Schwinger-DeWitt proper-time expansion and eikonal approximation

S. K. Kim
Department of Physics, Ewha Women's University, Seoul, Korea

Choonkyu Lee and D. P. Min
Department of Physics, Seoul National University, Seoul, Korea
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In the context of the Dirac equation, we show that the Schwinger-DeWitt proper-time expansion of the exact Green's function is useful for high-energy scattering and, in fact, provides a systematic generalization of the eikonal approximation. Because of its simplicity and its direct appeal to the coordinate-space scattering picture, the Schwinger-DeWitt expansion method should be valuable in studying corrections to the lowest-order eikonal approximation. A numerical comparison is made for an exponential potential. Within the same framework a systematic formalism is also developed to deal with large-angle scattering, and this yields a generalization of Schiff's large-angle formula. Applications to high-energy scattering problems in quantum field theories are indicated.

I. INTRODUCTION

Schwinger first proposed the use of the proper-time Green's function as a simple gauge-invariant regularization scheme in QED.[1] In Ref. 1, Schwinger considered vacuum polarization in the presence of an external electromagnetic field. His method has since developed into a useful field-theoretic tool, mainly through the hands of DeWitt.[2] DeWitt extensively used the proper-time technique to study one-loop renormalization counterterms in various quantum field theories with his background-field formalism.[2,3] Aside from finding renormalization counterterms, the proper-time technique is known to be very convenient in identifying the so-called chiral and trace anomalies (in flat or curved space-time).[4] Also, one of the present authors recently initiated an attempt to systemize the proper-time renormalization scheme, in conjunction with the background-field formalism, beyond one-loop order.[5]

The basic idea is simple. Consider a charged spin-$\frac{1}{2}$ field in the presence of an external (or background) electromagnetic field. The ordinary Feynman Green's function can be written as

$$S_F(x,y)=\left\langle x\left|\frac{1}{i\not{D}-m+i0+}\right|y\right\rangle$$
$$=-(i\not{D}_{(x)}+m)\int_0^\infty i\,d\tau\, e^{-\tau 0+}\langle x\tau|y\rangle\ , \qquad (1.1)$$

where $D_\mu=\partial_\mu+ieA_\mu$ (the gauge-covariant derivative), τ is Schwinger's proper-time variable,[1] and

$$\langle x\tau|y\rangle\equiv\langle x\,|\,e^{-i\tau(\not{D}+im)(\not{D}-im)}\,|\,y\rangle \qquad (1.2)$$

denotes the proper-time Green's function. In this paper, we use the γ-matrix and metric conventions of Bjorken and Drell,[6] and set $c=\hbar=1$. The function $\langle x\tau|y\rangle$ will obviously satisfy the Schrödinger-type equation

$$-\frac{1}{i}\frac{\partial}{\partial\tau}\langle x\tau|y\rangle=(\not{D}_{(x)}+im)(\not{D}_{(x)}-im)\langle x\tau|y\rangle$$
$$(\tau>0)\ . \qquad (1.3)$$

With proper-time representation (1.1), the short-distance (i.e., $x\approx y$) singularities of the ordinary Feynman Green's function transform into the small-τ singularities of $\langle x\tau|y\rangle$ (at the coincidence limit $x=y$), while not spoiling the gauge transformation character. Moreover, it is very easy to identify those small-τ singularities by using, with the "Schrödinger" equation (1.3), the following asymptotic series (often called the DeWitt WKB form[2]):

$$\tau\to 0+:\langle x\tau|y\rangle=\frac{-i}{(4\pi\tau)^2}\exp\left[-i\frac{(x-y)^2}{4\tau}-im^2\tau\right]$$
$$\times\left[\sum_{n=0}^{\infty}a_n(x,y)\tau^n\right]\ . \qquad (1.4)$$

(For mathematical investigations on this series, see Ref. 7.) There exist simple recurrence relations for the coefficient functions $a_n(x,y)$, and for renormalization problems it is only the coincidence limits $a_0(x,x)$, $a_1(x,x)$, and $a_2(x,x)$ which are important. They can be easily obtained, as well-defined local functions of a background electromagnetic field, through successive differentiations and then by taking coincidence (i.e., $x=y$) limits with those recurrence relations.[2] For other possible applications of the DeWitt WKB form (1.4), we suggest readers read the introduction of Ref. 5.

So far, in all known applications of the small-proper-time expansion, only the coincidence limits $a_n(x,x)$ [or at most the coincidence limits of the space-time derivatives of $a_n(x,y)$] have been considered. Hence one might think that the DeWitt expansion is useful only for problems like renormalization (this appears to be also a prevailing feeling among experts on the subject, despite the word "WKB" of the DeWitt WKB form). That is quite misleading. In fact, contrary to what one may guess, finding the coefficient functions $a_n(x,y)$ for arbitrary values of x and y is also simple. With such explicit solutions for $a_n(x,y)$, we find that the small-proper-time expansion can generate a systematic approximation for high-energy scattering problems. The scattering amplitude involves the coefficient function $a_n(x,y)$ with one of the arguments taken to be timelike infinity, i.e., $x^2 \to +\infty$ with $x^0 \to +\infty$ or $y^2 \to +\infty$ with $y^0 \to -\infty$. In the present work, we shall concentrate on the external-field problem. In the near future we plan to report our general approximation procedure for high-energy scattering amplitudes in quantum field theories, utilizing the DeWitt expansion together with the (path integral) background-field method. It goes without saying that the analysis in the present paper will be a prerequisite for this latter investigation.

In a localized external electromagnetic potential, the expression for the scattering amplitude which we shall derive in this paper has the series form

$$(\text{const}) \sum_{n=0}^{\infty} \int d^4y \, e^{i(p_f - p_i)\cdot y} \times \bar{u}(p_f, s_f) B_n(p_f; y) \not{A}(y) u(p_i, s_i) \tag{1.5a}$$

or, equivalently,

$$(\text{const}) \sum_{n=0}^{\infty} \int d^4y \, e^{i(p_f - p_i)\cdot y} \times \bar{u}(p_f, s_f) \not{A}(y) \mathscr{B}_n(p_i; y) u(p_i, s_i) , \tag{1.5b}$$

where p_f, s_f (p_i, s_i) designate the final (initial) state and

$$B_n(p_f;y) = \lim_{\substack{x^2 \to +\infty \\ [\text{with } m(x_\mu/\sqrt{x^2}) = p_f^\mu(\text{fixed})]}} \left[\left[\frac{\sqrt{x^2}}{2m} \right]^n a_n(x,y) \right] , \tag{1.6}$$

$$\mathscr{B}_n(p_i;y) = \lim_{\substack{x^2 \to +\infty \\ [\text{with } -m(x^\mu/\sqrt{x^2}) = p_i^\mu(\text{fixed})]}} \left[\left[\frac{\sqrt{x^2}}{2m} \right]^n a_n(y,x) \right] . \tag{1.7}$$

The right-hand sides of Eqs. (1.6) and (1.7) can be shown to be well-defined for a sufficiently localized potential A_μ, and are given by a relatively simple, explicit, expression for small n.

For a moderate potential, we find that the series (1.5a) or (1.5b) is a valid expansion if

$$|\mathbf{p}| b \gg 1, \quad |\mathbf{q}| b \lesssim 1 \tag{1.8}$$

where $\mathbf{p} = \mathbf{p}_f$ or $\mathbf{p}_i$, $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$ is the momentum transfer and b specifies the rough range of the external potential. Under these restrictions, it may be used for ultrarelativistic ($|\mathbf{p}| \gg m$), relativistic ($|\mathbf{p}| \sim m$), for high-energy nonrelativistic ($|\mathbf{p}| \ll m$) scattering problems. The range (1.8) corresponds to that for which the so-called eikonal approximation[8–11] is useful, and in fact, keeping only the lowest-order term proportional to $B_0(p_f;y)$ or $\mathscr{B}_0(p_i;y)$ in our series (1.5a) or (1.5b) produces the well-known eikonal formula. Higher-order terms proportional to $B_n(p_f;y)$ or $\mathscr{B}_n(p_i;y)$ $(n = 1,2,3,\ldots)$ in the series are suppressed by powers of $1/|\mathbf{p}|b$ as n increases, thus giving systematic corrections to the lowest-order eikonal result. Practically, these correction terms are expected to be particularly important when the magnitude of $|\mathbf{q}|b$ is not small compared to 1. (See our numerical illustration in Sec. VI.) With the value of $|\mathbf{q}|b$ significantly larger than 1 (i.e., large-angle scattering), the series (1.5a) or (1.5b) may not be reliable since

(i) the degree of spatial *variation* of the functions $B_n(p_f;y)$ or $\mathscr{B}_n(p_i;y)$, besides their magnitudes, becomes also relevant and there is no guarantee of smoothness of these functions as n increases, and

(ii) with a relatively smooth potential the scattering amplitude itself will become extremely small at large angle. But, when the external potential has a rather strong short-range core or the Fourier transform of the given external potential has sizable large momentum components, there will be a strong interest also in large-angle scattering. To deal with such a situation we have given a further improved, and still quite elegant, formalism in Sec. VII, which basically incorporates the idea of the distorted-wave Born approximation[12] into our framework. This yields a systematic generalization of Schiff's large-angle formula.[10]

We now summarize the contents of this paper. In Sec. II, we solve the recurrence relations for the coefficient functions $a_n(x,y)$ for arbitrary values of x and y. In Sec. III, we develop a systematic approximation for S-matrix elements on the basis of the Schwinger-DeWitt expansion. Here we reformulate the scattering theory with the Dirac equation such that S-matrix elements may be obtained directly from the large-timelike-distance behavior of the exact Feynman Green's function. This coordinate-space description of scattering is the most natural one to use for our investigation, and it also simplifies calculations considerably.

In Sec. IV, we discuss the validity range of the expansion and establish that the series (1.5a) or (1.5b) provides a useful approximation scheme for high-energy scattering. In Sec. V we specialize ourselves to the case of static external potentials. Here we give the explicit formula for high-energy differential scattering cross sections, keeping up to the terms $B_1(p_f;y)$ or $\mathscr{B}_1(p_i;y)$ in the series (i.e., including the leading-correction term to the lowest-order eikonal result). A numerical illustration is given with a simple exponential potential in Sec. VI. In Sec. VII, we present our improved approximation method for S-matrix elements which is suitable for large-angle scattering as well. Section VIII is devoted to discussions of our work.

II. THE SCHWINGER-DeWITT EXPANSION

With the exact Feynman Green's function written in the form (1.1), let us use the DeWitt WKB expansion (1.4) for the proper-time Green's function $\langle x\tau | y \rangle$. If we are allowed to perform the τ integration term by term, we will then obtain the following series expansion for $S_F(x,y)$:

$$S_F(x,y)=(i\not{D}_{(x)}+m)\left[\sum_{n=0}^{\infty} a_n(x,y)\left[i\frac{\partial}{\partial m^2}\right]^n \Delta_0(x-y;m^2)\right], \tag{2.1}$$

where

$$\Delta_0(x-y;m^2)=-\left\langle x\left|\frac{1}{\partial^2+m^2-i0+}\right|y\right\rangle$$
$$=-\int_0^\infty i\,d\tau\, e^{-\tau 0+}\left\langle x\left|e^{-i\tau(\partial^2+m^2)}\right|y\right\rangle \underset{m^2(x-y)^2\to+\infty}{\sim} \frac{m^{1/2}e^{(3/4)\pi i}}{2^{5/2}\pi^{3/2}[(x-y)^2]^{3/4}}e^{-im[(x-y)^2]^{1/2}} \tag{2.2}$$

denotes the free scalar Green's function. Moreover, noting that

$$-(i\not{D}_{(x)}+m)\left\langle x\left|\frac{1}{\not{D}\not{D}+m^2-i0+}\right|y\right\rangle=-\left\langle x\left|\frac{1}{\not{D}\not{D}+m^2-i0+}\right|y\right\rangle(i\overleftarrow{\not{D}}_{(y)}+m) \tag{2.3}$$

with $\overleftarrow{\not{D}}_{(y)}\equiv-\overleftarrow{\not{\partial}}_{(y)}+ie\not{A}_{(y)}$, it is also possible to expand $S_F(x,y)$ in the form

$$S_F(x,y)=\left[\sum_{n=0}^{\infty} a_n(x,y)\left[i\frac{\partial}{\partial m^2}\right]^n \Delta_0(x-y;m^2)\right](i\overleftarrow{\not{D}}_{(y)}+m)\ . \tag{2.4}$$

Note that the series (2.1) or (2.4) for $S_F(x,y)$ with *finite* values of x and y may be regarded as a systematic heavy-mass (i.e., large-m) expansion, but the nature of the series may be different when one of the arguments approaches timelike infinity. (See the last paragraph of Sec. IV.)

It is the series (2.1) or (2.4) which we shall use to evaluate S-matrix elements. For scattering problems, we are interested in certain specific asymptotic regions with coefficient functions $a_n(x,y)$ (see Sec. III). The functions $a_n(x,y)$ for general values of x,y should be of course determined by using the Schrödinger equation (1.3). Here note that Eq. (1.3) can also be written as

$$i\frac{\partial}{\partial\tau}\langle x\tau\,|\,y\rangle=(D^2+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}+m^2)\langle x\tau\,|\,y\rangle$$
$$\left[\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right] \tag{2.5}$$

with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and $\langle x\tau\,|\,y\rangle$ should obey the boundary condition

$$\tau\to 0+:\langle x\tau\,|\,y\rangle\to\delta^4(x-y)I\ . \tag{2.6}$$

Now, inserting the small proper-time expansion (1.4) into Eq. (2.5) and then matching various powers of τ yields the following recurrence relations:[2]

$$(x-y)^\mu D_\mu^{(x)}a_0=0\ , \tag{2.7}$$

$$ina_n=-i(x-y)^\mu D_\mu^{(x)}a_n+[D_{(x)}{}^2+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(x)]a_{n-1}\ . \tag{2.8}$$

For the boundary condition (2.6) to be fulfilled, we must here require that

$$a_0(x,x)=I\ . \tag{2.9}$$

We shall solve the recurrence relations (2.7) and (2.8) below.

