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Nonperturbative many-body techniques applied to a Yang-Mills field theory

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Starting from the cutoff version of a field-theoretical Hamiltonian derived from an SU(n) Yang-Mills theory in the Coulomb gauge, we investigate the structure of the emerging many-body problem within a Bogoliubov approximation for the ground state (= physical vacuum) and by considering suitable quasiparticle excitations for glueball states. The idea of the formation of a bag can be incorporated into this scheme. The energy expectation values are approximated by a cluster expansion. The (formal) results, allowing the numerical computation of the glueball spectrum at a later stage, are presented and the emerging structure is discussed. Special attention is hereby paid to the significance of the Gribov ambiguity and to the consequences of the singularity of the Hamiltonian at the Gribov horizon. It is suggested that the possibility of a rising potential between gluons be investigated, as such a potential could be a signal of confinement.

I. INTRODUCTION

Using a finite-lattice approximation for the Feynman path integrals, it seems now possible to obtain a rather reliable prediction for the spectrum of non-Abelian gauge theories.[1] However, in view of the numerical complexity of such "lattice Monte Carlo" calculations it should be desirable to have an alternative, if simpler, insight into the structure of the eigenfunctions of such gauge theories, even if the method used is less rigorous. A great advantage of the existence of the lattice results would be the possibility to test other approaches. The most promising alternative method to deal with the spectrum of a gauge theory seems to be the treatment of the corresponding *field-theoretical Hamiltonian* with *nonperturbative techniques.*

In the past, several attempts have been made in this direction: Pottinger and Warner[2] and Carson[3] tried to gain insight into the structure of the QCD spectrum within a "one-mode approximation" for the gluons, a technique that has been successfully applied to a nucleon-boson Hamiltonian by Bolsterli.[4] Using a hyperspherical formalism, Cutkosky[5] investigated the spectrum of a Yang-Mills theory including a larger number of modes. A complementary approach is that of Lüscher and Münster.[6,7] Within a YM theory, they treat the dynamics of the zero-momentum modes of the gluons rigorously. They treat the nonzero-momentum modes perturbatively, which is justified if the system is contained in a small volume. The extrapolation to a large volume poses the main problem of the method and is not reliably solved.

It is the purpose of this paper to propose an alternative, completely nonperturbative treatment of the field-theoretical Hamiltonian of an SU(n) ($n \geq 2$) YM theory in the Coulomb gauge, regularized by a phase-space cutoff.[8] Our treatment is given by approximation techniques borrowed from nonrelativistic many-body theory. The great advantage of our method will be that all final calculations can be performed in the infinite-volume limit—quite analogous to the case of infinite nuclear matter. Our techniques are easily generalized for an application to QCD which is our ultimate goal. We shall discuss here the "pure" YM case because it seems simpler and it may be used as a testing ground for our procedure. The proposed, nonperturbative approximation method may be summarized by two prescriptions.

(i) For the ground state (= physical vacuum) and low-lying excited states (glueballs) use a *Bogoliubov ansatz* whose parameters ought to be determined by the *Ritz* variational principle.

(ii) Since energy expectation values cannot be determined rigorously, because of the complexity of the YM Hamiltonian, for the calculation of these quantities use a *cluster expansion* (quite in the spirit of the hypernetted-chain approximation in Yastrow theory[9]).

The general structure of the Bogoliubov theory for bosons has been discussed recently by Birse *et al.*[10] In the framework of the YM theory this technique was applied by Gribov in order to construct a (possible) connection between the Coulomb-gauge horizon and confinement.[11] Nojiri considered this approximation for the ground state within the axial gauge.[12]

In the lowest order of this cluster expansion, our approximation prescriptions yield well-defined expressions for the energy expectation values in terms of the Bogoliubov parameters which have to be determined by minimization (the Ritz variational principle). In this paper, we shall be concerned with explaining the details of our approximation and with a presentation of the expressions for the energy functionals. Numerical applications will be given in future publications. Our calculations show, however, some structural results which should make our formal investigation interesting.

(1) The singular and, for the ground state, repulsive behavior of the Hamiltonian at the Gribov horizon makes it plausible that the canonical quantization of the YM theory in the Coulomb gauge is justified—at least in a good approximation. Thus the obstacle for defining this

quantization given by the Gribov ambiguity seems to be much mitigated. Here, the essential point is that the "Bogoliubov freedom" in the quantization procedure (see Sec. III) has to be taken into account.

(2) The same repulsive structure is also expected to prevent the ghost propagator and the effective Coulomb potential (see Sec. VI) from developing any singularity (except for the standard Coulomb pole). In this sense, our results do not support the idea that the singularity of the theory at the Gribov horizon yields directly a confining potential.

(3) The contributions of the Coulomb interaction to the glueball mass have an interesting sign structure: One has an attractive term in the irreducible two-particle matrix element and a repulsive term in the single-particle matrix elements. This structure makes a rising slope for the gluon-gluon effective potential (defined with wave packets in an adiabatic sense) plausible and, therefore, yields a signal for confinement.

(4) The Bogoliubov scheme allows for an extension of the ansatz for the glueball state which incorporates the idea of the formation of a bag. The significance of this structure can be tested by using the variational principle. The glueball mass displays in this case a decomposition familiar from the bag models.

Our paper is organized as follows. In Sec. II, we describe how to introduce the phase-space cutoff and how to deal with renormalization. (The main point is to study the cutoff dependence of the relation between observables for consistency.) In Sec. III, it is shown how the Bogoliubov theory is most easily incorporated by choosing a suitable ansatz for the $t=0$ field operators. The well-known ideas leading to the construction of the YM Hamiltonian in the Coulomb gauge are briefly presented in Sec. IV. In order to prepare this Hamiltonian for the application of our many-body techniques, a convenient expansion of this operator in momentum space is presented in Sec. V. The basic ideas of the cluster expansion for energy expectation values are described in Sec. VI, more details are given in the Appendix. The vacuum problem is then formulated. The emerging structure and its connection with the Gribov problems is elucidated in Sec. VII. The energy functionals for the glueball masses are presented and discussed in Sec. VIII. In Sec. IX, it is shown how the study of the (momentum) cutoff dependences of the results for the different energies is simplified through scale invariance. Finally, the formulation of a bag formation is introduced in Sec. X.

II. RENORMALIZATION

Haag's theorem[13] states that the only quantum field theory definable in a Fock space and fulfilling the Wightman axioms is the free field theory. Consequently, a rigorous definition of a relativistic field theory with interaction is not possible within that space. Since the many-body techniques we want to use are only applicable in Fock space we propose to use the following cutoff procedure for the treatment of the field-theoretical Hamiltonian.

(a) Assume a phase-space cutoff (Ω,M) for the definition of the $(t=0)$ field operators.[8] We specify the definition of this cutoff for the spinless case, the generalization for the YM case is obvious (see Sec. IV). Ω stands for the volume cutoff, so that a basis of single-particle states is given by discrete plane waves

$$f_k(x)=e^{ikx}/\sqrt{\Omega}\ ,\quad x,k\in R^3\ ,$$
$$k=2\pi\Omega^{-1/3}(n_1,n_2,n_3),\quad n_j \text{ an integer}\ , \tag{1}$$

and we have the usual continuum limit

$$\frac{1}{\Omega}\sum_k \underset{\Omega\to\infty}{\to} (2\pi)^{-3}\int d^3k\ . \tag{2}$$

Relating to each function f_k creation and annihilation operators $a_k^\dagger, a_k$ obeying

$$[a_k,a_{k'}^\dagger]=\delta_{kk'}\ , \tag{3}$$

the canonically conjugate cutoff field operators (for $t=0$) of a scalar theory are defined by introducing a momentum cutoff M for the possible range of k values:

$$\phi(x)=\sum_{|k|<M}(2\omega_k)^{-1/2}[f_k(x)a_k+f_k^*(x)a_k^\dagger]\ ,$$
$$\Pi(x)=\sum_{|k|<M}-i(\omega_k/2)^{1/2}[f_k(x)a_k-f_k^*(x)a_k^\dagger]\ , \tag{4}$$
$$\omega_k=(k^2+m^2)^{1/2}\ .$$

We mention that the Fock-space vacuum $|0\rangle$ is determined through the condition $a_k|0\rangle=0$ for all k and that the full Fock space is generated from states of the type:

$$a_{k_1}^\dagger a_{k_2}^\dagger\cdots a_{k_n}^\dagger|0\rangle\ ,\quad n=0,1,2,\ldots\ .$$

(b) Inserting such cutoff field operators into the expression for the (classical) energy density yields, after integration, a cutoff field-theoretical Hamiltonian $H(g,\Omega,M)$ which, in the case of the YM theory, depends on the cutoff and on the "unrenormalized" coupling constant g. We assume that $H(g,\Omega,M)$ is a well-behaved operator in the Fock space so that we can consider its spectrum. As derived most easily from the *formal* structure of perturbation theory, this spectrum has properties quite analogous to that of nuclear matter (or liquid He^4): For large Ω the ground-state energy $E_0(g,\Omega,M)$ is proportional to the volume, i.e.,

$$E_0(g,\Omega,M)/\Omega=\text{independent of }\Omega\ . \tag{5}$$

For a field theory, clearly the ground state itself has to be interpreted as the *physical vacuum* state.

