbol{\psi} ,$$
$$\boldsymbol{\psi}=-(\nabla-ig\boldsymbol{\phi})\psi ,$$

where ρ and j_i are formally defined by

$$\rho=\psi^\dagger\psi , \qquad (7)$$

$$j_i=\frac{1}{2mi}(\psi_i^\dagger\psi-\psi^\dagger\psi_i) . \qquad (8)$$

The forms (7) and (8) are, of course, familiar in wave mechanics as the usual Schrödinger-equation expressions for the probability density and current. No assumption need be made at this time concerning the relative ordering of matter field and gauge field operators.

The equations for the operators ϕ_i and ϕ can be solved explicitly in terms of ρ and j_i. In the radiation gauge these are precisely the same as Eqs. (4) and (5) provided that ϕ^0 and j^0 are replaced by ϕ and ρ. Since the non-Abelian version of (6) is not readily quantized in the radiation gauge,[6] it is of considerable interest to consider in parallel with that formulation the corresponding results in the axial gauge. There one finds[6] that the expressions for ϕ_i and ϕ become

$$\phi_i(x)=-g\epsilon_{ij}n_j\int d^2x'd_n(x-x')\rho(x') , \qquad (9)$$

$$\phi(x)=g\int d^2x'd_n(x-x')\mathbf{n}\times\mathbf{j}(x') , \qquad (10)$$

where the function $d_n(x)$ is defined by

$$-n\cdot\nabla d_n(x)=\delta(\mathbf{x})$$

and clearly has the explicit form

$$d_n(x)=-\tfrac{1}{2}\epsilon(\mathbf{x}\cdot\mathbf{n})\delta(\mathbf{n}\times\mathbf{x}) ,$$

where $\epsilon(x)$ is the alternating function

$$\epsilon(x)\equiv x/|x| .$$

Although Poincaré invariance has not been established for the non-Abelian axial gauge theory,[6] the less rigid structure imposed by the Galilean group will be seen to allow a Galilei-covariant theory for both Abelian and non-Abelian couplings. It is in fact this particular feature which principally motivates the simultaneous consideration of the axial gauge.

In order to demonstrate that (6) is Galilei covariant as alleged, one must find a set of seven operators (P_i, K_i, H, J, and M) such that the commutation relations

$$\begin{aligned}&[J,K_i]=i\epsilon_{ij}K_j,\ \ [K_i,H]=iP_i ,\\ &[J,P_i]=i\epsilon_{ij}P_j,\ \ [K_i,K_j]=0 ,\\ &[J,H]=[J,M]=0,\ \ [K_i,P_j]=i\delta_{ij}M ,\\ &[P_i,P_j]=[P_i,H]=[H,M]=[K_i,M]=[P_i,M]=0\end{aligned} \qquad (11)$$

corresponding to a one-dimensional extension of the Galilei group are satisfied. It is possible to consider in the case of two spatial dimensions a two-dimensional central extension of the group by replacing the commutator of K_i with itself by

$$[K_i,K_j]=i\epsilon_{ij}K ,$$

where K commutes with all other operators of the set. However, it is easy to see that the definition

$$K_i'\equiv K_i+\tfrac{1}{2}KM^{-1}\epsilon_{ij}P_j$$

allows one to transform to a set of operators (P_i,K_i', H, J, M) which satisfy the commutation relations (11). Thus the existence of a set which satisfies (11) is essential.

Since the basic equal-time commutator of the model is simply

$$[\psi(x),\psi^\dagger(x')]=\delta(\mathbf{x}-\mathbf{x}') ,$$

one readily infers that the mass operator M should have the form

$$M=m\int\psi^\dagger\psi\,d^2x$$

and that

$$[M,\psi(x)]=-m\psi(x)\ ,$$
$$[M,\phi_i]=[M,\phi]=0\ ,$$

which incidentally displays the claim made here that the set $\phi_i(x)$ and $\phi(x)$ are in essence massless gauge fields. The operators P_i are also found in the standard manner to have the form