The regular solution to Eq. (2.7), satisfying the boundary condition (2.9), is well known; it is given by

$$a_0(x,y)=\exp\left[-ie\int_y^x A_\mu(z)dz^\mu\right], \tag{2.10}$$

where $⨍_y^x$ represents the line integral along the straight-line path from y to x. For $a_n(x,y)$ with $n\geq 1$, we may integrate the recurrence relations (2.8) successively as follows. First, we define the functions $\bar{a}_n(x,y)$ by setting

$$a_n(x,y)=\exp\left[-ie⨍_y^x A_\mu(z)dz^\mu\right]\bar{a}_n(x,y)\ . \tag{2.11}$$

Then, Eq. (2.8) may be cast into the form

$$in\bar{a}_n(x,y)+i(x-y)^\mu\partial_\mu^{(x)}\bar{a}_n(x,y)=\bar{q}_n(x,y) \tag{2.12}$$

with

$$\bar{q}_n(x,y)\equiv\exp\left[ie⨍_y^x A_\lambda(z)dz^\lambda\right][D_{(x)}{}^2+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(x)]$$
$$\times\exp\left[-ie⨍_y^x A_\delta(z)dz^\delta\right]\bar{a}_{n-1}(x,y)\ . \tag{2.13}$$

To solve the linear partial differential equations (2.12), we may imagine the new functions

$$A_n(s;x,y)\equiv\bar{a}_n(s(x-y)+y,y)\ ,$$
$$Q_n(s;x,y)\equiv\bar{q}_n(s(x-y)+y,y)\ , \tag{2.14}$$

which are respectively obtained from $\bar{a}_n(x,y)$, $\bar{q}_n(x,y)$ through the substitution

$$(x^\mu,y^\mu)\to(s(x-y)^\mu+y^\mu,y^\mu)\ .$$

In terms of these new functions, Eq. (2.12) simply reads

$$s^{-n+1}\frac{\partial}{\partial s}(s^nA_n)=-iQ_n\ , \tag{2.15}$$

which is trivial to solve. Moreover, it follows from Eqs. (2.12) and (2.14) that

$$A_n(s=0;x,y)=\bar{a}_n(y,y)=-\frac{i}{n}\bar{q}_n(y,y)\ . \tag{2.16}$$

Now the solution to Eq. (2.15), satisfying the boundary condition (2.16), is given by

$$A_n(s;x,y)=\frac{-i}{s^n}\int_0^s ds'(s')^{n-1}Q_n(s';x,y)\ . \qquad (2.17)$$

Then, all that is needed is to set $s=1$ in Eq. (2.17), to obtain[13]

$$\bar{a}_n(x,y)=-i\int_0^1 ds\, s^{n-1}\bar{q}_n(s(x-y)+y,y)\ . \qquad (2.18)$$

Using Eqs. (2.13) and (2.18) iteratively, one can in principle determine all coefficient functions $\bar{a}_n(x,y)$ explicitly. For such calculations, we find the following identity particularly useful:

$$D_\mu^{(x)}\left[\exp\left[-ie\,⨍_y^x A_\delta(z)dz^\delta\right]\right]=\exp\left[-ie\,⨍_y^x A_\delta(z)dz^\delta\right]\left[\partial_\mu^{(x)}-ie(x-y)^\nu\int_0^1 ds s F_{\mu\nu}(s(x-y)+y)\right]\ . \qquad (2.19)$$

If we make use of Eq. (2.19) twice in Eq. (2.13), we may now write

$$\begin{aligned}\bar{q}_n(x,y)=\Bigg[&\partial_{(x)}{}^2-2ie(x-y)^\nu\left[\int_0^1 ds\, sF_{\mu\nu}(s(x-y)+y)\right]\partial^\mu_{(x)}-ie(x-y)^\nu\left[\int_0^1 ds\, s^2(\partial^\mu F_{\mu\nu})(s(x-y)+y)\right]\\&-e^2(x-y)^\nu(x-y)^\lambda\left[\int_0^1 ds\, sF_{\mu\nu}(s(x-y)+y)\right]\left[\int_0^1 ds' s' F^\mu{}_\lambda(s'(x-y)+y)\right]\\&+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(x)\Bigg]\bar{a}_{n-1}(x,y)\ . \end{aligned}\qquad (2.20)$$

[Here, $(\partial^\mu F_{\mu\nu})(s(x-y)+y)$ is equal to $[\partial^\mu F_{\mu\nu}(x)]\,|_{x^\mu\to s(x-y)^\mu+y^\mu}$]. Note that exponential phase factors no longer appear in Eq. (2.20). To obtain $\bar{q}_1(x,y)$ for instance, we may set $n=1$ and $\bar{a}_0(x,y)=I$ in Eq. (2.20); this gives the expression

$$\begin{aligned}\bar{q}_1(x,y)=&-ie(x-y)^\nu\left[\int_0^1 ds\, s^2\partial^\mu F_{\mu\nu}(s(x-y)+y)\right]\\&-e^2(x-y)^\nu(x-y)^\lambda\left[\int_0^1 ds\, sF_{\mu\nu}(s(x-y)+y)\right]\left[\int_0^1 ds' s' F^\mu{}_\lambda(s'(x-y)+y)\right]+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(x)\ .\end{aligned}\qquad (2.21)$$

From Eqs. (2.18) and (2.21), we then find

$$\begin{aligned}\bar{a}_1(x,y)&=-i\int_0^1 du\Bigg[-\frac{ie}{u^2}(x-y)^\nu\left[\int_0^u ds\, s^2(\partial^\mu F_{\mu\nu})(s(x-y)+y)\right]+\tfrac{1}{2}e\sigma^{\mu\nu}F_{\mu\nu}(u(x-y)+y)\\&\qquad-\frac{e^2}{u^2}(x-y)^\nu(x-y)^\lambda\left[\int_0^u ds\, sF_{\mu\nu}(s(x-y)+y)\right]\left[\int_0^u ds' s' F^\mu{}_\lambda(s'(x-y)+y)\right]\Bigg]\\&=-e(x-y)^\nu\int_0^1 ds\, s(1-s)\partial^\mu F_{\mu\nu}(s(x-y)+y)-\frac{i}{2}e\sigma^{\mu\nu}\int_0^1 ds\, F_{\mu\nu}(s(x-y)+y)\\&\quad+2ie^2(x-y)^\nu(x-y)^\lambda\int_0^1 ds(1-s)F_{\mu\nu}(s(x-y)+y)\int_0^s ds' s' F^\mu{}_\lambda(s'(x-y)+y)\ ,\end{aligned}\qquad (2.22)$$

where we have changed the order of integration to obtain the second expression. Inserting the expression (2.22) for $\bar{a}_1(x,y)$ into Eq. (2.20) will give the function $\bar{q}_2(x,y)$ explicitly, and the result can be in turn used to determine $\bar{a}_2(x,y)$ via Eq. (2.18), etc.

III. APPLICATION TO SCATTERING PROBLEMS

In an electromagnetic potential A_μ, the Dirac equation for a charged spin-$\frac{1}{2}$ particle reads

$$(i\not{D}-m)\Psi=0\quad (D_\mu=\partial_\mu+ieA_\mu)\ . \qquad (3.1)$$

For an external potential A_μ, we will assume the general form

$$A_\mu(x)=e^{-\eta|x^0|}\tilde{A}_\mu(\mathbf{x},x^0)\ . \qquad (3.2)$$

Here, $e^{-\eta|x_0|}$ (η is a very small, but finite, positive number) is the adiabatic switching factor and has been introduced explicitly such that $A_\mu(x)$ may be always imagined [even with a static potential $\tilde{A}_\mu(\mathbf{x},x_0)=\tilde{A}_\mu(\mathbf{x})$] to be *localized in the finite,* albeit very long along the time direction, *space-time region around $x^\mu=0$.* Note that, in our approach, the external potential A_μ is treated as being always time dependent. This will contribute significantly to simplifying our formula for S-matrix elements, making it possible to treat space and time coordinates on an equal footing.[14] Moreover, with a suitable reinterpretation, this formalism with a general time-dependent potential A_μ can be utilized in studying bremsstrahlung from a charged particle.[15] S-matrix elements in a static external potential can be obtained as a special case by letting $\eta\to 0+$ in the final stage (see Sec. V).

To describe scattering in the presence of an external potential A_μ, let ψ_f,ψ_i denote the final and incident plane-wave solutions of the free Dirac equation, i.e.,

$$\bar{\psi}_f(x)=\left[\frac{m}{E_f V}\right]^{1/2}\bar{u}(p_f,s_f)e^{ip_f\cdot x}\ , \qquad (3.3a)$$

$$\psi_i(x)=\left[\frac{m}{E_i V}\right]^{1/2} e^{-ip_i\cdot x}u(p_i,s_i)\ . \tag{3.3b}$$

Then, S-matrix elements for particle scattering can be obtained from[16]

$$S_{fi}=\delta_{fi}-ie\int d^4x\,\bar{\psi}_f(x)A\!\!\!/(x)\Psi^{(+)}(x)\ , \tag{3.4}$$

where $\Psi^{(+)}(x)$ represents the scattering solution to Eq. (3.1), satisfying the asymptotic condition:

$$(\text{positive}-\text{frequency part) of }\Psi^{(+)}(x)\underset{x^0\to-\infty}{\sim}\psi_i(x)\ ,$$

and

$$\Psi^{(+)}(x)\underset{x^0\to+\infty}{\sim}(\text{positive}-\text{frequency, outgoing, scattering waves only}).$$

With the help of the exact Feynman Green's function $S_F(x,y)$, $\Psi^{(+)}(x)$ can be expressed as

$$\Psi^{(+)}(x)=\psi_i(x)+e\int d^4y\,S_F(x,y)A\!\!\!/(y)\psi_i(y)\ . \tag{3.5}$$

Alternatively, S-matrix elements can be obtained also from

$$S_{fi}=\delta_{fi}-ie\int d^4x\,\bar{\Psi}^{(-)}(x)A\!\!\!/(x)\psi_i(x)\ , \tag{3.6}$$

where $\Psi^{(-)}(x)$ represents the scattering solution now satisfying the asymptotic condition,

$$(\text{positive}-\text{frequency part) of }\Psi^{(-)}(x)\underset{x^0\to+\infty}{\sim}\psi_f(x)$$

and

$$\Psi^{(-)}(x)\underset{x^0\to-\infty}{\sim}(\text{positive}-\text{frequency, incoming scattering waves only}).$$

This function $\Psi^{(-)}$ is given by

$$\bar{\Psi}^{(-)}=\bar{\psi}_f(x)+e\int d^4y\,\bar{\psi}_f(y)A\!\!\!/(y)S_F(y,x)\ . \tag{3.7}$$

The exact time-dependent scattering solutions $\Psi^{(\pm)}(x)$ will obey the following integral equations:

$$\Psi^{(+)}=\psi_i(x)+e\int d^4y\,S_F^0(x,y)A\!\!\!/(y)\Psi^{(+)}(y)\ , \tag{3.8a}$$

$$\bar{\Psi}^{(-)}=\bar{\psi}_f(x)+e\int d^4y\,\bar{\Psi}^{(-)}(y)A\!\!\!/(y)S_F^0(y,x)\ , \tag{3.8b}$$

where the $S_F^0(x,y)$ is the free Feynman Green's function

$$S_F^0(x,y)=\left\langle x\left|\frac{1}{i\partial\!\!\!/-m+i0+}\right|y\right\rangle\ . \tag{3.9}$$

Our approximation scheme is based on the series expansion (2.1) or (2.4) for $S_F(x,y)$ which has its simplest form naturally in the coordinate space. This is in contrast to the usual Born series for which momentum-space calculations are simplest. Hence there follows a significant calculational advantage if the S-matrix formula (3.4) or (3.6) is recast into the one which uses the coordinate-space scattering description in a more direct way, i.e., asymptotic behaviors of the exact scattering solutions $\Psi^{(\pm)}(x)$. The latter can be related to the appropriate asymptotic behaviors of the exact Feynman Green's function $S_F(x,y)$ as follows. First with $\bar{\Psi}^{(-)}(x)$ specified by the right-hand side of Eq. (3.7), we may look at the region

$$R^{(+)}:\left[\begin{array}{l}|\mathbf{x}|\to\infty,x^0\to+\infty\ (\text{such that }\eta x^0\gg1)\text{ with}\\ x^2\to+\infty\ (\text{i.e., timelike) and the ratio }\dfrac{x^0}{|\mathbf{x}|}\text{ fixed}\end{array}\right]. \tag{3.10}$$