The energy $E_n(g,\Omega,M)$ of a low-lying excited state behaves even simpler, if this energy is measured relative to that of the ground state, namely,

$$\epsilon_n=E_n-E_0=\text{independent of }\Omega\ . \tag{6}$$

The set of the quantities $\epsilon_n(g,M)$ has to be interpreted as the spectrum of the observed particles. We remark that such a volume independence is valid for the expectation value of the excited state with respect to any reasonable operator, if a corresponding vacuum expectation value is subtracted.

(c) "Renormalization" should now be performed ac-

cording to the following principle. For a *given* momentum cutoff M, $g(M)$ should be determined by adjusting one physical datum [e.g., if $\epsilon_1(g,M)$ is the smallest occurring mass, fit this quantity to the experiment]. This fixes $H(g(M),\Omega,M)$ (except for the Ω dependence which is trivial since observables are independent of Ω for large Ω) and allows—in principle—the prediction of any physical property.

(d) The crucial consistency check of this renormalization procedure lies in the study of the "continuum limit." If one physical datum is fixed, do other observables become independent of the cutoff M? When perturbation theory is applied to the approximate determination of the spectrum of the Hamiltonian, Dirac has shown that the standard results are obtained,[14] the consistency of this scheme being guaranteed for a limited range of cutoffs. [The problem is that $g(M)$ increases with M, invalidating perturbation theory for large M.] When nonperturbative techniques are applied, one may hope to increase the range of cutoff values where the prediction for observables is independent of M, though a reliable estimate of the structure for $M\to\infty$ might be very difficult to obtain. The situation here is quite analogous to that of the continuum limit in lattice calculations and (as in that case) we propose to investigate this problem numerically. Within a scalar ϕ^4 theory in $1+1$ dimensions (Haag's theorem bans the Fock space even in this case) the existence of the cutoff limit has been proven by Wiesbrock.[15]

III. THE BOGOLIUBOV TRANSFORMATION

The standard procedure for the "canonical quantization" of a relativistically invariant classical field theory fixes only the commutation relations between the canonically conjugate $t=0$ field operators. In fact, these commutation relations are sufficient in order to prove (on a formal basis) that when quantization is applied to all generators of the Poincaré group given by Noether's theorem (translations, rotations, boosts), one obtains a (formal) representation of this group (the ultimate goal of relativistic quantum field theory).

Inspection into the definition (4) for the field operators ϕ and Π clearly shows that the commutation relations are not changed when the replacement $\omega_k\to\lambda_k{}^{-1}$ is made, when λ_k is an arbitrary, positive, nonzero function of k. In fact, such a replacement can be achieved within the same Fock space by a Bogoliubov transformation:

$$b_k=u_k a_k+v_k a^\dagger_{-k}\ ,\quad (u_k)^2-(v_k)^2=1 \tag{7}$$

by setting (we assume that the real functions u_k,v_k,λ_k depend only on the modulus $|k|$ of the three-vector k)

$$\lambda_k=(u_k-v_k)^2/\omega_k\ . \tag{8}$$

The "quasiparticle operators" $b_k,b_k^\dagger$ obey the same commutation relations $[b_k,b_{k'}^\dagger]=\delta_{kk'}$. Eliminating the operators $a_k,a_k^\dagger$ in Eq. (4) by inverting Eq. (7),

$$a_k=u_k b_k-v_k b^\dagger_{-k}\ ,$$

we arrive at the b representation of the field operators:

$$\begin{aligned}\phi(x)&=\sum(\lambda_k/2)^{1/2}f_k(x)(b_k+b^\dagger_{-k})\ ,\\ \Pi(x)&=-i\sum(2\lambda_k)^{-1/2}f_k(x)(b_k-b^\dagger_{-k})\ ,\end{aligned} \tag{9}$$

which has the structure of Eq. (4) with the replacement $\omega_k\to\lambda_k{}^{-1}$.

The "quasiparticle" vacuum ψ, fixed by the condition $b_k\psi=0$ for all k, is related to the Fock space vacuum by

$$\psi=\exp\left[\tfrac{1}{2}\sum_{|k|<M}\frac{v_k}{u_k}a_k^\dagger a^\dagger_{-k}\right]|0\rangle\ . \tag{10}$$

ψ can be interpreted as a Bose BCS state. In the b representation, a convenient basis for the Fock space is generated by the states

$$b^\dagger_{k_1}b^\dagger_{k_2}\cdots b^\dagger_{k_n}\psi\ ,\quad n=0,1,2,\ldots\ .$$

For the purpose of approximating the ground state of a (cutoff) field-theoretical Hamiltonian it is convenient to leave open the function λ_k entering into the definition of the field operators and to determine λ_k by minimizing $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$. (For a general discussion of the application of the variational principle in field theory, see also Ref. 16.)

Note that since λ_k is assumed to be spherically symmetric, the state ψ has the same symmetries as the Fock-space vacuum $|0\rangle$ (trivial momentum, angular momentum). The generalization of the ansatz (10) for the YM case is given in Sec. V.

IV. THE YM HAMILTONIAN IN THE COULOMB GAUGE

We first specify our notations for the YM theory which we assume to be SU(n) ($n\geq 2$) gauge invariant. We denote the YM (vector) potentials by

$$A^a_\mu(\mathbf{x})\quad\begin{matrix}a=1,\ldots,(n^2-1)\ ,\quad \mathbf{x}=(x_0=t,\mathbf{x})\\ \mu=0,\ldots,3\ .\end{matrix} \tag{11}$$

As usual, we use indices i,j,l ($=1,2,3$) for denoting spatial parts of four-vectors. The SU(n) structure constants f^{abc} we assume to be antisymmetric and normalized to (a,b,c are color indices)

$$f^{abc}f^{dbc}=\delta^{ad}\ . \tag{12}$$

Here, and throughout the paper we use the *Einstein convention* for abbreviating sums. The YM fields $F^a_{\mu\nu}(\mathbf{x})$ are then given by

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^{abc}A^b_\mu A^c_\nu\ ,\quad \partial_\nu=\partial/\partial x_\mu \tag{13}$$

and the YM Lagrangian density has the form

$$\begin{aligned}L&=-\tfrac{1}{4}F^a_{\mu\nu}F^a_{\sigma\tau}g^{\mu\sigma}g^{\nu\tau}\ ,\\ g^{\mu\nu}&=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}\ .\end{aligned} \tag{14}$$

Due to the local SU(n) gauge invariance[17] of the YM

theory, two YM potentials have to be considered as equivalent if they may be transformed into each other by a gauge transformation. In order to deal with such "equivalence classes" of YM potentials it is convenient to work with sets of "representatives" which may be defined by a suitable gauge-fixing condition. The problem then is to define the quantization of the YM theory within a given gauge in such a way that the spectrum of the Hamiltonian is independent from the choice of the gauge (condition of gauge invariance of physical predictions). This problem has been solved for a large class of gauge-fixing conditions by Christ and Lee.[17] The most convenient gauge (because the gauge-fixing condition is rotational invariant) is the Coulomb gauge given by the transversality condition

$$\partial_j A_j^a = 0 \ , \quad a = 1, \ldots, (n^2-1) \ . \tag{15}$$

We neglect in the following the possible constraints between the Coulomb-gauge potentials which are due to the Gribov ambiguity.[11] A justification for this procedure will be discussed in Sec. VII. We first then have to eliminate (on the classical level) the dependent degrees of freedom from the Lagrangian, namely, A_0^a and the longitudinal part of the "electric" field ${}^L F_{0j}^a$. (The transverse fields ${}^T F_{0j}^a \equiv \Pi_j^a$, obeying $\partial_j \Pi_j^a = 0$, are canonically conjugate to the transverse potentials A_j^a.) This elimination is achieved (for details see Ref. 18) by taking into account those Euler Lagrange equations which represent constraints (i.e., contain no time derivatives). The constraint allowing the elimination of ${}^L F_{0j}^a$ can be written in the form[18]

$$\bar{\Delta}\phi^a = \rho^a \ , \quad {}^L F_{0j}^a = -\partial_j \phi^a \ , \tag{16}$$

where ρ^a is the "color-charge density" defined by

$$\rho^a = g f^{abc} A_j^b \Pi_j^c \tag{17}$$

(recall that Π_j^a is canonically conjugate to A_j^a). Equation (16) is very similar to the Poisson equation known from QED when quantized in the Coulomb gauge[19] (Dirac Schwinger quantization) which relates the electric potential ϕ to the charge density ρ and which yields the familiar (nonretarded) Coulomb interaction $\rho(1/\Delta)\rho$ between electrons in the QED Hamiltonian.