$$P_i=m\int j_i(x)d^2x$$

which satisfy

$$[P_i,\psi(x)]=i\nabla_i\psi(x)\ .$$

The rotation operator J is somewhat more interesting. It is given according to the canonical formalism by

$$J=m\int \mathbf{r}\times\mathbf{j}\,d^2x \qquad (12)$$

and can be decomposed[5] in the radiation gauge into the two parts

$$J=J_0+\frac{g^2}{4\pi}Q^2\ ,$$

where J_0 is the $g=0$ form of J and $Q\equiv M/m$. This implies the noncanonical form for the commutator of J with ψ

$$[J,\psi(x)]=i\mathbf{x}\times\nabla\psi(x)-\frac{g^2}{4\pi}\{Q,\psi(x)\} \qquad (13)$$

exactly as in the Poincaré-invariant version.[5,7] There is, however, the distinct difference here that J in the Galilean version can be redefined to be

$$J'=J-\frac{g^2}{4\pi}Q^2$$

without modifying the structure relations and restoring (13) to its canonical form. This cannot be done in special relativity since in that case the commutator $[K_1,K_2]$ must reproduce J. It will be the view here that this latter point is compelling and that J consequently should not be redefined so long as its relativistic counterpart necessarily includes the anomaly. Before leaving this discussion of the operator J it is appropriate to remark that its form is quite different in the axial gauge. Since the implications of this fact are, however, more appropriate to the following section of this paper, a more complete elaboration of this point will be deferred for the moment.

The operators K_i are also readily found and have the form

$$K_i=m\int x_i\rho(x)d^2x-tP_i\ .$$

The commutator with $\psi(x)$ is thus

$$[K_i,\psi(x)]=-mx_i\psi(x)-it\nabla_i\psi(x)$$

which implies after some calculation the gauge-field commutation relations

$$[K_i,\phi_j(x)]=-it\nabla_i\phi_j(x)\ ,$$
$$[K_i,\phi(x)]=i\phi_i-it\nabla_i\phi\ ,$$

which are valid in both radiation and axial gauges. This, together with the form of the Hamiltonian

$$H=\frac{1}{2m}\int\boldsymbol{\psi}^\dagger\cdot\boldsymbol{\psi}\,d^2x\ ,$$

allows one to verify that the commutator of Galilean boosts with H has the correct form. Direct calculation also yields the expected result

$$[H,\psi(x)]=-i\frac{\partial}{\partial t}\psi(x)\ .$$

Using the asserted form of the several global Galilei operators which have been given here, one can now proceed to determine whether they do indeed satisfy the structure relations of the group. By using the results quoted here for the commutators with the canonical fields it is entirely straightforward to verify that this is the case. The goal of constructing a Galilean-covariant gauge theory has thus been achieved and attention can now be directed to consequences implied by this model.

III. OPERATOR PRODUCTS

It has been recognized for some years that formal operator products are generally ambiguous and require a more detailed prescription than is generally provided by the usual naive definition. Schwinger, in particular, has pointed out that inconsistencies follow from the assumption that certain components of a conserved current operator commute with each other at a given time.[8] Since a formal calculation of such commutators in conventional spinor electrodynamics seems to imply commutativity of those operator components, one is forced to replace the usual local definition of the current by one in which the coordinates of the elementary field operators are taken to be different and then allowed to approach confluence by some rule to be applied at the end of a given calculation. This so-called point-splitting method eliminates the formal contradiction and can furthermore be extremely useful in the evaluation of anomalies.[9]

In the case of scalar electrodynamics it is less certain that point splitting is essential inasmuch as the noncommutativity of operator components emerges from the canonical formalism regardless of whether that device is used. Thus, for example, in the Galilean theory considered here one avoids the problem of Ref. 8 by noting that the local operator definitions imply

$$[\rho(x),\mathbf{j}(x')]=-\frac{i}{m}\nabla\rho(x)\delta(x-x')\ . \qquad (14)$$

It will nonetheless be argued here that (14) does not entirely eliminate the necessity for point splitting in this model. One will conclude from such a demonstration that also in relativistic theories [e.g., the theory described by (1)] operator definitions must be of concern even in cases for which $[j^0(x),j_i(x')]$ is manifestly nonzero.