Note that the free Feynman Green's function has the asymptotic behavior $x\in R^{(+)}$ *and* y^μ *finite:*

$$S_F^0(x,y)\to\frac{m^{1/2}e^{(3/4)\pi i}e^{-im\sqrt{x^2}}}{2^{5/2}\pi^{3/2}(x^2)^{3/4}}\left[\left[m\gamma^\mu\frac{x_\mu}{\sqrt{x^2}}+m\right]e^{im(x^\mu/\sqrt{x^2})y_\mu}+O\left[\frac{1}{|\mathbf{x}|}\right]\right]\ . \tag{3.11}$$

In the same asymptotic limit, we may express the exact Feynman propagator in the form

$$\begin{aligned}S_F(x,y)&=S_F^0(x,y)+\int d^4z\,S_F^0(x,z)eA\!\!\!/(z)S_F(z,y)\\&\to\frac{m^{1/2}e^{(3/4)\pi i}e^{-im\sqrt{x^2}}}{2^{5/2}\pi^{3/2}(x^2)^{3/4}}\left[m\gamma^\mu\frac{x_\mu}{\sqrt{x^2}}+m\right]\left[e^{im(x^\mu/\sqrt{x^2})y_\mu}+\int d^4z\,e^{im(x^\mu/\sqrt{x^2})z_\mu}eA\!\!\!/(z)S_F(z,y)\right]\ .\end{aligned} \tag{3.12}$$

Now suppose we identify (note that $p_f{}^2=m^2$)

$$m\frac{x^\mu}{\sqrt{x^2}}=p_f^\mu \tag{3.13}$$

in Eq. (3.12). We then find the following very useful connection:

$$\left[\frac{m}{E_f V}\right]^{1/2}\bar{u}(p_f,s_f)S_F(x,y)\to\frac{m^{3/2}e^{(3/4)\pi i}e^{-im\sqrt{x^2}}}{(2\pi)^{3/2}(x^2)^{3/4}}\bar{\Psi}^{(-)}(y;p_f,s_f)\ . \tag{3.14}$$

Inserting Eq. (3.14) into Eq. (3.6), we find

$$S_{fi}=\delta_{fi}+\left[\frac{(2\pi)^{3/2}e^{-(3/4)\pi i}}{m^{1/2}V(E_fE_i)^{1/2}}\right]\lim_{\substack{x^2\to\infty\\ (\text{with } m\frac{x^\mu}{\sqrt{x^2}}=p_f^\mu)}}(x^2)^{3/4}e^{im\sqrt{x^2}}\int d^4y\,\bar{u}(p_f,s_f)S_F(x,y)[-ie\not{A}(y)]e^{-ip_i\cdot y}u(p_i,s_i)\;. \tag{3.15}$$

Alternatively, with the scattering solution $\Psi^{(+)}$ specified by Eq. (3.5) or Eq. (3.8a), we may look at the asymptotic region

$$R^{(-)}:\left[\begin{array}{l}|\mathbf{x}|\to\infty,\ x^0\to-\infty\ (\text{such that} +\eta\,|x^0|\gg 1)\ \text{with}\\ x^2\to+\infty\ (\text{i.e., timelike})\ \text{and the ratio}\ \dfrac{x^0}{|\mathbf{x}|}\ \text{fixed}\end{array}\right]. \tag{3.16}$$

In this region, identifying $-m(x^\mu/\sqrt{x^2})$ with p_i^μ, we find

$$\left[\frac{m}{E_iV}\right]^{1/2}S_F(y,x)u(p_i,s_i)\to\frac{m^{3/2}e^{(3/4)\pi i}e^{-im\sqrt{x^2}}}{(2\pi)^{3/2}(x^2)^{3/4}}\Psi^{(+)}(y;p_i,s_i) \tag{3.17}$$

and

$$S_{fi}=\delta_{fi}+\left[\frac{(2\pi)^{3/2}e^{-(3/4)\pi i}}{m^{1/2}V(E_fE_i)^{1/2}}\right]\lim_{\substack{x^2\to+\infty\\ (\text{with } -m\frac{x^\mu}{\sqrt{x^2}}=p_i^\mu)}}(x^2)^{(3/4)}e^{im\sqrt{x^2}}\int d^4y\,\bar{u}(p_f,s_f)e^{ip_f\cdot y}[-ie\not{A}(y)]S_F(y,x)u(p_i,s_i)\;. \tag{3.18}$$

Here we just note that, due to our identification $m(x^\mu/\sqrt{x^2})=p_f^\mu$ [or $-m(x^\mu/\sqrt{x^2})=p_i^\mu$], different asymptotic regions of $S_F(x,y)$ [or $S_F(y,x)$ with Eq. (3.18)] correspond to different kinematical regions of given scattering experiments; using $m(x^\mu/\sqrt{x^2})=p_f^\mu$,

$$\begin{aligned}
&(\text{i})\ |\mathbf{v}_f|=\frac{|\mathbf{p}_f|}{E_f}\to 1,\ \ \frac{m}{E_f}\to 0\ (\text{ultrarelativistic}):\frac{|\mathbf{x}|}{x^0}\to 1,\ \ \frac{\sqrt{x^2}}{x^0}\to 0\;,\\
&(\text{ii})\ |\mathbf{v}_f|=O(1)<1,\ \ \frac{m}{E_f}=O(1)<1\ (\text{relativistic}):\frac{|\mathbf{x}|}{x^0}\ \text{and}\ \frac{\sqrt{x^2}}{x^0}\ \text{are}\ O(1)\;,\\
&(\text{iii})\ |\mathbf{v}_f|\ll 1,\ \ E_f\sim m\ (\text{nonrelativistic}):\frac{|\mathbf{x}|}{x^0}(\sim|\mathbf{v}_f|)\ll 1,\ \ \frac{\sqrt{x^2}}{x^0}\to 1\;.
\end{aligned} \tag{3.19}$$

We now consider the application of the DeWitt WKB form for evaluating S-matrix elements approximately. First we look at the formula (3.15), for which the expansion (2.1) for $S_F(x,y)$ is especially useful. Note that the expansion (2.1) can be rewritten in the form

$$S_F(x,y)=\exp\left[-ie\,⨍_y^x A_\delta(z)dz^\delta\right]\sum_{n=0}^{\infty}\left[\bar{a}_n(x,y)(i\not{\partial}_{(x)}+m)\left[i\frac{\partial}{\partial m^2}\right]^n\Delta_0(x-y;m^2)+K_n(x,y)\right] \tag{3.20}$$

with

$$K_n(x,y)=\left[i\not{\partial}_{(x)}\bar{a}_n(x,y)+\left[e\gamma^\mu(x-y)^\nu\int_0^1 ds\,sF_{\mu\nu}(s(x-y)+y)\right]\bar{a}_n(x,y)\right]\left[i\frac{\partial}{\partial m^2}\right]^n\Delta_0(x-y;m^2)\;, \tag{3.21}$$

where we have used Eqs. (2.11) and (2.19). We may then insert the series (3.20) into the formula (3.15). At this stage, for a localized potential, it can be shown that

$x^2\to+\infty$ $\left[\textit{with } m\frac{x^\mu}{\sqrt{x^2}}=p_f^\mu\right]$ *and* y^μ *finite:*

$$(\sqrt{x^2})^n\bar{a}_n(x,y)\to(\text{an } x^2\text{-independent function})\;, \tag{3.22}$$

$$\left[i\frac{\partial}{\partial m^2}\right]^n\Delta_0(x-y;m^2)\to\frac{m^{1/2}e^{(3/4)\pi i}e^{-im\sqrt{x^2}}}{2^{5/2}\pi^{3/2}(x^2)^{3/4}}\left[\frac{\sqrt{x^2}}{2m}\right]^n e^{ip_f\cdot y}\;, \tag{3.23}$$

$$(i\not{\partial}_{(x)}+m)\left[i\frac{\partial}{\partial m^2}\right]^n\Delta_0(x-y;m^2)\to\frac{m^{1/2}e^{(3/4)\pi i}e^{-im\sqrt{x^2}}}{2^{5/2}\pi^{3/2}(x^2)^{3/4}}\left[\frac{\sqrt{x^2}}{2m}\right]^n e^{ip_f\cdot y}(\not{p}_f+m)\;. \tag{3.24}$$

Since the behavior (3.22) will be evident in Sec. IV, we shall here excuse ourselves from providing its justification [see Eqs. (4.6) and (4.12), for instance]. On the other hand, the behaviors (3.23) and (3.24) are direct consequences of the asymptotic behavior shown in Eq. (2.2).

Accepting the behavior (3.22) implies that

$$x^2 \to +\infty \left[\textit{with } \frac{x^\mu}{\sqrt{x^2}} = p_f^\mu \right] \textit{ and } y^\mu \textit{ finite}:$$

$$(\sqrt{x^2})^n \left[i\partial_{(x)}\bar{a}_n(x,y) + \left[e\gamma^\mu(x-y)^\nu \int_0^1 ds\, sF_{\mu\nu}(s(x-y)+y) \right] \bar{a}_n(x,y) \right] \to O\left[\frac{1}{\sqrt{x^2}} \right] . \tag{3.25}$$

The situation became quite clear. According to the formula (3.15) and Eqs. (3.22)—(3.24), the $K_n(x,y)$ pieces in Eq. (3.21) do not contribute to S-matrix elements at all. And the formulas (3.14) and (3.15) now become

$$\bar{\Psi}^{(-)}(y;p_f,s_f) = \left[\frac{m}{E_f V} \right]^{1/2} \bar{u}(p_f,s_f) \exp[ip_f \cdot y - ieH(p_f;y)] \left[\sum_{n=0}^{\infty} \bar{B}_n(p_f;y) \right] , \tag{3.26}$$

$$S_{fi} - \delta_{fi} = \frac{(-ie)m}{V(E_f E_i)^{1/2}} \sum_{n=0}^{\infty} \int d^4y \exp[i(p_f - p_i)\cdot y - ieH(p_f;y)] \bar{u}(p_f,s_f) \bar{B}_n(p_f;y) A\!\!\!/(y) u(p_i,s_i) , \tag{3.27}$$

where

$$\bar{B}_n(p_f;y) \equiv \lim_{\substack{x^2 \to +\infty \\ (\text{with } m(x^\mu/\sqrt{x^2}) = p_f^\mu)}} \left[\left[\frac{\sqrt{x^2}}{2m} \right]^n \bar{a}_n(x,y) \right] , \tag{3.28}$$

and we have also set

$$H(p_f;y) \equiv \lim_{\substack{x^2 \to +\infty \\ (\text{with } m\frac{x^\mu}{\sqrt{x^2}} = p_f^\mu)}} \left[⨍_y^x A_\mu(z) dz^\mu \right] . \tag{3.29}$$

The connection from Eq. (3.28) to the formula (1.5a) given in the Introduction can be made as soon as one notes that

$$B_n(p_f;y) = e^{-ieH(p_f;y)} \bar{B}_n(p_f;y) \tag{3.30}$$

with the quantity $B_n(p_f;y)$ defined by Eq. (1.6).

With the S-matrix formula (3.18), it is the expansion (2.4) which is more convenient to use. Interchanging x and y in Eq. (2.4), we may rewrite it as

$$S_F(y,x) = \exp\left[-ie ⨍_x^y A_\delta(z) dz^\delta \right] \sum_{n=0}^{\infty} \left[\bar{a}_n(y,x) \left\{ \left[\left[i\frac{\partial}{\partial m^2} \right]^n \Delta_0(x-y;m^2) \right] (-\overleftarrow{\partial}_{(x)} + m) \right\} + K_n'(y,x) \right] \tag{3.31}$$

with

$$K_n'(y,x) = \left[-i\bar{a}_n(y,x)\overleftarrow{\partial}_{(x)} + \left[e\gamma^\mu(x-y)^\nu \int_0^1 ds\, sF_{\mu\nu}(s(x-y)+y) \right] \bar{a}_n(y,x) \right] \left[i\frac{\partial}{\partial m^2} \right]^n \Delta_0(x-y;m^2) . \tag{3.32}$$

Then we may follow the same reasoning as above to show that the $K_n'(y,x)$ pieces in Eq. (3.32), when inserted into Eqs. (3.17) and (3.18), yield no contribution to S-matrix elements and the scattering solution $\Psi^{(+)}$. We thus obtain alternative formulas to Eqs. (3.26) and (3.27):

$$\Psi^{(+)}(y;p_i,s_i) = \left[\frac{m}{E_i V} \right]^{1/2} \exp[-ip_i \cdot y - ie\mathscr{H}(p_i;y)] \left[\sum_{n=0}^{\infty} \overline{\mathscr{B}}_n(p_i;y) \right] u(p_i,s_i) , \tag{3.33}$$

$$S_{fi} - \delta_{fi} = \frac{(-ie)m}{V(E_f E_i)^{1/2}} \sum_{n=0}^{\infty} \int d^4y \exp[i(p_f - p_i)\cdot y - ie\mathscr{H}(p_i;y)] \bar{u}(p_f,s_f) A\!\!\!/(y) \overline{\mathscr{B}}_n(p_i;y) u(p_i,s_i) \tag{3.34}$$

with

$$\overline{\mathscr{B}}_n(p_i;y) \equiv \lim_{\substack{x^2 \to +\infty \\ (\text{with } -m\frac{x^\mu}{\sqrt{x^2}} = p_i^\mu)}} \left[\left[\frac{\sqrt{x^2}}{2m} \right]^n \bar{a}_n(y,x) \right] \tag{3.35}$$

and

$$\mathscr{H}(p_i;y) \equiv \lim_{\substack{x^2 \to +\infty \\ (\text{with } -m\frac{x^\mu}{\sqrt{x^2}} = p_i^\mu)}} \left[⨍_x^y A_\mu(z) dz^\mu \right] . \tag{3.36}$$

Since the quantity $\mathscr{B}_n(p_i;y)$ defined by Eq. (1.7) can be written as

$$\mathscr{B}_n(p_i;y)=e^{-ie\mathscr{H}(p_i;y)}\overline{\mathscr{B}}_n(p_i;y) , \tag{3.37}$$

we see that Eq. (3.34) coincides with Eq. (1.5b).