The important, and for the whole structure of the YM theory crucial, new ingredient, however, is that the operator $\bar{\Delta}$ is a "covariant" Laplacian containing an additional term involving the (transverse) potential A_j^a (note that $\bar{\Delta}$ is also a matrix in color space):

$$(\bar{\Delta})^{ab} = \Delta \delta^{ab} - g f^{abc} A_j^c \partial_j \ . \tag{18}$$

As derived, e.g., by Abers and Lee,[18] the (still classical) YM energy density in the Coulomb gauge is then given by

$$\mathscr{H} = \frac{1}{2}\left[\Pi_j^a \Pi_j^a + B_j^a B_j^a - \rho^a \frac{1}{\bar{\Delta}} \Delta \frac{1}{\bar{\Delta}} \rho^a \right] \ , \qquad B_j^a = \tfrac{1}{2} \epsilon_{jil} F_{il}^a \ , \tag{19}$$

where the last term corresponds to the Coulomb interaction of QED.

Before quantizing this Hamiltonian density one has to worry about gauge invariance. As shown by Christ and Lee,[17] standard canonical quantization in the *temporal* gauge leads, within perturbation theory, to the correct, gauge and Lorentz-invariant Feynman rules. Consequently, the quantized Hamiltonian in any other gauge has to be constructed from the temporal-gauge Hamiltonian according to the rules connecting different (Cartesian and curvilinear) coordinate systems. The result for the Coulomb-gauge Hamiltonian (now to be quantized in the standard way) is (we use the form of Lüscher[6])

$$\begin{aligned} H &= \tfrac{1}{2} \int_{t=0} d^3x \left[F^{-1} \Pi_j^a F^2 \Pi_j^a F^{-1} + B_j^a B_j^a \right. \\ &\qquad \left. - F^{-1} \rho^a F \frac{1}{\bar{\Delta}} \Delta \frac{1}{\bar{\Delta}} F \rho^a F^{-1} \right] \\ &= H_\pi + H_B + H_c \ , \\ F &= [\det(-\bar{\Delta})]^{1/2} \ . \end{aligned} \tag{20}$$

The "Faddeev-Popov determinant" F is a functional Jacobi determinant originating from the mentioned coordinate transformation. The explicit construction of the theory from the temporal gauge also leaves a remnant of Gauss's law eliminating unphysical states from the Fock space, namely, the (weak) condition of global SU(n) invariance of the states. Therefore, only states with *total* SU(n) color *zero* belong to the physical Hilbert space.[6] Note that this does not yet guarantee confinement (see Sec. VIII). The $t=0$ quantized field operators Π_j^a and A_j^a, entering into the definition of the quantities ρ^a, $\bar{\Delta}$, F, and B_j^a, are now constructed as in Dirac-Schwinger QED.[19] The expansion in terms of the plane waves f_k, with cutoff (Ω, M) and including the Bogoliubov freedom (7) reads

$$\begin{aligned} A_j^a(x) &= \sum_{|k|<M} (\lambda_k/2)^{1/2} e_j(k,r) [f_k(x) b_{kra} + \text{H.c.}] \ , \\ \Pi_j^a(x) &= -i \sum_{|k|<M} (2\lambda_k)^{-1/2} e_j(k,r) [f_k(x) b_{kra} - \text{H.c.}] \ , \quad [b_{kra}, b_{k'r'a'}^\dagger] = \delta_{kk'} \delta_{aa'} \delta_{rr'} \ . \end{aligned} \tag{21}$$

Here, the polarization vectors $e_j(k,r)$ ($r=1,2$ for $k \neq 0$, $r=1,2,3$ for $k=0$) make the operators A_j^a and Π_j^a transverse. This is guaranteed, if these vectors fulfill the relation

$$h_{ij}(k) \equiv \sum_r e_i(k,r) e_j(k,r) = \begin{cases} \delta_{ij} - k_i k_j / |k|^2 \ , & k \neq 0 \ , \\ \delta_{ij} \ , & k = 0 \ , \end{cases} \qquad k = (k_1, k_2, k_3) \ . \tag{22}$$

We recall that the function λ_k has to be determined by minimizing the expectation value $\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle$, where $b_{kra}\psi=0$ for all (kra). In the case of the YM theory, the BSC state ψ is related to the "bare" Fock-space vacuum (in the sense of Sec. III) through

$$\psi=e^S|0\rangle\ ,\quad S=\tfrac{1}{2}\sum\frac{v_k}{u_k}s_r a^\dagger_{kra}a^\dagger_{-kra}\ , \tag{23}$$

where the phase factor s_r is defined by[19]

$$e_j(k,r)=s_r e_j(-k,r)\ . \tag{24}$$

The operator S conserves momentum, angular momentum, parity, color, and transversality, so that the state ψ has the same trivial quantum numbers as the vacuum $|0\rangle$.

We mention that, in principle, one could simplify the expression for the YM Hamiltonian by working in the temporal gauge.[17] However, in this case, canonical quantization yields a Hilbert space with spurious degrees of freedom which have to be projected out by an auxiliary condition (Gauss's law). We have found that the fulfillment of this condition leads to at least the same complications as those connected with the Coulomb gauge (see also Ref. 20).

Another alternative would be the use of the axial gauge.[12] Though the Hamiltonian is formally much simpler in this case, the correct treatment of the Coulomb singularity is problematic.[21] Also the construction of glueball states with definite quantum numbers is more involved since the gauge-fixing condition is not rotation invariant. [The construction of "rotated states" in this gauge is analogous to that of boosted states in the Coulomb gauge, (see Ref. 19), for the QED case.] In order to check the independence of the results from the choice of the gauge-fixing condition, we think, however, that one should investigate more details of the structure of the YM theory in the axial gauge in the future.

V. MOMENTUM-SPACE EXPANSION OF THE HAMILTONIAN

In order to deal with the complicated structure of the Coulomb term and the Faddeev-Popov determinant it is convenient to write the YM Hamiltonian in (our discrete) momentum space. Therefore, we first introduce Fourier expansions for the field operators:

$$\begin{aligned}&A^a_j(x)=\sum_k A^{ak}_j e^{ikx}\ ,\quad A^{ak}_j=\sum_r(\lambda_k/2\Omega)^{1/2}e_j(k,r)(b_{kra}+s_r b^\dagger_{-kra})\ ,\\ &\Pi^a_j(x)=\sum_k \Pi^{ak}_j e^{ikx}\ ,\quad \Pi^{ak}_j=\sum_r -i(2\Omega\lambda_k)^{-1/2}e_j(k,r)(b_{kra}-s_r b^\dagger_{-kra})\ .\end{aligned} \tag{25}$$

The quantized covariant Laplacian $\bar\Delta$ then becomes an operator-valued color-momentum matrix $\bar\Delta^\mu_\nu$ where the index $\mu=(a,k)$ stands as an abbreviation for the color-momentum quantum numbers. The matrix $\bar\Delta^\mu_\nu$ is fixed by the condition

$$\bar\Delta\phi^a(x)=\bar\Delta^{ak}_{bq}\phi^{bq}e^{ikx}$$

if $\phi^a=\phi^{ak}e^{ikx}$ and has the form (recall the Einstein convention for doubly appearing indices)

$$\begin{aligned}&-\omega_\mu\omega_\nu\Delta^\mu_\nu=\delta^\mu_\nu-R^\mu_\nu\ ,\quad \omega_\mu=\omega_{ak}=\omega_k=|k|\ ,\\ &R^{ak}_{bq}=igf^{acb}A^{c(k-q)}_j k_j/(\omega_k\omega_q)\quad (k,q\neq 0)\ .\end{aligned} \tag{26}$$

The color charge density is expanded analogously,

$$\rho^a(x)=\rho^{ak}e^{ikx}\ ,\quad \rho^{ak}=gf^{abc}A^{bq}_j\Pi^{c(k-q)}_j\ . \tag{27}$$

The operators in Eq. (20) involving the Faddeev-Popov determinant obtain then the form

$$\begin{aligned}&H_\pi=\frac{\Omega}{2}(e^{-d}\Pi^{ak}_j e^{2d}\Pi^{a-k}_j e^{-d})\ ,\\ &H_c=\frac{\Omega}{2}e^{-d}\frac{\rho^{-\mu}}{\omega_\mu}e^d D^\mu_\sigma D^\sigma_\nu e^d\frac{\rho^\nu}{\omega_\nu}e^{-d}\ ,\quad -\mu=(a,-k)\ .\end{aligned} \tag{28}$$