The argument proceeds most directly from the necessity that the eigenvalues of gauge-invariant operators must not depend on the gauge in which the calculation is performed. If, for example, one considers the single-particle zero-momentum state

$$|p=0\rangle=\int d^2x\,\psi^\dagger(x)\,|0\rangle\,,$$

it is easy to see that the expression quoted earlier for J implies that

$$J\,|p=0\rangle=\frac{g^2}{4\pi}\,|p=0\rangle\,, \tag{15}$$

i.e., that this is an eigenstate of J with eigenvalue $g^2/4\pi$. Since the operator J which has been used here is the one appropriate to the radiation gauge, it is of interest to determine whether identical results are assured for the axial gauge.

Upon using the same starting point (12) for J, but now using axial-gauge expressions for $\phi_i(x)$, one obtains in place of (15)

$$\begin{aligned}J\,|p=0\rangle&=-g^2\int x\cdot n\rho(x)\rho(x')\\&\qquad\times d_n(x-x')d^2x\,d^2x'\,|p=0\rangle\\&=\frac{g^2}{4}\int|(\mathbf{x}-\mathbf{x}')\cdot n\,|\,\delta[(\mathbf{x}-\mathbf{x}')\times\mathbf{n}]\\&\qquad\times\rho(x)\rho(x')d^2x\,d^2x'\,|p=0\rangle\,.\end{aligned}$$

If one insists upon a strictly local form of $\rho(x)$, this leads to the conclusion that the eigenvalue of J is the indeterminate form

$$\frac{g^2}{4}\,|\,0\,|\,\delta(0)\,.$$

This ambiguity can be circumvented by the redefinition of $\rho(x)$ as

$$\rho(x)=\lim_{\epsilon\to 0}\psi^\dagger\left[x+\frac{\epsilon}{2}\right]\psi\left[x-\frac{\epsilon}{2}\right]\,,$$

where the two-dimensional vector ϵ is to be symmetrically averaged before taking the limit. The J eigenvalue now becomes

$$\lim_{\delta\to 0}\frac{g^2}{4}\frac{1}{(\pi\delta^2)^2}\int_{|\epsilon|,|\epsilon'|<\delta}d^2\epsilon\,d^2\epsilon'\,|(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}')\cdot\mathbf{n}\,|\times\delta((\boldsymbol{\epsilon}-\boldsymbol{\epsilon}')\mathrm{x}\mathbf{n}) \tag{16}$$

which indicates that the averaging is to be done over a circle of radius δ which is then allowed to go to zero. It is straightforward to show that upon completion of the calculation prescribed in (16) the equality of the angular momentum eigenvalues in the two different gauges is verified.

This particular application of the point splitting method is probably the most interesting argument for its incorporation in the Galilean model considered here since it requires the performance of a specific calculation. There are, however, other places in which the necessity of point splitting is made abundantly clear. Since they generally involve somewhat lengthy manipulations, only a relatively short summary is presented here. Specifically, one finds that in the absence of a point-splitting approach (a) the axial-gauge current is generally not conserved and (b) there is an anomaly in the $[H,\psi(x)]$ commutator in the axial-gauge formulation for a non-Abelian gauge group (i.e., the Lagrangian and Hamiltonian equations of motion are inconsistent with each other).

IV. THE N-PARTICLE STATES

One of the areas of particular interest in this model is the construction of the complete set of states. This task is rendered much more tractable in the Galilean case than in special relativity because of the Bargmann superselection rule on the mass. In a single-mass theory such as that being considered here this means that the N-particle state does not couple to any other state and consequently has the form

$$\begin{aligned}|\,N\,\rangle=\int d^2x_1\cdots d^2x_N\phi^\dagger(x_1)\cdots\phi^\dagger(x_N)\\\times f(x_1,\ldots,x_N)\,|\,0\,\rangle\,,\end{aligned} \tag{17}$$

where, because of the Bose statistics, the function $f(x_1,\ldots,x_N)$ is a symmetrical function of its arguments. Evidently the state (17) is an eigenstate of the mass operator M with eigenvalue Nm.