As will be shown in the next section, the expansions (3.27) or (3.34) can be used to obtain a systematic approximation in high-energy scattering problems. Which form between Eqs. (3.27) and (3.34) one uses is really a matter of convenience; they are equivalent within application range. The *full* right-hand side of Eq. (3.27) is equal to the *full* right-hand side of Eq. (3.34), but not necessarily in each order or when the series are truncated. Probably the most preferable would be to take the symmetric average of the two, thus making the symmetry of the S matrix (concerning the incoming and outgoing states) manifest in *each order.*[17] Note that if we neglect the distorted phase factors $e^{-ieH(p_f;y)}$ or $e^{-ie\mathscr{H}(p_i;y)}$ and keep only the first term in the series proportional to

$$\overline{B}_0(p_f;y)=\overline{\mathscr{B}}_0(p_i;y)=I , \tag{3.38}$$

the expansions given above reduce to the usual lowest-order Born approximation result.

IV. VALIDITY CRITERIA

In this section, we shall investigate the validity of the series (3.27) as a useful approximation for exact S-matrix elements. Conclusion for the series (3.34) will then be obvious, too. The potential $A_\mu(x)$ is assumed to be localized both in space and in time [see Eq. (3.2)]. Let b and τ_0 represent the spatial and temporal (or time duration) ranges of the external potential, respectively[18] and, just for the sake of making discussions simple, we may here assume that $b<\tau_0$. To study convergence of the series (3.27), we shall need a careful estimate of the ratio

$$r_{n+1}(p_f;y)=\frac{\overline{B}_{n+1}(p_f;y)}{\overline{B}_n(p_f;y)} \tag{4.1}$$

for y values restricted by [due to the presence of the $\mathcal{A}(y)$ factor in the integrand]

$$|\mathbf{y}|\lesssim b,\quad |y^0|\lesssim\tau_0 . \tag{4.2}$$

The case of a long-range potential will be treated separately.

We first look at the ratio r_1 in some detail. Noting that $\overline{B}_0(p_f;y)=I$ and using Eq. (2.22), we may write

$$r_1(p_f;y)=|\overline{B}_1(p_f;y)|=|X_1(p_f;y)+Y_1(p_f;y)+Z_1(p_f;y)| \tag{4.3}$$

with

$$X_1(p_f;y)=\lim_{x\to P}(-e)\frac{\sqrt{x^2}}{2m}(x-y)^\nu\int_0^1 du\frac{1}{u^2}\int_0^u ds\, s^2(\partial^\mu F_{\mu\nu})(s(x-y)+y) , \tag{4.4a}$$

$$Y_1(p_f;y)=\lim_{x\to P}\frac{(-ie)}{2}\sigma^{\mu\nu}\frac{\sqrt{x^2}}{2m}\int_0^1 duF_{\mu\nu}(u(x-y)+y) , \tag{4.4b}$$

$$Z_1(p_f;y)=\lim_{x\to P}(ie^2)\frac{\sqrt{x^2}}{2m}(x-y)^\nu(x-y)^\lambda\int_0^1 du\frac{1}{u^2}\left[\int_0^u ds\, sF_{\mu\nu}(s(x-y)+y)\right]\left[\int_0^u ds'\, s'F^\mu{}_\lambda(s'(x-y)+y)\right] . \tag{4.4c}$$

As a shorthand notation, we have here set (and assumed hereafter)

$$\lim_{\substack{x^2\to+\infty\\ [\text{with } m(x^\mu/\sqrt{x^2})=p_f^\mu]}}\equiv\lim_{x\to P} , \tag{4.5}$$

for y values constrained by the condition (4.2). For a localized potential, it is easy to see that the limits specified by Eqs. (4.4a)—(4.4c) are in fact well defined, with nonzero contributions due entirely from integration over very small $[\sim O(1/|\mathbf{x}|)]$ u, s, and s' regions. This also implies that

$$\lim_{x\to p}\sqrt{x^2}\,\overline{a}_1(x,y)=(\text{finite}) . \tag{4.6}$$

One may expect that relative order of magnitudes for $X_1(p_f;y)$, $Y_1(p_f;y)$, and $Z_1(p_f;y)$ depend strongly on the specific kinematical region in consideration [see Eq. (3.19)]. However, when we introduce the dimensionless variable

$$G\equiv\frac{eF_{\mu\nu}b^2}{|\mathbf{v}_f|} \tag{4.7}$$

with $F_{\mu\nu}$ representing the rough strength of external electromagnetic fields, we have obtained the following estimates with the expressions (4.4a)—(4.4c):[19]

$$X_1(p_f;y)\sim O\left[\frac{G}{|\mathbf{p}_f|b}\right],\quad \text{for all kinematical regions}$$

$$Y_1(p_f;y)\sim\begin{cases}O\left[\dfrac{G}{|\mathbf{p}_f|b}\right], & \text{if } |\mathbf{v}_f|=O(1) \text{ (including the limit } |\mathbf{v}_f|\to 1)\\ O\left[\dfrac{G}{|\mathbf{p}_f|b}\right]|\mathbf{v}_f|, & \text{if } |\mathbf{v}_f|\ll 1\end{cases} \tag{4.8}$$

$$Z_1(p_f;y)\sim O\left[\frac{G^2}{|\mathbf{p}_f|b}\right], \text{ for all kinematical regions .}$$

Note that $G\sim eF_{\mu\nu}b^2$ for (ultra) relativistic [i.e., $|\mathbf{v}_f|=O(1)$] scattering, and for nonrelativistic scattering the validity range of the usual Born series[20] in fact coincides with $G=eF_{\mu\nu}b^2/|\mathbf{v}_f|\ll 1$.

Having completed the estimate of $r_1(p_f;y)=\overline{B}_1(p_f;y)$, we may now turn to the estimation of $|\overline{B}_n(p_f;y)|$ $(n\geq 2)$. Here, due to the iterative nature of the very way the functions $\overline{a}_n(x,y)$ are constructed, the estimate for $|\overline{B}_2(p_f;y)|$, once made, can be readily generalized to draw conclusions about $\overline{B}_n(p_f;y)$ with $n\geq 3$, too. [In this regard, $r_1(p_f;y)$ is a bit special due to the fact that $\overline{a}_0(x,y)=I$ for all x,y]. Using Eqs. (2.18) and (2.20), $\overline{B}_2(p_f;y)$ can be expressed in terms of $\overline{a}_1(x,y)$. We organize the result as

$$|\overline{B}_2(p_f;y)|=|L_2(p_f;y)+M_2(p_f;y)+X_2(p_f;y)+Y_2(p_f;y)+Z_2(p_f;y)| , \tag{4.9}$$

where $L_2(p_f;y)$, $M_2(p_f;y)$, . . . , $Z_2(p_f;y)$ are special cases (i.e., $n=2$) of the functions defined by

$$L_n(p_f;y)=\lim_{x\to P}(-i)\left[\frac{\sqrt{x^2}}{2m}\right]^n\int_0^1 du\, u^{n-1}[\partial_{(x)}{}^2\overline{a}_{n-1}(x,y)]\,|_{x\to u(x-y)+y} , \tag{4.10a}$$

$$M_n(p_f;y)=\lim_{x\to P}(-2e)\left[\frac{\sqrt{x^2}}{2m}\right]^n(x-y)^\nu\int_0^1 du\, u^{n-2}(\partial^\mu_{(x)}\overline{a}_{n-1}(x,y))\,|_{x\to u(x-y)+y}\int_0^u ds\, sF_{\mu\nu}(s(x-y)+y) , \tag{4.10b}$$

$$X_n(p_f;y)=\lim_{x\to P}(-e)\left[\frac{\sqrt{x^2}}{2m}\right]^n(x-y)^\nu\int_0^1 du\, u^{n-3}\overline{a}_{n-1}(u(x-y)+y,y)\int_0^u ds\, s^2(\partial^\mu F_{\mu\nu})(u(x-y)+y) , \tag{4.10c}$$

$$Y_n(p_f;y)=\lim_{x\to P}\frac{(-ie)}{2}\sigma^{\mu\nu}\left[\frac{\sqrt{x^2}}{2m}\right]^n\int_0^1 du\, u^{n-1}\overline{a}_{n-1}(u(x-y)+y,y)F_{\mu\nu}(u(x-y)+y) , \tag{4.10d}$$

$$Z_n(p_f;y)=\lim_{x\to P}(ie^2)\left[\frac{\sqrt{x^2}}{2m}\right]^n(x-y)^\nu(x-y)^\lambda\int_0^1 du\, u^{n-3}\overline{a}_{n-1}(u(x-y)+y,y)$$
$$\times\int_0^u ds\, sF_{\mu\nu}(s(x-y)+y)\int_0^u ds'\, s'F^\mu{}_\lambda(s'(x-y)+y) . \tag{4.10e}$$

In fact, $\overline{B}_n(p_f;y)$ for $n\geq 2$ can also be written as

$$\overline{B}_n(p_f;y)=L_n(p_f;y)+M_n(p_f;y)+X_n(p_f;y)+Y_n(p_f;y)+Z_n(p_f;y) . \tag{4.11}$$

Using the behavior (4.6), it should not be very difficult to see that the functions defined by Eqs. (4.10a)–(4.10e) for $n=2$ are well defined, i.e.,

$$\lim_{x\to P}(\sqrt{x^2})^2\overline{a}_2(x,y)=(\text{finite}) . \tag{4.12}$$

Moreover, Eqs. (4.10a)–(4.10e) also imply that any nonvanishing contributions to $L_2(p_f;y)$, $M_2(p_f;y)$, . . . , $Z_2(p_f;y)$ are obtained entirely from the very small [$\sim O(1/|\mathbf{x}|)$] u, s, and s' regions; viz., to calculate $\overline{B}_2(p_f;y)$ by our iteration procedure, it is $\overline{a}_1(x,y)$ for finite values of x (more specifically $|\mathbf{x}|<b$) which becomes necessary. This may be compared to the calculation of $\overline{B}_1(p_f;y)$ for which only the $x\to P$ limit of $\overline{a}_1(x,y)$ is necessary. According to Eq. (2.22), we may here set

$$\overline{a}_1(x,y)\sim O\left[\frac{eF_{\mu\nu}(y)}{|\mathbf{v}_f|}\right][1+O(G)], \text{ for } |\mathbf{x}|\lesssim b \text{ and } |x^0|=\frac{|\mathbf{x}|}{|\mathbf{v}_f|} . \tag{4.13}$$

Based on these observations, we have estimated the magnitudes of $L_2(p_f;y)$, $M_2(p_f;y)$, . . . , $Z_2(p_f;y)$, obtaining the results

$$\left.\begin{aligned} &L_2(p_f;y)\sim O\left[\frac{G}{|\mathbf{p}_f|^2b^2}\right][1+O(G)]\\ &M_2(p_f;y),\ X_2(p_f;y)\sim O\left[\frac{G^2}{|\mathbf{p}_f|^2b^2}\right][1+O(G)]\\ &Z_2(p_f;y)\sim O\left[\frac{G^3}{|\mathbf{p}|^2b^2}\right][1+O(G)] \end{aligned}\right\}\text{ for all kinematical regions}$$

$$Y_2(p_f;y)\sim\begin{cases} O\left[\dfrac{G^2}{|\mathbf{p}|^2b^2}\right][1+O(G)], & \text{if } |\mathbf{v}_f|=O(1) \text{ (including the limit } |\mathbf{v}_f|\to 1)\\ O\left[\dfrac{G^2}{|\mathbf{p}|^2b^2}\right]|\mathbf{v}_f|[1+O(G)], & \text{if } |\mathbf{v}_f|\ll 1\ . \end{cases} \tag{4.14}$$

The analysis may be extended for $n=3$ and larger in an obvious fashion. For instance, the behavior (4.12) together with the formulas (4.10a)–(4.10e) for $n=3$ in turn leads to the conclusion that $L_3(p_f;y)$, $M_3(p_f;y),\ldots,$ $Z_3(p_f;y)$, and accordingly $\bar{B}_3(p_f;y)$ are well defined. In the formulas (4.10a)–(4.10e) for $n=3$, it is now $\bar{a}_2(x,y)$ for $|\mathbf{x}|<b$ which is responsible for nonvanishing results for those quantities. Here, using Eqs. (2.18) and (2.20) for $n=2$, one may find

$$\bar{a}_2(x,y)\sim O\left[\frac{eF_{\mu\nu}(y)}{b^2|\mathbf{v}_f|}\right][1+O(G)+O(G^2)+O(G^3)],\ \text{ for } |\mathbf{x}|\lesssim b \text{ and } |x^0|=\frac{|\mathbf{x}|}{|\mathbf{v}_f|}\ , \tag{4.15}$$

where the $O(eF_{\mu\nu}(y)/b^2|\mathbf{v}_f|)1$ contribution is due to the piece proportional to $\partial_x{}^2\bar{a}_1(x,y)$ from the right-hand side of Eq. (2.20). Then, estimates for the magnitudes of $L_3(p_f;y)$, $M_3(p_f;y),\ldots,Z_3(p_f;y)$ can be made using Eq. (4.15) in the formulas (4.10a)–(4.10e). For general n, repeating this procedure yields the following estimates:

$$\left.\begin{aligned} &L_n(p_f;y)\sim O\left[\frac{G}{(|\mathbf{p}_f|b)^n}\right][1+O(G)+\cdots O(G^{2n-3})],\\ &M_n(p_f;y),X_n(p_f;y)\sim O\left[\frac{G^2}{(|\mathbf{p}_f|b)^n}\right][1+O(G)\ \cdots\ O(G^{2n-3})],\\ &Z_n(p_f;y)\sim O\left[\frac{G^3}{(|\mathbf{p}_f|b)^n}\right][1+O(G)+\cdots+O(G^{2n-3})], \end{aligned}\right\}\text{ for all kinematical regions}$$

$$Y_n(p_f;y)\sim\begin{cases} O\left[\dfrac{G^2}{(|\mathbf{p}_f|b)^n}\right][1+O(G)+\cdots+O(G^{2n-3})], & \text{if } |\mathbf{v}_f|=O(1) \text{ (or } |\mathbf{v}_f|\to 1)\\ O\left[\dfrac{G^2}{(|\mathbf{p}_f|b)^n}\right]|\mathbf{v}_f|[1+O(G)+\cdots O(G^{2n-3})], & \text{if } |\mathbf{v}_f|\ll 1\ . \end{cases} \tag{4.16}$$