Here, the operator d is given by

$$-d=\tfrac{1}{2}\mathrm{tr}\ln(1-R)=\tfrac{1}{2}R^\mu_\nu R^\nu_\mu+\tfrac{1}{3}R^\mu_\nu R^\nu_\sigma R^\sigma_\mu+\cdots \tag{29}$$

and expands the Faddeev-Popov determinant (we have used $[\det(1-R)]^{1/2}=\exp[\frac{1}{2}\mathrm{tr}\ln(1-R)]$ and the fact that the c-number terms $\omega_\mu\omega_\nu$ in front of $\bar\Delta^\mu_\nu$ can be disregarded). The operator matrix D^μ_ν is the inverse of $-\omega_\mu\omega_\nu\bar\Delta^\mu_\nu$ in terms of a geometrical series

$$D^\mu_\nu=\delta^\mu_\nu+R^\mu_\nu+R^\mu_\sigma R^\sigma_\nu+R^\mu_\sigma R^\sigma_\tau R^\tau_\nu+\cdots\ . \tag{30}$$

As pointed out by Christ and Lee,[17] the effects of the determinant F can be accounted for by additional Faddeev-Popov potentials depending only upon the operator A^{ak}_j [denoted by $V_1(A)$ and $V_2(A)$ in Ref. 17]. Within our momentum-space expansion, these potentials are obtained by observing that the commutator expansion of $e^{-d}\Pi^\mu_j e^d$ has only two terms,

$$e^{-d}\Pi^\mu_j e^d=\Pi^\mu_j+[\Pi^\mu_j,d] \tag{31}$$

and that $[\Pi^\mu_j,d]$ may be expressed in terms of D^μ_ν,

$$[\Pi^\mu_j,d]=[\Pi^\mu_j,R^\nu_\sigma]D^\sigma_\nu\ . \tag{32}$$

Inserting this result into Eq. (28) yields

$$
\begin{aligned}
&H_\pi=\frac{\Omega}{2}\Pi_j^{-\mu}\Pi_j^{\mu}+V_1\ ,\quad V_1=V_1^1+V_1^2\ ,\\
&V_1^1=\frac{\Omega}{2}[\Pi_j^{-\mu},R_\sigma^\nu][D_\nu^\sigma,\Pi_j^\mu]\ ,\\
&V_1^2=\frac{\Omega}{2}[\Pi_j^{-\mu},R_\sigma^\nu]D_\nu^\sigma D_\tau^\rho[R_\rho^\tau,\Pi_j^\mu]\ ,\\
&H_c=V_c+V_2\ ,\quad V_c=\Pi_j^{-\mu}B_{ji}^{\mu\nu}\Pi_i^\nu\ ,\\
&B_{ji}^{ak,a'k'}=\frac{g^2\Omega}{2qq'}f^{abc}f^{a'b'c'}A_j^{b(k-q)}D_{dp}^{cq}D_{c'q'}^{dp}A_i^{b'(q'-k')}\ ,\\
&V_2=[\Pi_j^{-\mu},R_\sigma^\rho]([D_\rho^\sigma B_{ji}^{\mu\nu},\Pi_i^\nu]+D_\rho^\sigma D_\gamma^\tau[R_\tau^\gamma,\Pi_i^\nu]B_{ji}^{\mu\nu})\ .
\end{aligned}
\tag{33}
$$

The potentials V_1 and V_2 depend only on A_j^μ since the commutator of Π_j^μ with A_j^μ or R_ν^μ is a c number,

$$
[\Pi_j^{a(-k)},R_{b',q'}^{bq}]=f^{bb'a}\frac{g}{\Omega qq'}h_{ji}(k)q_i\delta_{k,q-q'}\ . \tag{34}
$$

With the help of the relation

$$
[D_\rho^\sigma,\Pi_j^\mu]=D_\nu^\sigma[R_\tau^\nu,\Pi_j^\mu]D_\rho^\tau\ , \tag{35}
$$

all terms of the Faddeev-Popov potentials can be expressed as functions of the operators A_j^μ and D_ν^μ. We quote the result only for the potential V_1^1, since the other terms can be shown not to contribute to the considered energy expectation values within our cluster approximation,

$$
\begin{aligned}
V_1^1=&-\frac{g^2f^{abc}f^{a'b'c}}{2kq\,|\,k-p\,|\,|\,q-p\,|\,\Omega}h_{ij}(k-q)k_i(q_j-p_j)\\
&\times D_a^{bq}(q-p)D_{ak}^{b'(k-p)}\ .
\end{aligned}
\tag{33a}
$$

The complete YM Hamiltonian in the Coulomb gauge is then given by

$$
\begin{aligned}
&H=H_0+V_1+V_2+V_3+V_4+V_c\ ,\\
&H_0=\frac{\Omega}{2}(k^2A_j^{a-k}A_j^k+\Pi_j^{a-k}\Pi_j^{ak})\ ,\\
&V_4=\frac{\Omega}{4}g^2f^{abc}f^{ab'c'}A_i^{b(k-q)}\\
&\quad\times A_j^{cq}A_i^{b'(p-k)}A_j^{c'(-p)}\ ,\\
&V_c=\frac{\Omega}{2}g^2\rho^{-\mu}D_\nu^\mu D_\sigma^\nu\rho^\sigma/(\omega_\mu\omega_\sigma)\ .
\end{aligned}
\tag{36}
$$

The Faddeev-Popov terms V_1 and V_2 are given in Eq. (33), which also contains an alternative form of the Coulomb interaction V_c. We did not specify the three-point interaction V_3 emerging from the expansion of the "magnetic energy" H_B [Eq. (20)], since it will never contribute to the matrix elements we want to consider.

VI. THE GROUND-STATE PROBLEM AND THE CLUSTER EXPANSION

Let us consider the vacuum expectation value

$$
E_0(\lambda_k)=\langle\psi\,|\,H\,|\,\psi\rangle/\langle\psi\,|\,\psi\rangle\ . \tag{37}
$$

The crucial observation for approximating $E_0(\lambda_k)$ is that the operator R_ν^μ is a linear expression in terms of the boson operators b_{kra} and $b_{kra}^\dagger$ (or the operators A_j^{ak}) so that for every term in the expansion of the Hamiltonian (36) [using Eq. (30)], the *Wick rule* is applicable. Hence the expectation value (37) is a functional of the elementary contractions, a convenient basis of which is obviously given by

$$
\begin{aligned}
&\underbrace{A_i^\mu A_j^{-\nu}}\equiv\langle\psi\,|\,A_i^\mu A_j^{-\nu}\,|\,\psi\rangle/\langle\psi\,|\,\psi\rangle\\
&\qquad=\delta^{\mu\nu}h_{ij}(k)\lambda_k/2\Omega\ ,\quad \mu=(a,k)\ ,\\
&\underbrace{\Pi_i^\mu\Pi_j^{-\nu}}=\delta^{\mu\nu}h_{ij}(k)/2\Omega\lambda_k\ ,\\
&\underbrace{\Pi_j^\mu A_i^{-\nu}}=-\underbrace{A_i^\nu\Pi_j^{-\mu}}=-i\delta^{\mu\nu}h_{ij}(k)/2\Omega\ .
\end{aligned}
\tag{38}
$$

To sum up all terms occurring due to the Wick rule is, of course, impossible. We propose a cluster expansion in dealing with an arbitrary term of the expansion of $(\bar{\Delta})^{-1}$ which is constructed as follows.

We first define for any given function η_k the "diagonal" contraction

$$
\eta_q\underbrace{R_{bq}^{ak}R_{a'k'}^{bq}}=Q_k(\eta)\delta_{a'k'}^{ak}\ . \tag{39}
$$

As easily derived from Eqs. (25) and (26), the function $Q_k(\eta)$ is related to η by a three-dimensional convolution integral (written still in our discrete version)

$$
\begin{aligned}
&Q_k(\eta)=\sum_q\frac{g^2}{2\Omega}\lambda_q h(q,k)\eta_{q-k}/\,|\,q-k\,|^2\ ,\\
&h(q,k)=h_{ij}(q)k_ik_j/\,|\,k\,|^2\\
&\qquad=1-(q\cdot k)^2/(\,|\,q\,|^2\,|\,k\,|^2)\ .
\end{aligned}
\tag{40}
$$

Considering now fourth-order terms in the operator R_ν^μ we have three contractions,

$$
\left.\begin{aligned}
&R_\mu^{ak}R_\nu^\mu R_\sigma^\nu R_{a'k'}^\sigma=\delta_{a'k'}^{ak}[Q_k(1)]^2\\
&R\ R\ R\ R=\delta_{a'k'}^{ak}Q_k[Q(1)]
\end{aligned}\right\}\ \text{noncrossing contraction}\ , \tag{41}
$$

$R\ R\ R\ R$ = new irreducible six-dimensional integral crossing contraction .