One desires, by means of the eigenvalue equation

$$H\,|\,N\,\rangle=E\,|\,N\,\rangle\,,$$

(where H has the form given in Sec. II) to derive a differential equation for $f(x_1,\ldots,x_N)$ which will in essence be the Schrödinger equation for the N-particle sector. In carrying out this reduction use is made of the fact that

$$[\phi_i(x),\psi(x)]=0$$

which eliminates the only ordering ambiguity in the structure of the Hamiltonian. In the case of the single-particle state the calculation is particularly simple with the result that

$$Ef(x)=-\frac{1}{2m}\nabla^2 f(x)\,,$$

or, in other words, the single-particle state is a free state of momentum p with $E=p^2/2m$. While there is nothing in principle to exclude a constant term from appearing in the energy-momentum relation (i.e., an internal energy), its absence must be welcome since dimensional arguments would require any nonvanishing result to be divergent.

In the evaluation of $f(x_1,\ldots,x_N)$ for $N\geq 2$ one needs only the structure of H and $\phi_i(x)$ together with the condition

$$\psi(x)\,|\,0\,\rangle=0$$

to obtain the differential equation which determines that function. Straightforward but tedious calculation leads to the result

$$Ef(x_1,\cdots,x_N)=-\frac{1}{2m}\sum_{i=1}^{N}\left[\nabla_i+ig^2\overline{\mathbf{n}}\sum_{j\neq i}d_n(x_i-x_j)\right]^2\times f(x_1,\cdots x_N)\,, \tag{18}$$

where

$$\bar{\mathbf{n}}_i \equiv \epsilon_{ij} n_j$$

and the axial-gauge formulation has clearly been employed. The corresponding Coulomb-gauge result is obtained by the replacement

$$\mathbf{n} d_n(x_i - x_j) \to \nabla_i \mathscr{D}(x_i - x_j) .$$

Since (18) as written contains terms of the form

$$[d_n(x_i - x_j)]^2 ,$$

it is essential to remark that the point-splitting device already discussed implies that such terms are more carefully evaluated as

$$\lim_{\epsilon \to 0} d_n(x_i - x_j) d_n(x_i - x_j + \epsilon)$$

which vanish by virtue of the structure of the function $d_n(x)$.

Before going on to display some of the physical consequences of the result (18) it may be well to carry out the extension of the model to the non-Abelian case. One starts from the Lagrangian

$$\begin{aligned}\mathscr{L} = {} & i\psi^\dagger \left[\frac{\partial}{\partial t} + igT^a\phi_a\right]\psi \\ & + \frac{1}{2m}[\boldsymbol{\psi}^\dagger \cdot \boldsymbol{\psi} + \boldsymbol{\psi}^\dagger \cdot (\nabla - igT^a\boldsymbol{\phi}_a)\psi \\ & - \psi^\dagger(\nabla - igT^a\boldsymbol{\phi}_a)\cdot\boldsymbol{\psi}] - \tfrac{1}{2}\phi[\nabla - \tfrac{1}{3}igt^a\boldsymbol{\phi}_a]x\boldsymbol{\phi} \\ & - \tfrac{1}{2}\boldsymbol{\phi} \times (\nabla - \tfrac{1}{3}igt^a\boldsymbol{\phi}_a)\phi - \tfrac{1}{2}\boldsymbol{\phi} \times \left[\frac{\partial}{\partial t} + \tfrac{1}{3}igt^a\phi_a\right]\boldsymbol{\phi} ,\end{aligned}$$

where the matrices t^a and T^a refer, respectively, to the regular and any other representation of the group under consideration. In direct analogy to the situation which one encounters in Minkowski space,[6] the fields $\boldsymbol{\phi}_a$ and ϕ_a can be explicitly evaluated only in the axial gauge with the results

$$\phi_{ia}(x) = -g\epsilon_{ij}n_j \int d^2x' d_n(x - x')\rho_a(x') ,$$

$$\phi_a(x) = g \int d^2x' d_n(x - x')\mathbf{n} \times \mathbf{j}_a(x')$$

and the formal definitions

$$\rho_a = \psi^\dagger T_a \psi ,$$

$$\mathbf{j}_a = \frac{1}{2mi}(\boldsymbol{\psi}^\dagger T_a \psi - \psi^\dagger T_a \boldsymbol{\psi}) .$$