We are now ready to discuss the physical domain for which the series (3.27) may be useful. First, look at the series (3.26) for the scattering solution. According to the estimate shown in Eq. (4.16), it is clear that the series (3.26) will be useful for

(i) $|\mathbf{p}_f|b\gg 1$,

and

$$\text{(ii)}\ \frac{G^2}{|\mathbf{p}_f|b}\ll 1\left[\text{with } G=\frac{eF_{\mu\nu}b^2}{|\mathbf{v}_f|}\right]. \tag{4.17}$$

Namely, the series (3.26) provides a systematic *high-energy* approximation for the scattering solution $\Psi^{(-)}(y;p_f,s_f)$ (for finite values of y) when the external potential is slowly varying in the distance scale defined by the particle three-momentum, $1/|\mathbf{p}_f|$. The ratio $r_n(p_f;y)$, defined in Eq. (4.1), measures the degree of convergence for the series (3.26), and we find here

$$r_n(y)=\begin{cases} O\left[\dfrac{1}{|\mathbf{p}_f|b}\right], & \text{for } G\lesssim 1\\ O\left[\dfrac{G^2}{|\mathbf{p}_f|b}\right], & \text{for } 1<G\ll(|\mathbf{p}_f|b)^{1/2}. \end{cases} \tag{4.18}$$

The criteria (4.17) and (4.18) for the expansion (3.26) are valid for all kinematical regions, covering from the ultrarelativistic [i.e., $|\mathbf{v}_f|\to 1$ or $(m/|\mathbf{p}_f|)\to 0$] case to the nonrelativistic (i.e., $|\mathbf{v}_f| \ll 1$) case. Also note that, unlike the Born series which is known to be useful for $G \ll 1$, one may use the series (3.26) at sufficiently high energy even with $G=O(1)$ or larger. [See (ii) in Eq. (4.17).]

When the expansion (3.26) is used to evaluate S-matrix elements via Eq. (3.27), we actually have another scale to worry about. It is due to the factor

$$e^{i(p_f-p_i)\cdot y}=\exp\{i(m^2+\mathbf{p}_f^2)^{1/2}y^0 -i[m^2+(\mathbf{p}_f-\mathbf{q})^2]^{1/2}y^0-i\mathbf{q}\cdot\mathbf{y}\} \tag{4.19}$$

with $\mathbf{q}=\mathbf{p}_f-\mathbf{p}_i$ denoting the three-momentum transfer, which introduces the additional distance scale, $1/|\mathbf{q}|$. Of course, for $|\mathbf{q}|b \lesssim 1$, it should be possible to take the criteria (4.18) for the series (3.26) also as the measure for the convergence of the expansion (3.27) for S-matrix elements. On the other hand, the situation is quite different for $|\mathbf{q}|b \gg 1$: the presence of the rapidly oscillating factor (4.19) then results in vanishingly small S-matrix elements if the remaining term in the integrand,

$$e^{-ieH(p_f;y)}\left[\sum_{n=0}^{\infty}\bar{B}_n(p_f;y)\right]A\!\!\!/(y) , \tag{4.20}$$

is slowly varying in the distance scale $1/|\mathbf{q}|$. According to our recurrence formula for $\bar{a}_n(x,y)$, it is quite conceivable that $\bar{B}_n(p_f;y)$ may involve more variations within the given y range (4.2) as n increases, simply because of the appearance of more $F_{\mu\nu}$'s with different arguments. Thus, although suppressed by the ratio given in Eq. (4.18), $\bar{B}_n(p_f;y)$ terms combined with the oscillating factor (4.19), for relatively large $|\mathbf{q}|b$, may yield larger contributions to the scattering amplitude when n is larger. This observation leads us to the conclusion that, as $|\mathbf{q}|b$ increases beyond 0(1) (i.e., for not-so-small-angle scatterings), we must taken into account more $\bar{B}_n(p_f;y)$ terms (than necessary for small-angle scatterings) to obtain an equally good approximation to corresponding S-matrix elements. When $|\mathbf{q}|b$ is much larger than 1, we do not expect the expansion (3.27) to be very useful.

The validity range of our expansion (3.27) clearly overlaps with that of the so-called eikonal approximation. In fact, keeping only zeroth-order term $\bar{B}_0(p_f;y)=I$ in Eq. (3.27) yields the well-known eikonal formula,[8–11] and thus terms coming from $\bar{B}_n(p_f;y)$ with $n\geq 1$ may be then regarded as giving systematic corrections to it (see Sec. V for a more explicit exposition on this with a general static potential). The series (3.26) for the scattering solution describes our *systematically improved eikonal wave function.* Here, simplicity of the functions $\bar{B}_n(p_f;y)$ at least for small n should be a big asset of our series (3.27). An analogous conclusion may be reached also with the expansion (3.34).

For a long-range potential with the behavior $\tilde{A}^0(\mathbf{x})\sim K/|\mathbf{x}|$ (K a constant) at large distance, some modifications are necessary with the above discussion. Following the same line of arguments as given above, it is easy to show that the quantities $\bar{B}_n(p_f;y)$ defined by Eq. (3.28) [or $\bar{\mathscr{B}}_n(p_i;y)$ defined by Eq. (3.35)] are still well defined. But the function $H(p_f;y)$ or $\mathscr{H}(p_i;y)$, defined by Eq. (3.29) or Eq. (3.36), diverges logarithmically. This divergence arising from $H(p_f;y)$ or $\mathscr{H}(p_i;y)$ only results in an overall (divergent) phase to S matrix and the scattering cross section is still well defined (in this regard, see also the end of Sec. V). In practice, to use our formalism, it is thus sufficient to regulate such long-range potential at large distance (e.g., Coulomb by a screened Coulomb) and remove the regularization at the level of scattering cross section. With this understanding, the condition (4.17) and the convergence ratio (4.18) are valid for the S-matrix formula (3.27) in such Coulomb type potential if one here identifies b with $1/|\mathbf{q}|$ and G with $eK/|\mathbf{v}_f|$.

Before closing this section, there is a remarkable fact worth mentioning with our series (3.27) or (3.34). It is that Eq. (3.27) continues to be useful for studying high-energy scattering [under the condition (4.17) and for $|\mathbf{q}|b$ not too large compared to 1] even with a *massless particle,* i.e., $m\rightarrow 0+$. [Note that, in the limit inside Eq. (3.28) for $\bar{B}_n(p_f;y)$, we may replace $\sqrt{x^2}/2m$ by $|\mathbf{x}|/2|\mathbf{p}_f|$. Also, the factor m at front in the right-hand side of Eq. (3.27) is just due to the specific normalization assumed for Dirac spinors $u(p,s)$.] This is in a marked contrast to the expansion (2,4) for $S_F(x,y)$ which becomes ill defined for $m\rightarrow 0+$, but consistent with the simple physical picture that the mass term may be ignored in ultrarelativistic scattering. Stated differently, our expansion (3.27) or (3.34) for S-matrix elements really corresponds to the high-energy (and not heavy-mass) expansion.

V. STATIC EXTERNAL POTENTIAL

Since most applications in the external-field problems involve static potentials, i.e.,

$$A^\mu(x)=e^{-\eta|x^0|}\tilde{A}^\mu(\mathbf{x})\equiv e^{-\eta|x^0|}(\tilde{A}^0(\mathbf{x}),\tilde{\mathbf{A}}(\mathbf{x})) , \tag{5.1}$$

we here consider our expansion for S-matrix elements in more explicit forms for this case. First let us look at the function $H(p_f;y)$ defined in Eq. (3.29). For a sufficiently localized function $\tilde{A}^\mu(x)$, it now reduces to

$$H(p_f;y)=\lim_{x\rightarrow P}(x-y)^\mu\int_0^1 ds\,\tilde{A}_\mu(s(\mathbf{x}-\mathbf{y})+\mathbf{y})e^{-\eta|s(x^0-y^0)+y^0|}=\frac{E_f}{|\mathbf{p}_f|}D^0(\hat{p}_f;\mathbf{y})-\hat{p}_f\cdot\mathbf{D}(\hat{p}_f;\mathbf{y}) \tag{5.2}$$

with

$$D^0(\hat{p}_f;\mathbf{y})\equiv\lim_{|\mathbf{x}|\rightarrow\infty}\left[|\mathbf{x}|\int_0^1 ds\,\tilde{A}^0(s|\mathbf{x}|\hat{p}_f+\mathbf{y})\right]=\int_0^\infty ds\,\tilde{A}^0(s\hat{p}_f+\mathbf{y}) , \tag{5.3a}$$

$$\mathbf{D}_f(\hat{p}_f;\mathbf{y})\equiv\lim_{|\mathbf{x}|\rightarrow\infty}\left[|\mathbf{x}|\int_0^1 ds\,\tilde{\mathbf{A}}(s|\mathbf{x}|\hat{p}_f+\mathbf{y})\right]=\int_0^\infty ds\,\tilde{\mathbf{A}}(s\hat{p}_f+\mathbf{y}) , \tag{5.3b}$$

where we have set $\mathbf{p}_f/|\mathbf{p}_f| \equiv \hat{p}_f$. Thanks to Eq. (5.2), the zeroth-order approximation with Eq. (3.27) reads

$$(S_{fi}-\delta_{fi})_{n=0}=\frac{(-ie)m}{VE_f}(2\pi)\delta(E_f-E_i)\int d^3\mathbf{y}\exp\left[-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{y}-ie\frac{E_f}{|\mathbf{p}_f|}D^0(\hat{p}_f;\mathbf{y})+ie\hat{p}_f\cdot\mathbf{D}(\hat{p}_f;\mathbf{y})\right]$$
$$\times\bar{u}(p_f,s_f)\tilde{A\!\!\!/}(\mathbf{y})u(p_i,s_i)\ . \tag{5.4}$$

Using Eqs. (5.3a) and (5.3b) for the functions $D^0(\hat{p}_f;\mathbf{y})$ and $\mathbf{D}(\hat{p}_f;\mathbf{y})$, the distorted phase factor present in the right-hand side of Eq. (5.4) may be expressed explicitly as ($\mathbf{v}_f\equiv\mathbf{p}_f/E_f$)

$$\exp\left[-ie\frac{E_f}{|\mathbf{p}_f|}D^0(\hat{p}_f;\mathbf{y})+ie\hat{p}_f\cdot\mathbf{D}(\hat{p}_f;\mathbf{y})\right]=\exp\left[-ie\int_0^\infty dt\,\tilde{A}^0(\mathbf{y}+\mathbf{v}_f t)+ie\int_0^\infty dt\mathbf{v}_f\cdot\tilde{\mathbf{A}}(\mathbf{y}+\mathbf{v}_f t)\right]\ . \tag{5.5}$$

The next order (i.e., $n=1$) term with the series (3.27) will be given by

$$(S_{fi}-\delta_{fi})_{n=1}=\frac{(-ie)m}{V(E_fE_i)^{1/2}}\int d^4y\exp\left[i(p_f-p_i)\cdot y-ie\frac{E_f}{|\mathbf{p}_f|}D^0(\hat{p}_f;\mathbf{y})+ie\hat{p}_f\cdot\mathbf{D}(\hat{p}_f;\mathbf{y})\right]$$
$$\times\bar{u}(p_f,s_f)\bar{B}_1(p_f;y)A\!\!\!/(y)u(p_i,s_i)\ . \tag{5.6}$$