The point is that noncrossing contractions can be computed solely with the help of the integral kernel Q reducing the determination of these terms to three-dimensional integrals and iterations of three-dimensional integrals, whereas the crossing contraction involves an irreducible six-dimensional integral.

Our cluster expansion is now defined by the prescription to sum up all noncrossing contractions. This approximation is analogous to the lowest-order hypernetted chain approximated in Yastrow theory,[9] where it is known to be rather successful. Since the inclusion of contractions with the color charge density ρ^a yields additional in-

tegrations, the terms left out in the final results are at least nine-dimensional, irreducible integrals. Of course, the importance of the neglected contributions has to be checked at a later stage.

Details of the derivation of the expression for $E_0(\lambda)$ within this cluster expansion are given in the Appendix. We quote here the result which states that the energy functional $E_0(\lambda)$ can be constructed by the following steps.

(i) Construct first the "ghost propagator," defined by

$$s_k/k^2=\frac{1}{k^2}\langle\psi\,|\,D^{ak}_{ak}\,|\,\psi\rangle/\langle\psi\,|\,\psi\rangle$$
$$=\langle\psi\,|\,(1/\bar{\Delta})^{ak}_{ak}\,|\,\psi\rangle/\langle\psi\,|\,\psi\rangle \tag{42}$$

within our cluster approximation. This yields for the function s_k the (nonlinear) integral equation

$$s_k[1-Q_k(s)]=1\ . \tag{43}$$

(ii) In a second step, compute the "effective Coulomb potential" σ_k/k^2 defined by

$$\sigma_k/k^2=\frac{1}{k^2}\langle\psi\,|\,D^{ak}_{bq}D^{ba}_{ak}\,|\,\psi\rangle/\langle\psi\,|\,\psi\rangle$$
$$=\langle\psi\,|\,\{(1/\bar{\Delta})\Delta(1/\bar{\Delta})\}^{ak}_{ak}\,|\,\psi\rangle\langle\psi\,|\,\psi\rangle\ . \tag{44}$$

Within the cluster expansion the function σ_k is found by solving the integral equation

$$\sigma_k=s_k{}^2[1+Q_k(\sigma)]\ . \tag{45}$$

(iii) Applying the cluster approximation to the expectation value (37), one then shows that $E_0(\lambda)$ is formally simply given by evaluating expectation values of two- and four-point interactions obtained from the Hamiltonian (36) by replacing $D^{ak}_{bq}\to\delta^a_b\delta^k_q s_k$ in V^1_1 [Eq. (33a)], $D^{ak}_{\mu}D^{\mu}_{bq}\to\sigma_k\delta^a_b\delta^k_q$ in V_c, and leaving out V^2_1, V_2, and V_3. This yields

$$E_0(\lambda)=(n^2-1)\Omega(e_0+e_1+e_4+e_c)\ ,$$
$$e_0=[1/(2\Omega)]\sum_k(k^2\lambda_k+1/\lambda_k)\ ,\quad e_1=[g^2/(2\Omega^2)]\sum_{kq}h(k-q,k)s_ks_q/q^2\ ,$$
$$e_4=[g^2/(6\Omega^2)]\left[\sum_k\lambda_k\right]^2\ ,\quad e_c=[g^2/(8\Omega^2)]\sum_{kq}(\lambda_k-\lambda_q)^2\bar{h}(k,q)\sigma_{k-q}/(\lambda_k\lambda_q\,|\,k-q\,|^2)\ , \tag{46}$$
$$\bar{h}(k,q)=1+(k\cdot q)^2/(k^2q^2)\ .$$

For the explicit evaluation of the sums over the momenta in Eqs. (40) and (46), it is always understood that the large volume limit is taken yielding the replacement

$$\frac{1}{\Omega}\sum_k\to\frac{1}{(2\pi)^3}\int_{|k|<M}d^3k\ .$$

In this limit, the zero-momentum modes represent only a set of measure zero (corresponding amplitudes go like $\Omega^{-1/3}$, see Ref. 6) so that the special structure of the projector $h_{ij}(k)$ [Eq. (22)] for $k=0$ can be disregarded. (Our approach is different from Refs. 6 and 7 in this respect where a finite, small volume is considered.)

We remark that the energy density of the vacuum E_0/Ω behaves smoothly in the limit $\Omega\to\infty$ and becomes independent of Ω, quite analogous to the corresponding structure of the nuclear matter.

VII. THE GRIBOV AMBIGUITY AND THE STRUCTURE OF THE GROUND-STATE FUNCTIONAL $E_0(\lambda)$

In this section we want to discuss the implications of the special structure of the Coulomb gauge which has been revealed by Gribov.[11] He was the first to point out that the set of YM potentials fixed by the transversality condition (15) are not independent (in the sense of the equivalence defined by gauge transformations). Hence already a subset of Coulomb-gauge YM potentials, which can be characterized as being bounded by a well-defined horizon (the set of smallest YM potentials yielding zero for the determinant F), can be shown to generate all possible YM potentials by gauge transformations. Expanding the (classical) YM potential $A^a_j(x)$ in momentum space [including the phase-space cutoff (Ω,M)],

$$A^a_j(x)=\sum_{kr}\alpha_{kar}e_j(k,r)e^{ikx}(\Omega)^{1/2} \tag{47}$$

(α_{kar}=complex numbers) Gribov shows that A^a_j lies "within" the horizon if the following condition holds (note that Ref. 11 uses a different normalization of the structure constants)

$$\{g^2/[6\Omega(n^2-1)]\}\sum_{kra}|\,\alpha_{kar}\,|^2/k^2<1\ . \tag{48}$$

After quantization [substituting

$$\alpha_{kar}\to(\lambda_k/2)^{1/2}(b_{kar}+s_rb^{\dagger}_{-kar})\ ,$$

see Eq. (25)] Gribov replaces the numbers $|\,\alpha_{kar}\,|^2$ by the corresponding ground-state expectation values yielding in the Bogoliubov approximation

$$|\,\alpha_{kar}\,|^2=\lambda_k\ . \tag{49}$$

The condition (48) is then equivalent to [compare Eq. (40)]

$$Q_{k\to0}(\eta=1)<1\ . \tag{48a}$$

Gribov derives from these structures a "confinement scenario" by assuming that the function λ_k makes the quantity $Q_0(1)$ fulfill the equality instead of the inequali-

ty in Eq. (48a),

$$Q_0(1)=1 \ . \tag{50}$$

The ghost propagator s_k/k^2, approximated by the first iteration of Eq. (43) $\{s_k=[1-Q_k(1)]^{-1}\}$, develops then a $1/k^4$ singularity for $k=0$, a signal of confinement. Using an assumption corresponding to Eq. (50) (rephrased in the hyperspherical formalism), Cutkoski[5] also obtains such a singular behavior for the "longitudinal" propagator σ_k/k^2.

Inspection of the structure of our results does not support the assumption Eq. (50) entering into this confinement scenario. The crucial point is that Eq. (50) is only valid if the vacuum state is *dominated* by gluon-field configurations very near to the horizon. The distribution of these field configurations is in our case described by the function λ_k. Within our concept, however, this function has to be determined *variationally* by minimizing the ground-state energy $E_0(\lambda)$ given by Eq. (46). Now all contributions to $E_0(\lambda)$ are *positive,* this holding especially for the Coulomb and the Faddeev-Popov terms e_c and e_1 which are neglected by Gribov. (Note that the functions s_k and σ_k are always positive.)

Assuming λ_k such that Eq. (50) holds or that s_k or σ_k become singular for $k=0$ would yield for the energy density

$$e_0(\lambda)=E_0(\lambda)/\Omega=+\infty$$

(because e_c and e_1 diverge), whereas a choice of λ_k giving nonsingular solutions of Eqs. (43) and (45) gave finite values for $e_0(\lambda)$. (All these considerations are valid for a finite cutoff M and for $\Omega\to\infty$.) Thus the variational principle and the repulsive nature of the Coulomb and Faddeev-Popov interactions at the Gribov horizon suggest that the vacuum expectation values s_k and σ_k will not become singular. In fact, it is expected that the minimum of $e_0(\lambda)$ will yield a solution λ_k where the condition (48a) is clearly fulfilled, making the quantization procedure (which disregards the Gribov ambiguity) consistent. (In a recent paper, Cutkosky[22] came to a quite similar conclusion.) A numerical confirmation of these conjectures is, of course, needed and will be sought in the near future.