The Galilei covariance of the system can be determined with only minor changes in the earlier discussion of the Abelian model. In contrast to the case of special relativity the much less restrictive structure of the Galilei group allows such a demonstration without difficulty.

Finally, the N-particle state can be defined in much the same way as in the Abelian case. The wave function now depends on the coordinates $x_1, \ldots, x_N$, and in addition carries N-group indices $a_1, \ldots, a_N$. The latter can be suppressed in general with no loss in clarity so that the Schrödinger equation now reads

$$Ef(x_1, \ldots, x_N) = \frac{-1}{2m}\sum_{i=1}^{N}\left[\nabla_i + ig^2\bar{\mathbf{n}}\sum_{j \neq i} T_i \cdot T_j d_n(x_i - x_j)\right]^2 f(x_1, \ldots, x_N) ,$$

it being understood that the matrix T_i acts upon the index a_i of the wave function f.

These results complete the demonstration that a c-number Schrödinger equation can indeed be obtained in both Abelian and non-Abelian cases. What is not yet clear is whether there exist any nontrivial observable effects in the model. It is to this question effort will now be directed.

V. THE $N=2$ SECTOR

As is usual in dealing with the N-body problem, it is only the case $N=2$ which combines tractability with nontriviality and it is consequently that sector which merits additional study. Working in the Coulomb gauge the Schrödinger equation takes the form

$$Ef(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2m}\left[p_1{}^2 + p_2{}^2 + \frac{g^4}{2\pi^2 |\mathbf{x}_1 - \mathbf{x}_2|^2} + \frac{g^2}{2\pi}\left\{(\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{p}_1 - \mathbf{p}_2), \frac{1}{|\mathbf{x}_1 - \mathbf{x}_2|^2}\right\}\right] f(\mathbf{x}_1, \mathbf{x}_2) ,$$

where $\mathbf{p} \equiv -i\nabla$. This equation is more transparent in the center-of-mass coordinates

$$\mathbf{p}_1 = \tfrac{1}{2}\mathbf{P} + \mathbf{p} ,$$

$$\mathbf{p}_2 = \tfrac{1}{2}\mathbf{P} - \mathbf{p} ,$$

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{x}_1 + \mathbf{x}_2) ,$$

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$$

for which one has

$$Ef(\mathbf{R}, \mathbf{r}) = \left[\frac{P^2}{4m} + \frac{p^2}{m} + \frac{g^4}{4\pi^2 m r^2} + \frac{g^2}{2\pi m}\left\{\mathbf{r} \times \mathbf{p}, \frac{1}{r^2}\right\}\right] \times f(\mathbf{R}, \mathbf{r}) . \quad (19)$$

Thus, in addition to the kinetic energy terms corresponding to center-of-mass motion with mass $2m$ and relative motion with reduced mass $m/2$, there appear g^2-dependent interaction terms.

Equation (19) can be separated in the standard way by taking

$$f(\mathbf{R},\mathbf{r})=\exp(iK\cdot R)g(\mathbf{r})$$

to yield

$$\epsilon g(\mathbf{r})=\left[\frac{p^2}{m}+\frac{g^4}{4\pi^2 mr^2}+\frac{g^2}{2\pi m}\left\{r\times p,\frac{1}{r^2}\right\}\right]g(\mathbf{r}) ,$$

where

$$\epsilon=E-\frac{K^2}{4m} .$$

A subsequent separation of the angular dependence by means of the substitution

$$g(\mathbf{r})=e^{iL\phi}h_L(r), \quad L=0,\pm 1,\pm 2,\ldots$$

yields the one-dimensional differential equation

$$\left[\frac{1}{r}\frac{d}{dr}r+m\epsilon-\frac{(L+g^2/2\pi)^2}{r^2}\right]h_L(r)=0 . \tag{20}$$

Equation (20) clearly displays the fact that the basic interaction in this Galilean model is the modification of the centrifugal barrier by means of the replacement

$$L\to L+g^2/2\pi .$$

It is worth noting that such a result could almost be arrived at by dimensional considerations since, except for possible δ-function contributions to the effective potential, only r^{-2} terms can be induced by the dimensionless coupling constant g.

