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Black hole in thermal equilibrium with a scalar field: The back-reaction

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The accurate approximation found by Page for the expectation value of the renormalized thermal equilibrium stress-energy tensor of a free conformal scalar field in a Schwarzschild black-hole background is used as the source in the semiclassical Einstein equation. The back-reaction and new equilibrium metric are found perturbatively in order $\hbar$. The new metric is not asymptotically flat unless the system is enclosed by a reflecting wall. Solutions are obtained for systems of finite radius using microcanonical (fixed energy) and canonical (fixed temperature) boundary conditions. Explicit effects of the back-reaction on the equilibrium temperature distribution inside the cavity are given. With microcanonical boundary conditions there is an asymptotically flat region where the temperature at infinity is defined. It is shown that this temperature does not have the Schwarzschild value $\hbar(8\pi M)^{-1}$ for a black hole of mass M. Curvature invariants are computed and the order-$\hbar^2$ correction to the conformal scalar-field trace anomaly originating from the back-reaction that this field produces is found. The principal qualitative features of the results should be valid for any quantum field at one loop in the Schwarzschild geometry.

I. INTRODUCTION

Ever since Hawking's discovery that a black hole in empty space radiates energy with a thermal spectrum,[1] it has been believed that a black hole can exist in (possibly unstable) thermal equilibrium with a heat bath possessing a characteristic temperature distribution. An appropriate heat bath could be composed, for example, of the quanta of a massless free field in the black-hole geometry. The gravitational effect of the heat bath is characterized by its gravitationally induced renormalized stress-energy tensor.[2] In the semiclassical approach to quantized spacetime geometry, one then asks for the relation of this tensor to the metric. The expectation value of the renormalized stress-energy tensor in an appropriate "vacuum" state is regarded as the source in the Einstein equation and one solves this equation self-consistently for the metric. This is the back-reaction problem. It is hoped that this metric gives a better approximation to the spacetime geometry associated with thermal equilibrium than one which satisfies the source-free Einstein equation.

Semiclassical back-reaction problems of the type outlined above are worthy of serious study because of the absence of a consistent quantum theory of gravity. The large amount of work[2] beginning in the middle 1970's that was devoted to finding appropriate sources for the semiclassical Einstein equation was motivated largely by Hawking's discovery of the quantum emission of energy by black holes.[1] One of the important questions, then, that one wants to answer concerns the effect of quantized matter on the geometry of black holes. In view of the far-reaching implications of the association of a temperature with black holes, a temperature whose existence is entirely quantum mechanical in origin, one of the most significant back-reaction problems concerns a black hole in thermal equilibrium with quantized matter. People have been aware of the significance of this problem for a number of years, but until now it has not been treated because of insufficient knowledge of a precise description of quantized matter in thermal equilibrium with a black hole and the difficulty of solving problems involving the nonlinear Einstein equation. However, if one regards the problem as involving sources that are the expectation values of renormalized stress-energy tensors computed at one loop on classical backgrounds and assumes that all the relevant spacetime scales are somewhat larger than the Planck scale, approximate solutions can be found without difficulty.

In this paper a back-reaction problem of the above type is solved to first order. The source is the expectation value of the renormalized stress-energy tensor of a free massless conformal scalar field appropriate to the spacetime of a classical Schwarzschild black hole in its equilibrium or Hartle-Hawking[3] state. I use Page's[4] closed-form expression for this tensor, which has been shown recently by Howard and Candelas[5] and by Howard[6] to be a highly accurate approximation to the exact result which they have computed. Such tensors are constructed by using renormalization techniques[2] on the real Euclidean section of the Schwarzschild geometry with its (Euclidean) time coordinate identified with period $\beta_0 = 8\pi M$ so as to eliminate conical singularities at $r = 2M$. Thus, in this approach, the background geometry is completely static and quantum fluctuations of the metric are ignored. One may wonder whether this procedure is justified because, it has been argued,[7] metric fluctuations are essential in enabling a black hole to undergo the two-way exchange of energy with the heat bath that is necessary for the existence of thermal equilibrium. Moreover, there must always be an effective stress-energy tensor for the gravitons that is comparable in order of magnitude and effect to that of any other field from which the heat bath is constructed.[2] However, it is reasonable to ignore the quantum fluctuations of the metric and the associated additional stress-energy source when one is computing the semiclassical metric back-reaction caused by the quantized scalar field alone to first order. A consequence of this approach is that the back-reaction Δg_{ab} is only meaningful in a perturbative sense. That is, the *effect* of the scalar field stress-energy tensor T^{ab} is regarded as a perturbation of the classical Schwarzschild geometry.