Clearly, a variation of the function λ_k, when according to Gribov parametrized by $\lambda_k=(k^2+\kappa^4/k^2)^{-1/2}$, has to start from large values of κ; the "free vacuum" given by $\kappa=0$ would violate the condition (48a) and hence cannot be considered as a physically significant state of the theory.[23]

We mention that the situation with the Gribov ambiguity can be compared to treatment of the (radial) problem of the nuclear vibrations in a molecule, when the (adiabatic) potential $V(r)$ is approximated by $\text{const}\times(r-r_0)^2$ and the eigenfunctions are assumed to be oscillator functions

$$f_n(r)=h_n(r-r_0)\exp[-\omega(r-r_0)^2] \ .$$

These eigenfunctions do not vanish at $r=0$ and "leak" therefore into the unphysical range of negative r values. But everybody agrees that this is a minor deficiency if r_0 is large enough and n small, so that $f_n(0)$ is very small. In the same sense there might be a small leaking of the states across the horizon in our case. It is planned to investigate this structure in more detail when doing numerical calculations.

VIII. THE MASSES OF GLUEBALLS

Observable properties derived from a field theory are related to excited states. Within our YM theory the simplest physical states are color-zero two-particle excitations (in the sense of the Bogoliubov theory they could be called also two-quasiparticle excitations). In order to elucidate the structure of the energy functional of such glueballs, it is convenient to consider first (unphysical) single-particle (sp) excitations,

$$|\, kar\,\rangle = b^\dagger_{kar}\,|\,\psi\,\rangle \ . \tag{51}$$

The energy expectation value for such gluonlike states may be written

$$\langle\, kar\,|\,H\,|\,kar\,\rangle/\langle\, kar\,|\,kar\,\rangle = E_0+\epsilon(k) \ , \tag{52}$$

where E_0, the vacuum energy (46), is proportional to the volume and $\epsilon(k)$, the sp energy, is independent of the volume. Within our cluster expansion, $\epsilon(k)$ gets only contributions from the operators H_0, V_4, and V_c, yielding

$$\epsilon(k)=\tfrac{1}{2}(k^2\lambda_k+1/\lambda_k)+[g^2/\Omega]\left[\tfrac{1}{3}\lambda_k\sum_q\lambda_q+\tfrac{1}{8}\sum_q\bar{h}(k,q)(\lambda_k+\lambda_q)^2\sigma_{k-q}/(\lambda_q\lambda_k\,|\,k-q\,|^2)\right] \ . \tag{53}$$

Note that, like E_0 also this sp energy is a sum of only positive terms. The physical glueball states we want to consider we choose to have total momentum zero, so that the energy expectation value can be interpreted as the glueball mass. Since the total color has to be zero these states have the structure

$$|\,G\,\rangle=[(n^2-1)\Omega]^{-1/2}S_{ij}e_i(k,r)e_j(k,r')\times\gamma(k)b^\dagger_{kar}b^\dagger_{-kar'}\,|\,\psi\,\rangle \ . \tag{54}$$

A suitable choice of the coefficients S_{ij} yields the correct coupling of the gluon spins to a total spin S:

$$\begin{aligned} &S_{ij}=\delta_{ij} \ , \quad \text{for } S=0 \ , \\ &S^1_{ij}=\epsilon_{1ij} \ , \quad \text{for } S=1 \ , \\ &S^{11'}_{ij}=\tfrac{1}{2}\delta_{1j}\delta_{j1'}+\tfrac{1}{2}\delta_{1j'}\delta_{j1} \\ &\qquad -\tfrac{1}{3}\delta_{ij}\delta_{11'} \ , \quad \text{for } S=2 \ . \end{aligned} \tag{55}$$

Assuming an orbital angular momentum L, the function $\gamma(k)$ has the structure

$$\gamma(k)=R(\,|\,k\,|\,)Y^L_M(k/|\,k\,|\,) \ . \tag{56}$$

Due to the Bose nature of the gluons we have the selection rule

$$S+L=\text{even} . \tag{57}$$

Coupling S and L to some angular momentum J, we obtain the following quantum numbers J^P ($P=$parity) for glueballs with small L:

$$\begin{aligned} &L=0:\ J^P=0^+,2^+ , \\ &L=1:\ 0^-,1^-,2^- , \\ &L=2:\ 0^+,1^+,2^+,2^+,3^+,4^+ . \end{aligned} \tag{58}$$

The function $R(k)$ has to be determined from the Ritz principle by minimizing the expectation value $\langle G \mid H \mid G\rangle/\langle G \mid G\rangle$, the function λ_k being kept fixed from the vacuum problem (note that the state G is always orthogonal to the vacuum state ψ). The expression for the glueball mass will be given here only for $S=0$, the important sign structure, however, which is discussed below, is valid for any SLJ. Assuming $S_{ij}=\delta_{ij}$, we obtain for the norm of the state $\mid G\rangle$

$$\langle G \mid G\rangle=\frac{4}{\Omega}\sum_k \mid \gamma(k) \mid^2 . \tag{59}$$

The energy matrix element becomes within our cluster approximation

$$\begin{aligned} &\langle G \mid H \mid G\rangle=E_0\langle G \mid G\rangle+\tilde{\epsilon}_{\rm sp}+\tilde{\epsilon}_{2p} , \\ &\tilde{\epsilon}_{\rm sp}=\frac{8}{\Omega}\sum_k \mid \gamma(k) \mid^2\epsilon(k) , \\ &\tilde{\epsilon}_{2p}=\epsilon_{2p}{}^4+\epsilon^c_{2p} , \\ &\tilde{\epsilon}_{2p}{}^4=[g^2/(2\Omega^2)]\sum_{kp}\gamma^*(k)\gamma(p)\lambda_k\lambda_p[4-\bar{h}(k,p)] , \\ &\tilde{\epsilon}^c_{2p}=-[g^2/(2\Omega^2)]\sum_{kp}\gamma^*(k)\gamma(p)\bar{h}(k,p) \\ &\qquad\times(\lambda_p+\lambda_k)^2\sigma_{k-p}/(\lambda_k\lambda_p \mid k-p \mid^2) , \end{aligned} \tag{60}$$

[the function $\bar{h}(k,p)$ is defined in Eq. (46)]. Note that this glueball energy has a structure well known from standard many-body theory which is characterized by the *contraction type*. We indicate this by a compact (but obvious) notation:

$$\begin{aligned} &G=b^\dagger b^\dagger\rangle , \\ &\langle G \mid G\rangle E_0=\langle b\,b\,H\,b^\dagger b^\dagger\rangle , \\ &\tilde{\epsilon}_{\rm sp}=\langle \underbrace{b\,b}\,\underbrace{H\,b^\dagger}\,b^\dagger\rangle , \\ &\tilde{\epsilon}_{2p}=\langle \underbrace{b\,b}\,\underbrace{H\,b^\dagger}\,b^\dagger\rangle . \end{aligned} \tag{61}$$

Thus $\tilde{\epsilon}_{\rm sp}$ and $\tilde{\epsilon}_{2p}$ have the interpretation of being the contributions to the glueball mass from the single-particle energies and from the "residual" two-particle interaction, respectively, $\tilde{\epsilon}_{2p}$ being again split up into contributions from the four-point and from the Coulomb interaction (assume $\langle G \mid G\rangle=1$ here and for the following considerations). Taking only the Coulomb contributions $\tilde{\epsilon}^c_{\rm sp}$ and $\tilde{\epsilon}^c_{2p}$ to $\tilde{\epsilon}_{\rm sp}$ and $\tilde{\epsilon}_{2p}$, where $\tilde{\epsilon}^c_{\rm sp}$ is defined by

$$\tilde{\epsilon}^c_{\rm sp}=[g^2/(\Omega^2)]\sum_{kp}\gamma(k) \mid^2\bar{h}(k,p)(\lambda_k+\lambda_0)^2 \times\sigma_{k-p}/(\lambda_p\lambda_k \mid k-p \mid^2) \tag{62}$$

—these contributions are supposed to be the most important ingredients of the glueball mass—we observe an important difference in the *sign structure:* The Coulomb interaction is *attractive* in the two-particle, *repulsive* in the sp matrix element. Hence the glueball mass appears as the difference of two contributions where, however, the individual terms are expected to become large for large values of the momentum cutoff M. This is because $\epsilon(k)$ [$=\epsilon(k,g(M),M)$ in the sense of Sec. II; see also Sec. IX] represents a "self-energy" which cannot be fixed in the cutoff limit [the gluon state is unphysical and hence $\epsilon(k)$ is not observable]. Thus it is plausible that $\epsilon(k)$ and $\tilde{\epsilon}^c_{\rm sp}$ will become large with increasing M, the same then being true for the (negative) $2p$ contribution $\tilde{\epsilon}^c_{2p}$ in order that the glueball mass itself is fixed. If one now constructs a set of wave-packet glueball states parametrized through the distance R of the two gluons,

$$\mid G(R)\rangle=\sum[\cos(k-p)\cdot R]f(k)f(p) \times e_j(k,r)e_j(p,s)b^\dagger_{kar}b^\dagger_{pas} \mid \psi\rangle \tag{63}$$

[a simple ansatz for $f(k)$ would be $f(k)\sim\exp(-\alpha k^2)$] and defines the "adiabatic potential" between two gluons,

$$V(R)=\langle G(R) \mid H \mid G(R)\rangle/\langle G(R) \mid G(R)\rangle-E_0 \tag{64}$$

[we omit here the rather lengthy explicit expression for $V(R)$], it is expected that this quantity rises with increasing R (for large finite M to some large, finite value) since the two-particle matrix element $\tilde{\epsilon}^c_{2p}$ will go to zero for large R (here it is important that the function σ_k defining the effective Coulomb potential is nonsingular, see Sec. VII) whereas the sp terms will stay constant (and large).