Since the modified centrifugal term has the form of a repulsive potential, there can clearly be no bound states, and one consequently must deal with the scattering effects of the coupling. To accomplish this it is necessary to adapt the usual three-dimensional scattering formalism to two dimensions, a task the principal points of which will be summarized here.

One seeks a solution of (20) such that an appropriate summation over L will yield a wave function

$$\psi(\mathbf{r})=e^{i\mathbf{k}\cdot\mathbf{r}}+\psi_{\rm sc}(\mathbf{r}) , \tag{21}$$

where

$$\psi_{\rm sc}(\mathbf{r}) \underset{r\to\infty}{\sim} r^{-1/2}e^{ikr}f(\phi) .$$

The general solution of (20) has the form

$$\psi(\mathbf{r})=\sum_{L=-\infty}^{\infty} e^{iL\phi}[A_L^{(1)}H^{(1)}_{L+g^2/2\pi}(kr) + A_L^{(2)}H^{(2)}_{L+g^2/2\pi}(kr)] , \tag{22}$$

where $A_L^{(1,2)}$ are coefficients to be determined and $k^2=m\epsilon$. Upon taking the vector $\mathbf{k}$ to be along the y axis (with ϕ measured as usual from the x axis) and using the Fourier-Bessel expansion

$$e^{iz\sin\phi}=\sum_{-\infty}^{\infty} e^{in\phi}J_n(z) ,$$

one finds that regularity at the origin requires

$$A_L^{(1)}=A_L^{(2)}\equiv A_L .$$

In addition, a comparison of the coefficients of e^{-ikr} in (21) and (22) yields

$$A_L=\tfrac{1}{2}e^{-ig^2/4}$$

whereupon $\psi(\mathbf{r})$ takes the form

$$\psi(\mathbf{r})=e^{-ig/4}\sum_{-\infty}^{\infty} e^{iL\phi}J_{L+g^2/2\pi}(kr) . \tag{23}$$

Evaluating (23) for large values of r, the scattered wave can be separated and the amplitude $f(\phi)$ evaluated giving

$$f(\phi)=(2i/\pi k)^{1/2}\sum_{-\infty}^{\infty} e^{iL(\phi-\pi/2)}e^{i\delta_L}\sin\delta_L ,$$

where the phase shift δ_L has the L-independent value of $-g^2/4$.

To this point no recognition of the underlying Bose statistics has been made. This requires in circular coordinates that

$$\psi(r,\phi)=\psi(r,\phi+\pi)$$

or equivalently that only the even values of L should appear in the summation. Using this fact and doing the L summation there results

$$\psi_{\rm sc}(\mathbf{r})=(4i\pi/kr)^{1/2}e^{ikr}e^{-ig^2/4}\sin(g^2/4) \times[\delta(\phi-\pi/2)+\delta(\phi+\pi/2)] .$$

This clearly displays the highly singular form of the scattering as the amplitude vanishes for all but the exact forward and backward directions.

The effective width Q (the analog of the cross section σ in two spatial dimensions) is obtained by integrating the squared modulus of the scattering amplitude over all angles. The formal result is

$$Q=\frac{8}{k}\sum_{L\ \rm even}\sin^2\delta_L \tag{24}$$

which for the computed L-independent value of the phase shift is either zero or infinite. The particular values at which vanishing effective width is obtained correspond to

$$g^2=4m\pi, \quad m=0,\pm 1,\pm 2,\cdots$$

and are readily recognized upon reference to (20) as the special cases for which one has a radial equation with integral angular momentum, but which correspond to the radial and angular parts being $2m$ units displaced relative to each other.