The quantity $\bar{B}_1(p_f;y)$ may be calculated using Eqs. (2.22) and (3.28). For an external potential (5.1), we immediately obtain

$$\bar{B}_1(p_f;y)=\lim_{x\to P}\left[\frac{|\mathbf{x}|}{2|\mathbf{p}_f|}\bar{a}_1(x,y)\right]$$
$$=-\frac{e}{2|\mathbf{p}_f|^2}p_f^\nu\int_0^\infty ds\,s\partial^i\tilde{F}_{i\nu}(s\hat{p}_f+\mathbf{y})-\frac{ie}{4|\mathbf{p}_f|}\sigma^{\mu\nu}\int_0^\infty ds\tilde{F}_{\mu\nu}(s\hat{p}_f+\mathbf{y})$$
$$+\frac{ie^2}{|\mathbf{p}_f|^3}p_f^\nu p_f^\lambda\int_0^\infty ds\tilde{F}_{\mu\nu}(s\hat{p}_f+\mathbf{y})\int_0^s ds'\,s'\tilde{F}^{\mu}{}_{\lambda}(s'\hat{p}_f+\mathbf{y})\ , \tag{5.7}$$

where we have denoted $\tilde{F}_{\mu\nu}=\partial_\mu\tilde{A}_\nu-\partial_\nu\tilde{A}_\mu$. {Note that the limit $x\to P$ here [see Eq. (4.5)] may be represented by $|\mathbf{x}|\to\infty$ with $m(x^\mu/\sqrt{x^2})=p_f^\mu$ (: fixed) and y^μ finite}. Inserting Eq. (5.7) into Eq. (5.6) and then integrating over the variable y^0 yields

$$(S_{fi}-\delta_{fi})_{n=1}$$
$$=\frac{(-ie)m}{VE_f}(2\pi)\delta(E_f-E_i)\int d^3\mathbf{y}\exp\left[-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{y}-ie\frac{E_f}{|\mathbf{p}_f|}D^0(\hat{p}_f;\mathbf{y})+ie\hat{p}_f\cdot\mathbf{D}(\hat{p}_f;\mathbf{y})\right]$$
$$\times\bar{u}(p_f,s_f)\left[-\frac{e}{2|\mathbf{p}_f|^2}p_f^\nu\int_0^\infty ds\,s\partial^i\tilde{F}_{i\nu}(s\hat{p}_f+\mathbf{y})-\frac{ie}{4|\mathbf{p}_f|}\sigma^{\mu\nu}\int_0^\infty ds\tilde{F}_{\mu\nu}(s\hat{p}_f+\mathbf{y})\right.$$
$$\left.+\frac{ie^2}{|\mathbf{p}_f|^3}p_f^\nu p_f^\lambda\int_0^\infty ds\tilde{F}_{\mu\nu}(s\hat{p}_f+\mathbf{y})\int_0^s ds'\,s'\tilde{F}^{\mu}{}_{\lambda}(s'\hat{p}_f+\mathbf{y})\right]\tilde{A\!\!\!/}(\mathbf{y})u(p_i,s_i)\ . \tag{5.8}$$

Explicit consideration beyond the $n=1$ term with the expansion (3.27), which is straightforward but undoubtedly more complicated, will not be given here.

The information given above on S-matrix elements can be readily transcribed into that for the differential scattering cross section, $d\sigma/d\Omega$. In the approximation keeping up to the term proportional to $\bar{B}_1(p_f;y)$ in the series (3.27), we have (before considering the spin sum)

$$\frac{d\sigma}{d\Omega}=\frac{m^2}{4\pi^2}\left|\int d^3\mathbf{y}\exp\left[-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{y}-ie\int_0^\infty dt\,\tilde{A}^0(\mathbf{y}+\mathbf{v}_f t)+ie\int_0^\infty dt\,\mathbf{v}_f\cdot\tilde{\mathbf{A}}(\mathbf{y}+\mathbf{v}_f t)\right]\right.$$
$$\left.\times\,\bar{u}(p_f,s_f)\left[Q^{\mathrm{I}}(p_f;\mathbf{y})+\sigma^{\mu\nu}Q^{\mathrm{II}}_{\mu\nu}(p_f;\mathbf{y})\right]e\tilde{A\!\!\!/}(\mathbf{y})u(p_i,s_i)\right|^2, \tag{5.9}$$

where we have used the notation (5.5) and the functions $Q^{\mathrm{I}}(p_f;\mathbf{y})$, $Q^{\mathrm{II}}(p_f;\mathbf{y})$ are given by

$$Q^{\rm I}(p_f;\mathbf{y})=1-\frac{e}{2m^2}(1-\mathbf{v}_f{}^2)p_f^\nu\int_0^\infty dt\, t\partial^i\widetilde{F}_{i\nu}(\mathbf{y}+\mathbf{v}_f t)+\frac{ie^2}{m^3}(1-\mathbf{v}_f)^{3/2}p_f^\nu p_f^\lambda\int_0^\infty dt\,\widetilde{F}_{\mu\nu}(\mathbf{y}+\mathbf{v}_f t)\int_0^t dt'\,t'\widetilde{F}^{\mu}{}_{\lambda}(\mathbf{y}+\mathbf{v}_f t')\,, \tag{5.10}$$

$$Q^{\rm II}_{\mu\nu}(p_f;\mathbf{y})=-\frac{ie}{4m}(1-\mathbf{v}_f{}^2)^{1/2}\int_0^\infty dt\,\widetilde{F}_{\mu\nu}(\mathbf{y}+\mathbf{v}_f t)\,. \tag{5.11}$$

Setting $Q^{\rm I}(p_f;\mathbf{y})=1$ and $Q^{\rm II}_{\mu\nu}(p_f;\mathbf{y})=0$ in Eq. (5.9) produces the result which would be obtained if only the $\overline{B}_0(p_f;y)$ were kept with the series (3.27); this gives (one version of) the usual eikonal formula.[8–11] Our expression for $d\sigma/d\Omega$ specified by Eqs. (5.9)–(5.11) also includes the leading corrections to it, with the quantity $Q^{\rm I}(p_f;y)-1$ giving the correction to charge scattering and the term proportional to $Q^{\rm II}_{\mu\nu}(p_f;y)$ generating important spin-dependent effects. Also, according to Eqs. (5.10) and (5.11), it may well be that those correction terms could be given some simple physical interpretations along the classical coordinate-space scattering picture.

Alternative expressions for the differential cross section may be obtained by using the series (3.34) instead of Eq. (3.27). We shall here only state the result when the series (3.34) is used with a static potential. With $A_\mu(x)$ given by the form (5.1), the function $\mathscr{H}(p_i;y)$ defined in Eq. (3.36) reduces to ($\hat{p}_i\equiv\mathbf{p}_i/|\mathbf{p}_i|$, $\mathbf{v}_i\equiv\mathbf{p}_i/E_i$)

$$\begin{aligned}\mathscr{H}(p_i;y)&=\frac{E_i}{|\mathbf{p}_i|}\int_{-\infty}^0 ds\,\widetilde{A}^{\,0}(s\hat{p}_i+\mathbf{y})-\hat{p}_i\cdot\int_{-\infty}^0 ds\,\widetilde{\mathbf{A}}(s\hat{p}_i+\mathbf{y})\\&=\int_{-\infty}^0 dt\,\widetilde{A}^{\,0}(\mathbf{y}+\mathbf{v}_i t)-\int_{-\infty}^0 dt\,\mathbf{v}_i\cdot\widetilde{\mathbf{A}}(\mathbf{y}+\mathbf{v}_i t)\,,\end{aligned} \tag{5.12}$$

and from Eqs. (2.22) and (3.35) the quantity $\overline{\mathscr{B}}_1(p_i;y)$ acquires the form

$$\begin{aligned}\overline{\mathscr{B}}_1(p_i;y)=&\frac{e}{2|\mathbf{p}_i|^2}p_i^\nu\int_{-\infty}^0 ds\,s\partial^i\widetilde{F}_{i\nu}(s\hat{p}_i+\mathbf{y})-\frac{ie}{4|\mathbf{p}_i|}\sigma^{\mu\nu}\int_{-\infty}^0 ds\,\widetilde{F}_{\mu\nu}(s\hat{p}_i+\mathbf{y})\\&-\frac{ie^2}{|\mathbf{p}_i|^3}p_i^\nu p_i^\lambda\int_{-\infty}^0 ds\,\widetilde{F}_{\mu\nu}(s\hat{p}_i+\mathbf{y})\int_s^0 ds'\,s'\widetilde{F}^{\mu}{}_{\lambda}(s'\hat{p}_i+\mathbf{y})\,.\end{aligned} \tag{5.13}$$

We thus find the following alternative expression for $d\sigma/d\Omega$:

$$\begin{aligned}\frac{d\sigma}{d\Omega}=\frac{m^2}{4\pi^2}\Bigg|\int d^3\mathbf{y}\exp\left[-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{y}-ie\int_{-\infty}^0 dt\,\widetilde{A}^{\,0}(\mathbf{y}+\mathbf{v}_i t)+ie\int_{-\infty}^0 dt\,\mathbf{v}_i\cdot\widetilde{\mathbf{A}}(\mathbf{y}+\mathbf{v}_i t)\right]\\\times\,\bar{u}(p_f,s_f)e\widetilde{\not{A}}(\mathbf{y})\left[T^{\rm I}(p_i;\mathbf{y})+\sigma^{\mu\nu}T^{\rm II}_{\mu\nu}(p_i;\mathbf{y})\right]u(p_i,s_i)\Bigg|^2\end{aligned} \tag{5.14}$$

with the functions $T^{\rm I}(p_i;\mathbf{y})$, $T^{\rm II}_{\mu\nu}(p_i;\mathbf{y})$ given by

$$\begin{aligned}T^{\rm I}(p_i;\mathbf{y})=&\,1+\frac{e}{2m^2}(1-\mathbf{v}_i{}^2)p_i^\nu\int_{-\infty}^0 dt\,t\partial^i\widetilde{F}_{i\nu}(\mathbf{y}+\mathbf{v}_i t)\\&-\frac{ie^2}{m^3}(1-\mathbf{v}_i{}^2)^{3/2}p_i^\nu p_i^\lambda\int_{-\infty}^0 dt\,\widetilde{F}_{\mu\nu}(\mathbf{y}+\mathbf{v}_i t)\int_t^0 dt'\,t'\widetilde{F}^{\mu}{}_{\lambda}(\mathbf{y}+\mathbf{v}_i t)\,,\end{aligned} \tag{5.15}$$

$$T^{\rm II}_{\mu\nu}(p_i;\mathbf{y})=-\frac{ie}{4m}(1-\mathbf{v}_i{}^2)^{1/2}\int_{-\infty}^0 dt\,\widetilde{F}_{\mu\nu}(\mathbf{y}+\mathbf{v}_i t)\,. \tag{5.16}$$

Again, setting $T^{\rm I}(p_i;\mathbf{y})=1$ and $T^{\rm II}_{\mu\nu}(p_i;\mathbf{y})=0$ in (5.14) gives another version of the usual eikonal formula.

For a given external potential, numerical analysis will usually be required to perform the integrals appearing in (5.9) and (5.14). In the next section, we will present numerical calculations of the high-energy scattering cross section for a simple exponential potential.

We make a brief comment on the case of a long-range potential. If $\widetilde{A}^{\,0}(x)\sim Z/|\mathbf{x}|$ for large $|\mathbf{x}|$ (i.e., Coulomb-type), the quantity $D^0(p_f;\mathbf{y})$ defined by (5.3a) is ill defined since the integral

$$\int_0^\infty ds\,\widetilde{A}^{\,0}(s\hat{p}_f+\mathbf{y})\sim Z\int_0^\infty ds\frac{1}{|s\hat{p}_f+\mathbf{y}|} \tag{5.17}$$

is logarithmically divergent. Here, suppose we regularize the potential by a Yukawa-type, i.e., $\widetilde{A}^{\,0}(x)\sim(Z/|\mathbf{x}|)e^{-\lambda|\mathbf{x}|}$ with a very small positive number λ. Then, for this regularized potential,

$$\begin{aligned}\int_0^\infty ds\,\widetilde{A}^{\,0}(s\hat{p}_f+\mathbf{y})&=Z\int_0^\infty ds\frac{e^{-\lambda|s\hat{p}_f+\mathbf{y}|}}{|s\hat{p}_f+\mathbf{y}|}\\&\underset{\lambda\to0+}{\longrightarrow}-Z\ln\lambda+(\text{finite terms})\,.\end{aligned} \tag{5.18}$$

But, this logarithmic divergence in $D_0(\hat{p}_f;\mathbf{y})$ only introduces a phase $e^{-ieZ(E_f/|\mathbf{p}_f|)\ln\lambda}$ into S-matrix elements [see Eq. (5.8)], and thus it will completely disappear in the scattering cross section formula (5.9). In this manner (due to Dalitz[21] originally), unambiguous results for the high-energy scattering cross section may be secured by our method even for a Coulomb-type potential at large distance.

VI. NUMERICAL ILLUSTRATION

We now want to see, through numerical calculations, the degree of improvements over the standard eikonal formula when we keep the first-order correction terms with our formula (3.27). This will be done by comparing our results and standard eikonal results on the scattering amplitude with those from the partial-wave analysis.

We shall consider the exponential potential, which has often been used to test various approximation methods of analyzing the scattering amplitude with the Schrödinger equation.[22] We thus take

$$V(\mathbf{y})[\equiv eA_0(\mathbf{y})] = -\frac{U_0}{2m}\exp\left[-\frac{|\mathbf{y}|}{b}\right], \tag{6.1}$$

with suitable values for $U_0(>0)$ and b. With this choice, the function $eH(\mathbf{p}_f;\mathbf{y})$ becomes

$$eH(\mathbf{p}_f,\mathbf{y}) = -\frac{\gamma U_0}{2|\mathbf{p}_f|}\int_0^\infty \exp(-|\mathbf{y}+s\hat{p}_f|/b)ds\ , \tag{6.2}$$

where $\gamma = E_f/m$.

Specific values taken for numerical calculation are

(i) $m=1$ GeV, $|\mathbf{p}_i|=1$ GeV,

$U_0/2m = 0.24$ GeV, $b = 1/1.45$ fm ,

(ii) $m=1$ GeV, $|\mathbf{p}_i|=3$ GeV,

$U_0/2m = 0.24$ GeV, $b = 1/1.45$ fm .