In this sense, we obtain here a "signal" for confinement; a detailed numerical investigation of this structure, especially of the cutoff dependence, is, of course, necessary to confirm this suggestion.

IX. CONSEQUENCES OF SCALE INVARIANCE

The investigation of the cutoff variation of observables is greatly simplified by the formal scale invariance of the original theory (taken without cutoff). In order to elucidate the emerging structure for the matrix elements under consideration, let us introduce the dependencies of the momentum cutoff M and the (unrenormalized) coupling constant g explicitly into our notations. (The limit $\Omega\to\infty$ is supposed always to be taken.) We then have

$$\begin{aligned} &e(\lambda,g,M)=E_0(\lambda)/\Omega , \text{ from Eq. (44)} , \\ &\epsilon(k,\lambda,g,M)=\epsilon(k) , \text{ from Eq. (51)} , \\ &\tilde{\epsilon}(\lambda,R,g,M)=\tilde{\epsilon}_{\rm sp}+\tilde{\epsilon}_{2p} , \text{ from Eq. (60)} \\ &\qquad\qquad\qquad\qquad\text{(quantum number } LSJ \\ &\qquad\qquad\qquad\qquad\text{suppressed)} . \end{aligned} \tag{65}$$

We denote by $\lambda_m(k,g,M)$ the minimum of $e(\lambda,g,M)$ and

by $R_m(k,g,M)$ that of $\tilde{\epsilon}(\lambda_m(g,M),R,g,M)$. Inspection into the structure of these functions then shows that we have

$$e(\bar{\lambda},g,\rho M)=e(\lambda,g,M)\rho^4 \text{ if } \bar{\lambda}(k)=\rho^{-1}\lambda(\rho^{-1}k) . \tag{66}$$

Here it is important to notice that the solutions $s_k=s(k,\lambda,M)$ and $\sigma_k=\sigma(k,\lambda,M)$ of the integral equations (42) and (43) have the scaling property

$$\begin{aligned} s(k,\bar{\lambda},\rho M)&=s(\rho^{-1}k,\lambda,M) ,\\ \sigma(k,\bar{\lambda},\rho M)&=\sigma(\rho^{-1}k,\lambda,M) . \end{aligned} \tag{67}$$

Consequently, the minima of $e(\lambda,g,M)$ are related by

$$\lambda_m(k,g,\rho M)=\rho^{-1}\lambda_m(\rho^{-1}k,g,M) \tag{68}$$

and the minimum values $e_m(g,M)=e(\lambda_m,g,M)$ scale like

$$e_m(g,\rho M)=e_m(g,M)\rho^4 . \tag{69}$$

Completely analogous relations hold for excitation energies:

$$\begin{aligned} &\epsilon(k,\bar{\lambda},g,\rho M)=\epsilon(\rho^{-1}k,\lambda,g,M)\rho ,\\ &\tilde{\epsilon}(\bar{\lambda},\bar{R},g,\rho M)=\tilde{\epsilon}(\lambda,R,g,M)\rho , \qquad \text{if } \bar{R}(k)=\rho^{-3/2}R(\rho^{-1}k) . \end{aligned} \tag{70}$$

Also, if $R(k)$ is a minimum of $\tilde{\epsilon}(\lambda,R,G,M)$, then $\bar{R}(k)$ is a minimum of $\tilde{\epsilon}(\bar{\lambda},\bar{R},g,\rho M)$ and the minimum values scale like

$$\tilde{\epsilon}_m(g,\rho M)=\tilde{\epsilon}_m(g,M)\rho \text{ if } \tilde{\epsilon}_m(g,M)=\tilde{\epsilon}(\lambda_m,R_m,g,M) . \tag{71}$$

Let $\tilde{\epsilon}_{\text{exp}}$ now be the lowest experimental glueball mass and choose *LSJ* determining $\tilde{\epsilon}_m(g,M)$ correspondingly. The cutoff dependence of the coupling constant $g(M)$ is then determined by fitting this observable (see Sec. II), i.e., by demanding $\tilde{\epsilon}_m(g(M),M)=\tilde{\epsilon}_{\text{exp}}$, which according to Eq. (71) now simplifies to

$$\tilde{\epsilon}_m(g(M),M_0)=M_0/M\tilde{\epsilon}_{\text{exp}} . \tag{72}$$

It is therefore sufficient to compute $\tilde{\epsilon}_m(g,M_0)$ for some *fixed* cutoff M_0 as a function of the coupling constant g and to use (72) to determine $g(M)$.

X. DESCRIPTION OF BAG FORMATION WITHIN THE BOGOLIUBOV SCHEME

Due to the attractive nature of the residual two-particle Coulomb interaction in Eq. (60) it might be energetically more favorable for the system to form a "bag" in the vacuum when considering a glueball.[24] It is conceivable that in this way essentially more "binding" for the glueball is obtained. Within our Bogoliubov theory, the formation of such a bag can be easily formulated. Whether it is an important structure or not could then be decided from the variational principle.

Clearly, the idea behind a bag formation is that of a vacuum polarization, i.e., if the glueball state (54) had the structure

$$|G\rangle=B|\psi\rangle , \quad B=\sum B_{rs}(k,p)b^\dagger_{kar}b^\dagger_{pas} , \tag{54a}$$

we now have to allow for the replacement of $|\psi\rangle$ by a new state $|\bar{\psi}\rangle$, including "core polarization" effects. This leads to the glueball ansatz

$$|\bar{G}\rangle=B|\bar{\psi}\rangle , \quad |\bar{\psi}\rangle=e^S|\psi\rangle . \tag{73}$$

If S is a two-particle operator

$$S=\sum S_{rs}(k,p)b^\dagger_{kar}b^\dagger_{pas} , \tag{74}$$

$\bar{\psi}$ is still a Bose BCS state, and the same techniques as before may be applied for evaluating expectation values. The simplest ansatz for the coefficients $S_{rs}(k,p)$ would be to assume a spherical bag characterized by a function $f(k)=f(|k|)$ yielding

$$S=\frac{1}{\Omega}\sum f(k)f(p)e_j(kr)e_j(p,s)b^\dagger_{kar}b^\dagger_{pas} . \tag{74a}$$

The important new structure here is that the vacuum-polarization effects, expressed through the operator S, break momentum conservation (as in any bag model). It can be shown that it is *necessary* (and sufficient) to allow for such a violation of translational invariance in order to get a nontrivial result for the function $S_{rs}(k,p)$. A translational-invariant ansatz for S, corresponding to allowing only for a replacement $\lambda_k\to\lambda'_k$, would yield $\lambda'_k=\lambda_k$ by the Ritz principle. The reason for this result can be traced back to the different volume dependencies of the energies of the ground state and of excited states. The effects of the operator e^S are only local since the norm $|S\psi|$ has to stay finite for $\Omega\to\infty$ [$f(k)$ in Eq. (74a) has to be square integrable]. The structure of the new glueball mass can be classified in analogy to Eq. (61),

$$\langle\tilde{G}|H|\bar{G}\rangle/\langle\bar{G}|\bar{G}\rangle-E_0=\bar{\epsilon}_{\text{bag}}+\bar{\epsilon}_{\text{sp}}+\bar{\epsilon}_{2p} , \tag{75}$$

where

$$\begin{aligned} \bar{\epsilon}_{\text{bag}}&=\langle\bar{\psi}|\underline{b\,b\,H\,b^\dagger}b^\dagger|\bar{\psi}\rangle-E_0\\ &=\langle\bar{\psi}|H|\bar{\psi}\rangle/\langle\bar{\psi}|\bar{\psi}\rangle-\langle\psi|H|\psi\rangle/\langle\psi|\psi\rangle \end{aligned} \tag{76}$$

is a "Casimir" energy corresponding to the pressure term of the bag models and

$$\begin{aligned} \bar{\epsilon}_{\text{sp}}&=\langle\bar{\psi}|\underline{b\,b}\,\underline{H}\,\underline{b^\dagger}b^\dagger|\bar{\psi}\rangle ,\\ \epsilon_{2p}&=\langle\bar{\psi}|\underline{b\,b}\,\underline{H}\,\underline{b^\dagger}b^\dagger|\bar{\psi}\rangle , \end{aligned} \tag{77}$$

have the interpretation of sp energies and irreducible two-particle interaction, respectively. Defining new elementary contractions in analogy to Eq. (38), it is straightforward to extend the cluster expansion for the expectation values of Eqs. (76) and (77), including the bag formation. We omit here the rather lengthy results; we only mention that the sign structure discussed in Sec. VIII remains unchanged, the new bag term $\bar{\epsilon}_{\text{bag}}$ becomes positive.