It is of some interest to make a comparison with the corresponding classical result. By formally taking $\hbar\to 0$ in (24) it can be argued that Q must be zero in that limit. A more convincing argument is based on the observation that in direct analogy to the electromagnetic case the force on a particle is proportional to the antisymmetrical derivative $\partial_\mu\phi_\nu-\partial_\nu\phi_\mu$ which by the equations of motion is zero everywhere except at the locations of the various charged particles. Thus the classical force is a δ-function contact interaction which leads to no scattering except for the extraordinary case of a direct collision. Since the calculation of scattering presupposes a uniformly distributed beam of incident particles, it follows that direct collisions never occur and that the effective width must be zero.

Finally, mention should be made of the possible implications with respect to quark binding mechanisms in this model. For the special case of $N=2$ one can, because of the fact that only $T_1 \cdot T_2$ appears in the non-Abelian generalization of (18), transform the Schrödinger equation to the Coulomb gauge by means of methods to be discussed in Sec. VI. Upon doing that one is effectively replacing g^2 in the $N=2$ Abelian case by g^2 times the eigenvalue of $T_1 \cdot T_2$. Potential support for quark binding mechanisms would arise for the SU(2) group if bound states could be found in the $N=2$ sector. Since that has been seen to be not possible one is forced to consider, for example, the possibility of N-particle bound states in an SU(N) model. This considerably more complex problem is beyond the scope of the present work.

VI. GAUGE TRANSFORMATIONS

In Sec. IV the Schrödinger equation for the N-body wave function $f(x_1,\ldots,x_N)$ has been derived. Since the Abelian model allows quantization in both Coulomb and axial gauges, it is, of course, possible to carry out the derivation by brute force in either gauge. More satisfying perhaps would be an approach in which the transition to other gauges is made merely by performing the appropriate gauge transformation on the N-particle wave function.

To accomplish this goal it is convenient to assume that one has begun with an axial gauge characterized by the vector $\mathbf{n}$. Since the Schrödinger equation (18) contains the combination

$$\nabla_i + ig^2 \bar{\mathbf{n}} \sum_{i\neq j} d_n(x_i - x_j)$$

it is clear that a transformation to the Coulomb gauge is effected by the relation

$$f_{\text{Coulomb}}(x_1,\ldots,x_N) = \exp\left[ig^2 \sum_{i>j} \Lambda(x_i - x_j)\right] f_{\text{axial}}(x_1,\ldots,x_N)$$

provided that

$$\nabla_i \Lambda(x) = \epsilon_{ij}[\nabla_j(x) - n_j d_n(x)] \; . \tag{25}$$

Standard manipulation allows one to solve (25) for the Fourier transform of $\Lambda(x)$, i.e.,

$$\Lambda(x) = \int \frac{dk}{(2\pi)^2} \frac{\mathbf{n}\times\mathbf{k}}{k^2 k\cdot n} e^{ikx}$$

which is evaluated by elementary methods to yield

$$\Lambda(x) = \frac{1}{2\pi} \tan^{-1} \frac{\mathbf{n}\times\mathbf{x}}{\mathbf{n}\cdot\mathbf{x}} \; . \tag{26}$$

In other words, one has the elegantly simple result that the two wave functions differ by a phase which is (up to a factor of $g^2/2\pi$) simply the sum of the angles made by all vectors $\mathbf{x}_i - \mathbf{x}_j$ with respect to $\mathbf{n}$.