The magntiudes of $1/|\mathbf{p}_i|b$ and $G^2/|\mathbf{p}_i|b$, which play an important role for the convergence of our series, are respectively,

Case (i): $1/|\mathbf{p}_i|b = 1/3.5$,

$G^2/|\mathbf{p}_i|b = \frac{2}{5}$ $(G=1.2)$,

Case (ii): $1/|\mathbf{p}_i|b = 1/10.5$,

$G^2/|\mathbf{p}_i|b = \frac{1}{12}$ $(G=0.9)$.

For these two cases, we plot the ratios

$$R_1 = \frac{(d\sigma/d\Omega)_{\text{eik}}}{(d\sigma/d\Omega)_{\text{phase shift}}}\ ,$$
$$R_2 = \frac{(d\sigma/d\Omega)_{\text{ours}}}{(d\sigma/d\Omega)_{\text{phase shift}}} \tag{6.3}$$

in Figs. 1 and 2. Here note that $(d\sigma/d\Omega)_{\text{ours}}$ is specified by our formula (5.9) while $(d\sigma/d\Omega)_{\text{eik}}$, the usual eikonal cross section, corresponds to the result obtained by setting $Q^{\text{I}}(p_f;y) = Q^{\text{II}}_{\mu\nu}(p_f;y) = 0$ in the same formula. The scattering cross section by the phase shift analysis, $(d\sigma/d\Omega)_{\text{phase shift}}$, is calculated following the standard procedure (see Ref. 23, for instance). All cross sections are spin-averaged ones. For the sake of more detailed comparison, we plot, in Fig. 3, the spin-nonflip part of the scattering amplitude for the case (i). The spin-nonflip

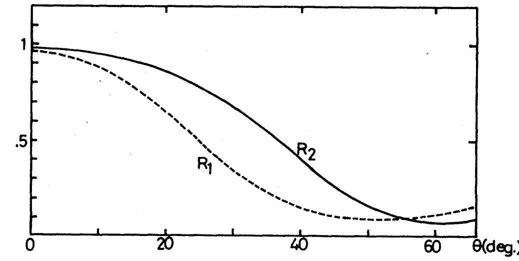

FIG. 1. The ratios R_1, R_2 defined in Eq. (6.3) for the case (i). The result of the phase shift is based on the partial-wave sum up to $l=69$.

part, $f(\theta)$, and the spin-flip part, $g(\theta)$, of the scattering amplitude are related to the spin-averaged cross section, $(d\sigma/d\Omega)_{\text{spin-ave}}$ by

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{spin-ave}} = |f(\theta)|^2 + |g(\theta)|^2\ . \tag{6.4}$$

Note that the spin-nonflip part comes not only from $Q^{\text{I}}(p_f;y)$ but also from $Q^{\text{II}}(p_f;y)$.

As is shown in Figs. 1 and 2, the improvement is significant when the scattering angle is not too large compared to the angular range fixed by the condition $|\mathbf{q}|b \lesssim 0(1)$, i.e.,

Case (i): $\sin(\theta/2) \lesssim \frac{1}{7}$,

Case (ii): $\sin(\theta/2) \lesssim \frac{1}{20}$.

The fact that, at large angle, the spin-averaged cross section obtained from the usual eikonal formula approaches that of the phase-shift analysis should be considered fortuitous. This can be immediately seen by looking at Fig. 3 where we compare the spin-nonflip parts $f(\theta)$ due to our method, eikonal and phase-shift analysis.

If we set $\gamma=1$ in Eq. (6.2), it reduces to the phase factor of the usual eikonal formula for the Schrödinger equation. With the Schrödinger equation, Berriman and Castillejo[22] used the exponential potential to compare various methods of improving the usual eikonal formula. In fact, one of their choices for parameters coincides with our case (i). Given their plots, it may be also interesting to see

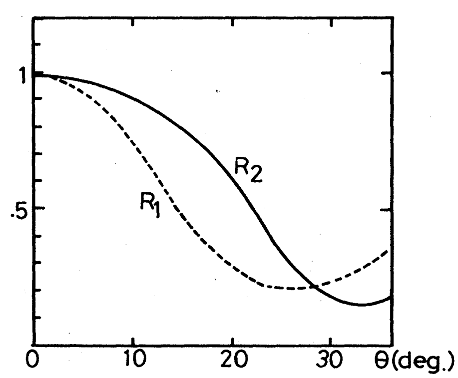

FIG. 2. The ratios R_1, R_2 defined in Eq. (6.3) for the case (ii). The result of the phase shift is based on the partial-wave sum up to $l=169$.

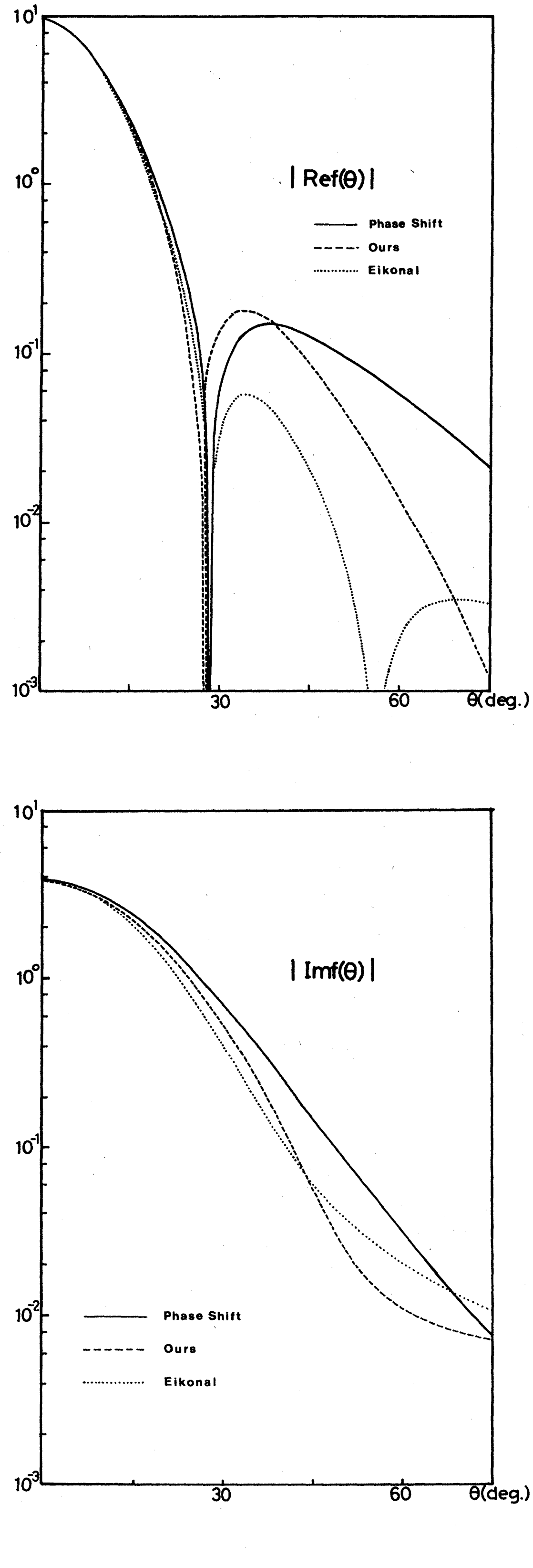

FIG. 3. Real and imaginary parts of the spin-nonflip part of the scattering amplitude $f(\theta)$ (given in the units of fermi) for the case (i).

the result obtained via our Eq. (5.9) but with γ set to 1. For the case (i), we have thus plotted

$$R_1^{\rm Sch}=\left[\frac{d\sigma}{d\Omega}\right]_{\rm eik}^{\gamma=1}\Bigg/\left[\frac{d\sigma}{d\Omega}\right]_{\rm phase\ shift}^{\rm Schrödinger},$$
$$R_2^{\rm Sch}=\left[\frac{d\sigma}{d\Omega}\right]_{\rm ours}^{\gamma=1}\Bigg/\left[\frac{d\sigma}{d\Omega}\right]_{\rm phase\ shift}^{\rm Schrödinger} \qquad (6.5)$$

in Fig. 4. Of course, $(d\sigma/d\Omega)^{\rm Schrödinger}_{\rm phase\ shift}$ here refers to the cross section obtained by partial wave analysis with the Schrödinger equation. The improvement, shown in Fig. 4, is again quite good. Comparing the results with those from Ref. 22, we see that our formula in this case yields the result which is better than that of Sugar and Blankenbecler[11] and comparable to that of Saxon and Schiff.[11]

VII. EXTENSION TO LARGE-ANGLE SCATTERING

We have shown that the scattering formula (3.27) or (3.34) generalizes the eikonal formula for small-angle scattering. However, according to our discussions in Sec. IV, this scattering formula may not be applicable for a strong potential or for sufficiently large-angle scattering. On the other hand, many potentials of physical interest possess a rather strong short-range core region which may give significant large-angle scattering. We now extend our work also to deal with such a case in a systematic way. The resulting formula will correspond to a generalization of Schiff's large-angle formula.[10,24]

For our present purpose, we find it convenient to assume that the external potential has the following general form:

$$A_\mu(x)=A_\mu^{(1)}(x;\Lambda)+A_\mu^{(2)}(x;\Lambda) \qquad (7.1)$$

with

$$A_\mu^{(1)}(x;\Lambda)=e^{-\eta|x^0|}\tilde{A}_\mu^{(1)}(\mathbf{x},x^0;\Lambda)\,,$$
$$A_\mu^{(2)}(x;\Lambda)=e^{-\eta|x^0|}\tilde{A}_\mu^{(2)}(\mathbf{x},x^0;\Lambda)\,. \qquad (7.2)$$

Here, Λ is an adjustable parameter to be explained shortly. In Eq. (7.2), the adiabatic switching factor $e^{-\eta|x^0|}$ has been introduced by the same reasoning as in Eq. (3.2). In separating a given external potential A_μ into two pieces as in Eq. (7.1), it will be assumed here that the usual Born series is applicable with $A_\mu^{(1)}$ set to zero while the eikonal

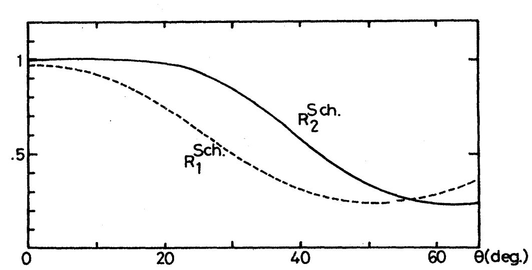

FIG. 4. The ratios $R_1^{\rm Sch},R_2^{\rm Sch}$ defined in Eq. (6.5) for the case (i) with $\gamma=1$.

approximation is valid with $A_\mu^{(2)}$ set to zero. In practice, depending on the shape of the given external potential A_μ and/or on the interested momentum-transfer range, such separation may be affected, smoothly, by adjusting a certain free parameter Λ. Our strategy is that, by combining the eikonal approximation with the (distorted-wave) Born series, we can deal with a quite broad class of external potentials and/or study high-energy scattering amplitudes over broad momentum-transfer range. Note that the underlying physics of the separation (7.1) is quite similar to the distorted-wave Born approximation[12] (when the potential is divided, in configuration space, into the short- and long-range components) or to the standard treatment of hard versus soft photons in QED (when the Fourier transform of any given potential A_μ is divided into two parts in comparison to the magnitude of the observed momentum transfer $\mathbf{q}=\mathbf{p}_f-\mathbf{p}_i$). We shall return to the question of suitably choosing Λ at the end, and until then the parameter Λ will be suppressed in the discussion.

In this section, we find it convenient to use the more symmetrically written form for the S-matrix elements:

$$S_{fi}=\delta_{fi}-i\int d^4x\, d^4y\, \bar\psi_f(x)(i\partial\!\!\!/_x-m)\times S_F(x,y)(-i\overleftarrow{\partial\!\!\!/}_y-m)\psi_i(y)\ . \tag{7.3}$$

Of course, this formula is equivalent to Eq. (3.4) or Eq. (3.6). We may now expand the exact Feynman propagator $S_F(x,y)$ as

$$S_F=S_F^{(1)}+S_F^{(1)}eA\!\!\!/^{(2)}S_F^{(1)}+S_F^{(1)}eA\!\!\!/^{(2)}S_F^{(1)}eA\!\!\!/^{(2)}S_F^{(1)}+\cdots \tag{7.4}$$

with

$$S_F^{(1)}(x,y)=\left\langle x\left|\frac{1}{i\partial\!\!\!/-eA\!\!\!/^{(1)}-m+i0+}\right|y\right\rangle\ . \tag{7.5}$$

Then, the formula (7.3) may be cast into the form

$$S_{fi}=\delta_{fi}+S_{fi}^{(\text{eik})}+\sum_{n=1}^{\infty}S_{fi,n}^{(\text{dist Born})}\ , \tag{7.6}$$

where

$$S_{fi}^{(\text{eik})}=-i\int d^4x\, d^4y\, \bar\psi_f(x)(i\partial\!\!\!/_x-m)S_F^{(1)}(x,y)(-i\overleftarrow{\partial\!\!\!/}_y-m)\psi_i(y)\ , \tag{7.7}$$

$$S_{fi}^{(\text{dist Born})}=-i\int d^4x d^4y\prod_{i=1}^{n}d^4z_i\, \bar\psi_f(x)(i\partial\!\!\!/_x-m)S_F^{(1)}(x,z_1)eA\!\!\!/^{(2)}(z_1)S_F^{(1)}(z_1,z_2)\cdots eA\!\!\!/^{(2)}(z_n)S_F^{(1)}(z_n,y)(-i\overleftarrow{\partial\!\!\!/}_y-m)\psi_i(y)\ . \tag{7.8}$$

Of course, all the formulas derived in Sec. III may be directly used for all quantities which involve the eikonal part $A_\mu^{(1)}$ only. The piece $S_{fi}^{(\text{eik})}$ can be immediately obtained from the formula (3.27) or (3.34) if we replace A_μ by $A_\mu^{(1)}$ there.