Summarizing, we may say that the application of nonperturbative many-body techniques to a cutoff YM Hamiltonian yields promising formal results. A decisive test of the validity of our approach will be given by a numerical calculation of the glueball spectrum and by a study of

the cutoff limit. We intend to perform such computations in the future.

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APPENDIX

In this appendix we will give some more details of the structure of the cluster expansion for the expectation value (37).

Let us first evaluate the "longitudinal propagator"

$$\langle\psi|\rho^{-\mu}D^{\mu}_{\nu}D^{\nu}_{\tau}\rho^{\tau}|\psi\rangle/\langle\psi|\psi\rangle\ . \tag{A1}$$

From the structure of Eqs. (27) and (30) of the operators ρ^{μ} and D^{μ}_{ν} we may symbolize a general term of (A1) in the form [each R from Eq. (30) yields one A]

$$\underset{\rho}{\Pi}\underbrace{A\ \underset{D}{A\,A\,A\,A\,A\,A\,A\,A}\ \underset{D}{A\,A\,A\,A\,A\,A\,A\,A}\ A}_{\rho}\Pi\ . \tag{A2}$$

We first consider the case where the operators appearing in ρ are contracted with themselves, as indicated in (A2).

The possible contractions of DD are then divided into classes according to the number of "overlapping" contractions,

$$\begin{array}{ll} \underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,A}}}}} \quad \underbrace{A\,\underbrace{A\,\underbrace{A\,A}}} & \text{no overlap}\ , \\ \underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,A}}}}}} \quad \underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,A}}}}} & \text{one overlap}\ , \\ \underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,A}}}}}}} \quad \underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,A}}}}} & \text{two overlaps, ...} \end{array}$$

Inspection of the structure of the integral equation (43), formally solved by iterations, shows that the solution s_k sums up all noncrossing contractions within one D. Let us demonstrate this explicitly up to third order. We have from Eq. (43)

$$S=1+Q(s)+Q^2(s)+Q^3(s)+\cdots\ , \tag{A3}$$

where we have dropped the argument k. If s_j is the jth iteration, we obtain keeping terms up to third order

$$\begin{aligned} &s_1=1+P+P^2+P^3\ ,\quad P=Q(1)\ , \\ &S_2=1+P+Q(P)+Q(P^2)+P^2+2PQ(P)+P^3\ , \\ &s_3=1+P+P^2+Q(P)+Q(P^2)+P^3 \\ &\quad +Q[Q(P)]+2PQ(P)\ . \end{aligned} \tag{A4}$$

We see that apart from the first- and second-order terms $[P+P^2+Q(P)]$ corresponding to the contractions (39) and (41), we have generated exactly all noncrossing third-order terms,

$$\begin{aligned} &\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A}}}}}}+\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A}}}}\ \underbrace{A\,\underbrace{A\,\underbrace{A}}}+\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A}}}}}\ \underbrace{A}\ \underbrace{A} \\ &\quad+\underbrace{A\,\underbrace{A}}\ \underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A}}}}+\underbrace{A\,\underbrace{A\,\underbrace{A\,\underbrace{A}}}}\ \underbrace{A\,\underbrace{A}}\ \underbrace{A}\ , \end{aligned}$$

contributing to the ghost propagator, Eq. (42). From the structure of the Coulomb term (A1) it is then clear that all nonoverlapping contractions in (A2) are given by $s_k{}^2$.

Iterating the integral equation (45) it is seen that the first term yields the contributions of nonoverlapping, the second term those of once-overlapping, the third term those of twice-overlapping, etc., contractions arriving thus at the result, Eq. (45).

It remains to be shown that contractions where ρ and D are combined do not contribute within our approximation. Therefore we first remark that contractions of the type

$$\underbrace{\Pi A\ A\,\underbrace{A\,\underbrace{A\,A\,A}}\ \underbrace{A\,A\,\underbrace{A\,\underbrace{A\,A\,A}}}\ A\Pi}_{\rho\qquad D\qquad\qquad D\qquad\qquad\rho} \tag{A5}$$

are "crossing" and yield higher-order cluster contributions. On the other hand, a noncrossing contraction of the type

$$\underbrace{\Pi A\ \underbrace{A\,\underbrace{A\,A\,A}}\ \underbrace{A\,A\,\underbrace{A\,A\,A}}\ A\Pi}_{\rho\qquad D\qquad\qquad D\qquad\qquad\rho} \tag{A6}$$

vanishes due to the transversality of the gluons. In fact, the contractions of the form (A6) can be summed up in the form

$$\sum h_{ij}(q)F_i(q)F_j(q)\sigma_q/(q^2\lambda_q)\ , \tag{A7}$$

where

$$\begin{aligned} F_j(q)&=\frac{1}{\Omega}\sum_k s_{q-k}\lambda_k h_{ij}(k)q_i/|q-k|^2 \\ &=q_j\tilde{F}(|q|)\ . \end{aligned} \tag{A8}$$

The last equation follows from the rotational invariance of the vector field $F_j(q)$ $[R_{ij}F_j(R^{-1}q)=F_i(q)$ for arbitrary rotations $R]$. Since $h_{ij}(q)q_j=0$, Eq. (A7) is zero.

Considering now the Faddeev-Popov potentials we may proceed as before and consider first nonoverlapping contractions (with respect to the different D^{μ}_{ν} terms), yielding the same function s_k. Taking then overlapping contractions, however, one easily demonstrates that these all give higher-order clusters, the same is even true for the nonoverlapping contractions of the terms $V_1{}^2$ and V_2. Thus only the nonoverlapping contractions of the potential $V_1{}^1$ contribute to $E_0(\lambda)$ yielding the term e_1 of Eq. (46).

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[1]See, e.g., J. Kuti, Nucl. Phys. **A416**, 25 (1984); K. Ishakawa *et al.*, Z. Phys. C **19**, 327 (1983).

[2]D. E. L. Pottinger and R. C. Warner, Phys. Lett. **114B**, 349 (1982).

[3]L. J. Carson, Ph.D. thesis, University of Washington, 1982.

[4]M. Bolsterli, Phys. Rev. D **27**, 349 (1983).

[5]R.E. Cutkosky, Phys. Rev. Lett. **51**, 538 (1983).
[6]M. Lüscher, Nucl. Phys. **B219**, 233 (1983).
[7]M. Lüscher and G. Münster, Nucl. Phys. **B232**, 445 (1984).
[8]D. Schütte, Nucl. Phys. **A411**, 369 (1983).
[9]See, e.g., S. Rosati and S. Fantoni, in *The Many-Body Problem*, Lecture Notes in Physics No. 138, edited by R. Guardiolo and J. Ros (Springer, Berlin, 1981), p. 1.
[10]M. Birse, E. M. Henley, G. Luebeck, and L. Wilets, in Proceedings of the Lewes Workshop on Solitons in Nuclear and Elementary Particle Physics, University of Delaware, 1984 (unpublished).
[11]V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).
[12]S. Nojiri, Z. Phys. C **22**, 245 (1984).
[13]R. Haag, Phys. Rev. **112**, 669 (1958).
[14]P. Dirac, *Lectures on Quantum Field Theory* (Academic, New York, 1966).
[15]G. Wiesbrock, Ph.D. thesis, University of Bonn, 1983.
[16]J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974), and references therein.
[17]N. H. Christ and T. D. Lee, Phys. Rev. D **22**, 939 (1980).
[18]E. S. Abers and B. W. Lee, Phys. Rep. **9C**, 1 (1973).
[19]J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[20]J. D. Bjorken and S. D. Drell, *Progress in Physics* (Birkhäuser, Boston, 1982), Vol. 4, p. 423.
[21]A. Burnel and M. Van der Rest-Jaspers, Phys. Rev. D **28**, 3121 (1983).
[22]R. E. Cutkosky, Phys. Rev. D **30**, 447 (1984).
[23]D. Zwanziger, Nucl. Phys. **B209**, 336 (1982).
[24]T. D. Lee, *Particle Physics and Introduction to Field Theory* (Harwood-Academic, New York, 1981).