It turns out that in order to regard the effect of T^{ab} as a perturbation, there are *two* mass or length scales that are relevant. It is not difficult to see how these scales come about. One is clearly the mass of the hole M compared to the Planck mass $M_P=\hbar^{1/2}$ (units $G=c=k_B=1$). For this purpose I define $\epsilon=\hbar M^{-2}$ and assume order-ϵ^2 terms have effects negligible compared to order-ϵ terms. The other scale comes about because, although the exact T^{ab} here is of order ϵ, T_{ab} is asymptotically *constant*. Hence, the corrected metric cannot be asymptotically flat. Thus, in order that $|\Delta g_{ab}|$ be small compared to the background metric one cannot consider the system to be of unbounded extent. One must introduce some radius r_0 such that the cumulative effect of T^{ab} on Δg_{ab} is not too large. One can do this in such a way that the perturbative solution is uniformly valid over a very wide range of parameters $M>M_p$ and r_0. A wide range of validity is possible because, in effect, the Hawking temperature is very low; the relation between the two scales is therefore mediated by a large constant that enters T^{ab}. It turns out that the radiant energy can be comparable to or considerably greater than the mass of the hole in this approach.

There is more to the introduction of the radius r_0 as an effective boundary at a finite distance than is indicated above because the back-reaction problem does not have definite solutions unless boundary conditions are specified at r_0. Two physically natural choices present themselves. One is to think of the spherical cavity as closed (ideal perfectly reflecting wall) and to specify the total energy E of the system at r_0, which corresponds to the black hole in a microcanonical ensemble. The other is to specify the temperature at r_0, which corresponds to a black hole in a canonical ensemble. The advantage of the microcanonical picture is that the equilibrium can be stable,[8] whereas the canonical ensemble for black holes is believed to be generically unstable.[8] Both types of solutions will be presented here and analysis of stability will be presented elsewhere.

There are several consequences that follow from the back-reaction in finite systems. One is that the spacetime geometry inside the cavity contains one parameter in addition to the mass $M_{\rm BH}$ of the black hole. This is not surprising because the new parameter can be taken to be the radius of the cavity containing the thermal radiation. It also happens that the surface gravity is not given in general by the Schwarzschild value $\kappa=(4M_{\rm BH})^{-1}$. This has physical significance in the microcanonical ensemble, where there is an asymptotically flat region outside the cavity, because it is shown that $T_\infty=\kappa\hbar(2\pi)^{-1}$, and therefore T_∞ suffers a correction because of the back-reaction in a finite cavity.

In addition, one finds that some of the curvature invariants of dimension $(\text{length})^{-4}$ acquire changes in $O(\epsilon)$. From this I show that one can easily compute through $O(\epsilon^2)$ the trace anomaly of the free conformal scalar field, where the correction arises from the stress-energy tensor used as the source. That is, $(\Delta T^a_a)(T^b_b)^{-1}=O(\epsilon)$ follows from knowing the metric back-reaction Δg_{ab} in order ϵ. I also show that the often-neglected invariant $\Box R$ is not negligible in order ϵ.

In this work I shall neglect explicit quantum effects of the wall of the cavity as well as the fact that a spherical wall at finite radius would alter somewhat the scalar mode functions used to construct the stress-energy tensor, which has been computed at present only in infinite space. However, the reader will observe that many of the results do not depend on any features of the stress-energy tensor other than that it represents inside the chosen cavity any kind of matter in static thermal equilibrium with an uncharged spherical hole, in the one-loop approximation on the Schwarzschild background.

II. BACK-REACTION PROGRAM

Suppose there is an external (nongravitational) free field ψ with a quadratic Lagrangian on a curved background spacetime with metric $\hat{g}_{ab}$ and that ψ is in a vacuum state $\langle\psi\rangle=0$ appropriate to the background. Quantum fluctuations of ψ give rise to $\langle\psi^2\rangle\neq 0$ and one can obtain the expectation value of a renormalized symmetric stress-energy tensor $\langle T^{ab}\rangle_{\rm ren}$, henceforth denoted simply by T^{ab}. For a free field ψ, the one-loop T^{ab} is the complete result[2] and is of order $\hbar$ because the quantum fluctuations of ψ are of order $\hbar^{1/2}$. (It is well known that loop expansions can be regarded as expansions in powers of $\hbar$, as I shall regard them here.[9])

In accordance with current methods,[2] we assume that the background spacetime is Ricci flat ($\hat{R}_{ab}=\hat{G}_{ab}=0$) and that T^{ab} satisfies the background "conservation law"

$$\hat{\nabla}_b T^{ab}=0\ , \tag{2.1}$$

where $\hat{\nabla}_b$ denotes the covariant derivative with respect to $\hat{g}_{ab}$ and its Levi-Civita connection $\hat{\Gamma}^a{}_{bc}$. At one loop there is also an effective stress-energy tensor $\tau^{ab}(\phi)=O(\hbar)$, comparable to $T^{ab}(\psi)$, that represents the effect of quantum fluctuations of the metric: $g_{ab}=\hat{g}_{ab}+\phi_{ab}$, where ϕ_{ab} is the *quantum* gravitational field satisfying $\langle\phi\rangle=0$ and $\langle\phi^2\rangle\neq 0$ on the background. Presumably one has that $\hat{\nabla}_b\tau^{ab}=0$ or at least $\hat{\nabla}_b(T^{ab}+\tau^{ab})=0$. Although no such $\tau^{ab}(\phi)$ is considered in the present work, in the discussion of this section I shall include it formally in order to obtain perspective on the back-reaction program.