Using Eqs. (3.5) and (3.7), the piece $S_{fi,n}^{(\text{dist Born})}$ can be rewritten as

$$S_{fi,n}^{(\text{dist Born})}=-i\int\prod_{i=1}^{n}dz_i\bar\Psi^{(-)}(z_1;p_f,s_f)eA\!\!\!/^{(2)}(z_1)S_F^{(1)}(z_1,z_2)\cdots eA\!\!\!/^{(2)}(z_n)\Psi^{(+)}(z_n;p_i,s_i)\ , \tag{7.9}$$

where $\Psi^{(\pm)}$ here denote the exact scattering solutions in the presence of the potential $A_\mu^{(1)}$ only. The expansion (7.9) has the form of distorted-wave Born series and thus its validity range may well be taken just as that for the usual Born series, i.e.,

$$\frac{e\tilde F_{\mu\nu}^{(2)}[b_2]^2}{|\mathbf{v}|}\ll 1\quad\left[|\mathbf{v}|=\frac{|\mathbf{p}_f|}{p_f^0}\ \text{or}\ \frac{|\mathbf{p}_i|}{p_i^0}\right] \tag{7.10}$$

where $\tilde F_{\mu\nu}^{(2)}$ represents the average value of $\tilde F_{\mu\nu}^{(2)}(x)=\partial_\mu\tilde A_\nu^{(2)}(x)-\partial_\nu\tilde A_\mu^{(2)}(x)$ and b_2 is the range [actually the smaller between the temporal and spatial ranges of the potential $A_\mu^{(2)}(x)$].

For the right-hand side of Eq. (7.9), we may use our improved eikonal series (3.26) or (3.33) (with A_μ replaced by $A_\mu^{(1)}$) for the scattering solution $\Psi^{(\pm)}$ and the expansion (2.1) for $S_F^{(1)}$. Explicitly, for $n=1$, this gives the result

$$\begin{aligned}S_{fi,n=1}^{(\text{dist Born})}&=-ie\int d^4y\,\bar\Psi^{(-)}(y;p_f,s_f)A\!\!\!/^{(2)}(y)\Psi^{(+)}(y;p_i,s_i)\\&=\frac{(-ie)m}{V(E_fE_i)^{1/2}}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\int d^4y\exp[i(p_f-p_i)y-ieH(p_f;y)-ie\mathscr{H}(p_i;y)]\\&\qquad\times\bar u(p_f,s_f)\bar B_n(p_f;y)A\!\!\!/^{(2)}(y)\overline{\mathscr{B}}_l(p_i;y)u(p_i,s_i)\ .\end{aligned} \tag{7.11}$$

Owing to the presence of $A\!\!\!/^{(2)}(y)$ inside the integrand, this piece can, especially, provide information on high-energy scattering at large-momentum transfer. In fact, keeping only the ($n=0$, $l=0$) term in this series yields the so-called Schiff's large-angle scattering formula;[10] viz., our series (7.11) for $S_{fi,n=1}^{(\text{dist Born})}$ corresponds to a systematic generalization of Schiff's result. It is straightforward to express $S_{fi,n}^{(\text{dist Born})}$ with $n\geq 2$ [see Eq. (7.8)] in an analogous manner. But, one must also use here Eq. (3.20) or (3.31) for $S_F^{(1)}(x,y)$ and not just the series (3.26) or (3.33), and accordingly the expressions will be more involved. For practical purposes one may limit oneself to the consideration of $S_{fi}^{(\text{eik})}$ and $S_{fi,n=1}^{(\text{dist Born})}$ first, and after that think about the $S_{fi,n=2}^{(\text{dist Born})}$ term, etc. For practical calculations, we give here the explicit expression for

$S_{fi,n=1}^{(\text{dist Born})}$, assuming static potentials and keeping only the $n,l=0,1$ terms in Eq. (7.11). This leads to

$$S_{fi,n=1}^{(\text{dist Born})}=\frac{(-ie)m}{VE_f}(2\pi)\delta(E_f-E_i)\int d^3\mathbf{y}\,\bar{u}(p_f,s_f)\Big[\Big[Q^{\text{I}}(p_f,y)+T^{\text{I}}(p_i,y)-1\Big]\tilde{\mathcal{A}}^{(2)}(y)+\sigma^{\mu\nu}Q^{\text{II}}_{\mu\nu}(p_f;\mathbf{y})\tilde{\mathcal{A}}^{(2)}(y)$$
$$+\tilde{\mathcal{A}}^{(2)}(y)\sigma^{\mu\nu}T^{\text{II}}_{\mu\nu}(p_i;\mathbf{y})\Big]u(p_i,s_i)$$
$$\times\exp\Big[-i(\mathbf{p}_f-\mathbf{p}_i)\cdot\mathbf{y}-ie\int_0^\infty dt[\tilde{A}^{(1)0}(\mathbf{y}+\mathbf{v}_ft)-\mathbf{v}_f\cdot\tilde{\mathbf{A}}^{(1)}(\mathbf{y}+\mathbf{v}_ft)]$$
$$-ie\int_{-\infty}^0 dt[\tilde{A}^{(1)0}(\mathbf{y}+\mathbf{v}_it)-\mathbf{v}_i\cdot\tilde{\mathbf{A}}^{(1)}(\mathbf{y}+\mathbf{v}_it)]\Big], \tag{7.12}$$

where the quantities Q^{I}, Q^{II}, T^{I}, and T^{II} are given by Eqs. (5.10), (5.11), (5.15), and (5.16) with $\tilde{A}_\mu$ set to $\tilde{A}_\mu^{(1)}$.

We finally return to the question of how one may divide a given potential $A_\mu(x)$ into two pieces as assumed in Eq. (7.1). The free parameter Λ may be naturally identified with b_2 [i.e., the range of the potential $A_\mu^{(2)}(x)$], while b_1 represents the range of the given potential $A_\mu(x)$ from the long-range side. By high-energy scattering we mean the region $|\mathbf{p}_i|b_1\gg1$, $|\mathbf{p}_f|b_1\gg1$. To include short-range (possibly singular) or large-momentum components of a given potential A_μ in the part $A_\mu^{(2)}$ while reserving its long-range or relatively small-momentum components for $A_\mu^{(1)}$, Λ may be naturally chosen to fulfill the following conditions:

(i) $\Lambda\ll b_1$, $\dfrac{e\tilde{F}_{\mu\nu}^{(2)}\Lambda^2}{|\mathbf{v}|}\ll1$ [see Eq. (7.10)]

and

(ii) the Fourier transform of $A_\mu^{(1)}(x)$, say, $\bar{A}_\mu^{(1)}(k^0,\mathbf{k})$, should be strongly suppressed outside the momentum range $k^0\Lambda\lesssim O(1)$, $|\mathbf{k}|\Lambda\lesssim O(1)$. Note that, in view of the condition (i), the magnitude of $\tilde{F}_{\mu\nu}^{(2)}$ might be significantly larger than the average value of $\tilde{F}_{\mu\nu}^{(1)}$. For high-energy scattering at relatively small-momentum transfer [i.e., $|\mathbf{q}|b_1\lesssim O(1)$], the dominant role will be played by the eikonal part $S_{fi}^{(\text{eik})}$ while $S_{fi,n=1}^{(\text{dist Born})}$ can be useful to study contributions, say, due to a strong short-range term in the potential. On the other hand, for the case of large-momentum transfer (i.e., $|\mathbf{q}|b_1\gg1$), Λ may be chosen such that

$$\frac{1}{|\mathbf{q}|}\ll\Lambda\ll b_1 \tag{7.13}$$

and this will make the part $S_{fi,n=1}^{(\text{dist Born})}$ dominant over the eikonal part. It is the case of intermediate momentum transfer where contributions due to *both* parts (and possibly the pieces $S_{fi,n}^{(\text{dist Born})}$ with $n\geq2$) should be carefully assessed. In this last case, considerable arbitrariness may show up in the results (with a truncated series) depending on the choice of Λ and for the best result one may adopt a suitable optimization scheme.[25] To examine such features further, we feel that numerical analysis of high-energy scattering (with some typical potentials), on the basis of our general scheme, will be extremely valuable. This is under consideration.

VIII. DISCUSSIONS

The Schwinger-DeWitt proper-time expansion, proved to be powerful in dealing with renormalization problems, is marked by its simplicity and elegance. In this work, we have found another important aspect of the same expansion in that it can be used to generate a systematic high-energy approximation to the scattering amplitude—it generalizes the eikonal formula. The lowest-order coefficient function $a_0(x,y)$ in the Schwinger-DeWitt expansion, which plays the role of a covariantizing phase factor in renormalization problems, is also responsible for the usual eikonal formula. Correction terms to the usual eikonal result can then be related to the suitable asymptotic limit of the coefficient functions $a_n(x,y)$ $(n\geq1)$. All quantities are calculated in the coordinate space directly. The leading correction terms are given by especially simple expressions, and we have seen that numerical calculations including those terms yield quite encouraging results.

We believe that our correction terms share the same physical origin as the Saxon-Schiff (or Sugar-Blankenbecler) correction terms,[11] but ours appear simpler. Also, it may be possible to reproduce our method (which is based on the DeWitt WKB expansion) by making a systematic approximation with the standard WKB approach. The precise connection, which we do not have at this moment, is desirable since it will further illuminate the physical source of our various correction terms to the leading eikonal formula. [In this regard, see expecially Yennie, Boos, and Ravenhall.[11] These authors obtain a systematic (eikonal-type) high-energy approximation starting from the standard WKB method.]

Even though our formula for the not-so-small-angle scattering developed in Sec. VII is a bit involved, it is straightforward and can be systematically improved within the validity range of DWBA. For the subject of the fixed-angle high-energy scattering, it seems worth mentioning an interesting recent paper by Cheng, Coon, and Zhu.[26] With a Yukawa-type potential as an example, these authors obtained the fixed-angle scattering amplitude by using the WKB method together with a rather elaborate analysis near the origin. Within the validity range of DWBA, our formula should also give results consistent with theirs. But, we believe that the detailed comparison should be made with the help of extensive numerical calculations—this is being pursued by Park.[27]

Although we have concentrated on the Dirac equation in this paper, our method can be used, with certain straightforward changes, to study high-energy scattering of particles with different spins, too. It is not necessary to restrict the external potential to an Abelian background-gauge field, either; high-energy particle scattering in a non-Abelian background-gauge field and/or Lorentz-scalar potential may also be considered. (A physical example is the 't Hooft-Polyakov monopole solution[28] as the background potential.) When a non-Abelian gauge field matrix $A = A^a T^a$(T^a: generator matrices in the given fermion representation) is used with the Dirac equation (3.1), the function $a_0(x,y)$ in the proper-time expansion is given by the path-ordered phase factor

$$a_0(x,y) = P\left\{\exp[-ie \textstyle\oint_y^x A_\mu^a(z) T^a dz^\mu]\right\}$$

and correspondingly $a_n(x,y)$ for $n \geq 1$ become also more involved. But, recurrence relations are still solved by a straightforward generalization of the method described in Sec. II. Our analysis may also be carried over to the case with a background-gravitational field (e.g., in the Schwarzschild metric).

As mentioned in the Introduction, we believe that the present investigation on the external-field problem can be really turned into a part of a useful quantum-field-theoretic tool. Just like the Born series in the ordinary quantum mechanics, the usual perturbation series in quantum field theories have a rather limited range of applications. In certain cases (e.g., bremsstrahlung of soft particles, Sudakov form factor,[29] and various high-energy exclusive or inclusive scattering processes), this has been circumvented by summing an infinite number of Feynman diagrams of certain particular types together with a specific approximation made for each chosen Feynman diagram. But it will be nice to have a systematic machinery which may provide us with correct physical amplitudes more directly. One possible approach may be obtained by a judicious combination of

(i) the Feynman path-integral formulation of quantum field theories,

(ii) the background-field method,[2] and

(iii) the Schwinger-DeWitt proper-time expansion for the Green's functions in the presence of background fields. (See Ref. 30 for a related attempt.) Note that background fields in this approach will be dynamically generated ones. But, much further work will be required to effect this idea in a concrete way (and also to see its limitation).

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[18]One may here imagine, for instance, a Yukawa-type potential ($\sim e^{-|\mathbf{x}|/b}$) for the spatial dependence. Also, for the case of a static potential, i.e., with $\tilde{A}_\mu(\mathbf{x},x^0) = \tilde{A}_\mu(\mathbf{x})$, we may set $\tau_0 = 1/\eta$ ($\eta =$ adiabatic switching factor).

[19]As mentioned earlier, we are here assuming $b < \tau_0$. For general cases, the parameter b appearing in Eqs. (4.7) and (4.8) may be replaced by the *smaller* between the spatial range b and time-duration range τ_0 of the external potential.

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