In view of the fact that the vector $\mathbf{n}$ is not restricted in any way, it must be possible to transform between axial gauges which are specified by two different choices (e.g., $\mathbf{n}$ and $\mathbf{n}'$). Although this could be accomplished by transforming first to the Coulomb gauge as an intermediate gauge choice and subsequently to a different axial gauge using in each case the result (26), it is more instructive to proceed directly. In this instance the gauge function $\Lambda'(x)$ must be found to satisfy

$$\nabla_i \Lambda' = \epsilon_{ij}[n_j' d_{n'}(x) - n_j d_n(x)]$$

from which one derives

$$\Lambda'(x) = -(\mathbf{n}\times\mathbf{n}') \int d_n(x-x') d_{n'}(x') d^2x' \; .$$

This integral can be evaluated explicitly with the result

$$\Lambda'(x) = -\tfrac{1}{4} E(\mathbf{n},\mathbf{n}',\mathbf{x}) \; ,$$

where $E(\mathbf{n},\mathbf{n}',\mathbf{x})$ is defined by

$$E(\mathbf{a},\mathbf{b},\mathbf{c}) = \epsilon(\mathbf{a}\times\mathbf{b})\epsilon(\mathbf{b}\times\mathbf{c})\epsilon(\mathbf{c}\times\mathbf{a}) \; .$$

The function E makes the symmetry properties of the gauge function $\Lambda'(x)$ manifest since it is odd (even) under odd (even) permutations of its arguments and invariant under all sign changes of these arguments. The fact that $E(\mathbf{n},\mathbf{n}',\mathbf{x})$ can only take on the values ± 1 is also evident and has remarkable implications for the special values of g^2 which correspond to zero effective width. This is evident from the relation

$$f_{n'}(x_1,\ldots,x_N) = \exp\left[-\frac{ig^2}{4} \sum_{i>j} E(\mathbf{n},\mathbf{n}',\mathbf{x}_i - \mathbf{x}_j)\right] \times f_n(x_1,\ldots x_N)$$

which for the case of $g^2 = 4m\pi$ becomes

$$f_{n'}(x_1,\cdots x_N) = (\pm 1) f_n(x_1,\cdots x_N) \; . \tag{27}$$

The result (27) is a statement that the N-particle wave function in the axial gauge is independent of the particular quantization direction chosen. This would be easily understandable if one could show generally that for these special values of g^2 the model becomes a free-field theory. At this point, however, it has only been shown that such values of g^2 lead to zero effective width for two-body scattering.

A final observation concerning gauge transformations arises from noting that although quantization of the non-Abelian theory has not been carried out in the Coulomb gauge, one can at least inquire as to the result of formally attempting the transformation of the N-body wave function from the axial gauge to the Coulomb gauge. It is clear that this cannot be trivially effected in general merely by inserting a factor of $T_i \cdot T_j$ into $\Lambda(x_i - x_j)$ since the relevant matrices do not commute. There is the special case of $N=2$, however, which does allow this since only a single matrix form is involved. Thus one has in that very special instance the possibility of a Coulomb gauge propagator despite the absence of a consistent quantization in that gauge. It is not clear whether the limited success in the $N=2$ sector could be useful in the effort to effect a Coulomb gauge quantization for non-Abelian couplings.

VII. CONCLUDING REMARKS

The study presented here of a Galilean-invariant model field theory has shown that the concept of local gauge invariance can be reconciled in certain special circumstances to the Bargmann superselection rule on the mass. It was also found that the model contains many of the features of its Poincaré-invariant version including, for example, fractional spin and the nonlinearities which prevented a Coulomb-gauge quantization of the non-Abelian model. On the other hand, the axial gauge totally circumvented covariance problems as a consequence of the less rigid structure of Galilean relativity.

Aside from the matter of the existence of a Galilean gauge theory there is reason to believe that the aspects of the model touched upon here can shed light on a number of features of field theory. For example, the fact that a generally divergent effective width was found which then had the property of vanishing for favored values of the coupling is intriguing. Does the same phenomenon occur in the Minkowski space version? Also, is it an isolated oddity or can similar phenomena be found in more realistic theories?

Finally, note should be made of the fact that thus far only spinless particles have been coupled to the gauge field. It is to be expected that at least some qualitative changes must occur for other spin values. One can only conjecture at this point whether aspects such as possible quark-binding mechanisms are more profitably explored with particles which possess an intrinsic spin.

ACKNOWLEDGMENT

This work was supported in part by the U.S. Department of Energy under Contract No. DE-AC02-76ER13065.

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