The back-reaction problem is to solve the semiclassical Einstein equation

$$G^{ab}(g)=8\pi[T^{ab}(\psi)+\tau^{ab}(\phi)] \tag{2.2}$$

for a *classical* metric $g_{ab}=\hat{g}_{ab}+\Delta g_{ab}$. We write

$$G^{ab}=\hat{G}^{ab}+\Delta G^{ab}=\Delta G^{ab}=\delta G^{ab}+H^{ab}\ ,$$

where δG^{ab} is the linear part of ΔG^{ab}. In view of the fact that the right-hand side of (2.2) satisfies the Bianchi identity (2.1) in the background metric, we can only solve (2.2) for Δg_{ab} in the first approximation:

$$\delta G^{ab}(\hat{g},\Delta g)=8\pi[T^{ab}(\psi)+\tau^{ab}(\phi)]\ , \tag{2.3}$$

where $\hat{\nabla}_b \delta G^{ab}=0$ follows from (2.1). If M is the Schwarzschild mass of the background, the linearization is carried out with respect to $\epsilon=\hbar M^{-2}$ and we seek a solution Δg_{ab} of the inhomogeneous equation (2.3). I do not consider solutions of the homogeneous problem $\delta G^{ab}=0$, as these have been widely studied and are not of direct interest in the problem of the back-reaction caused by $T^{ab}(\psi)$.

In the following, I ignore τ^{ab} because it is not presently known. Alternatively, one could imagine a model theory in which the number of free conformal scalar fields is sufficiently large to dominate the thermal radiation. For T^{ab}, I will use Page's[4] closed-form expression. Because the background is asymptotically flat, while T^{ab} is asymptotically constant, the question of boundary conditions is of primary physical importance.

III. BACKGROUND GEOMETRY AND STRESS TENSOR

The background spacetime is given by the Schwarzschild metric

$$ds^2=-\left[1-\frac{2M}{r}\right]dt^2+\left[1-\frac{2M}{r}\right]^{-1}dr^2+r^2d\omega^2\,,$$
$$d\omega^2=d\theta^2+\sin^2\theta\,d\phi^2\,. \tag{3.1}$$

The real Euclidean (Riemannian) section is found[3,10] from the "Wick rotation" $t\rightarrow -it$, which changes the sign of g_{tt}. The Euclidean time t is then given a period β and the topology is $R^2\times S^2$. There is no conical singularity at the "axis" $r=2M$ provided that in the R^2 (fixed θ and ϕ), the proper circumference of a circle of constant r is related to its proper radius in the usual ratio 2π, in the limit as $r\rightarrow 2M$ (elementary flatness). This requires that the regularity period β be given by

$$\beta=\lim_{r\rightarrow 2M}4\pi(g_{rr}g_{tt})^{1/2}\left[\frac{d}{dr}g_{tt}\right]^{-1}\,, \tag{3.2}$$

which is identically $2\pi/\kappa$, $\kappa=$ surface gravity of the event horizon in the Lorentzian geometry. We find the usual result $\beta_0=8\pi M$ from (3.1). The Hawking temperature[1] is $T_H=\hbar(8\pi M)^{-1}$ so $\beta_0=\hbar T_H{}^{-1}$. It should be noted that the Hawking temperature is the "temperature at infinity" of the background geometry. The temperature associated with a static observer at $r=r_0$ is well known to be given by

$$T_{\mathrm{loc}}(r_0)=T_H\,|g_{tt}(r_0)|^{-1/2}$$
$$=T_H\left[1-\frac{2M}{r_0}\right]^{-1/2}\,. \tag{3.3}$$

Because the Schwarzschild geometry is asymptotically flat in spatial directions, as exemplified in (3.3), we see that T_{loc} and T_H become identical as $r_0\rightarrow\infty$. There is, of course, an evident distinction between the period β that defines regularity at $r=2M$ and the local inverse temperature at finite radius. This distinction will become important later when we must immerse the black hole dressed by its heat bath into a spherical cavity of finite radius r_0.

The thermal stress tensor is computed on the real Euclidean section and is static. Its nonzero components in either Lorentzian or Euclidean spacetime are, in Page's approximation,[4]

$$T_t^t=-3\frac{\epsilon}{\lambda M^2}(f-h),\quad T_r^r=\frac{\epsilon}{\lambda M^2}(f+h)\,,$$
$$T_\theta^\theta=T_\phi^\phi=\frac{\epsilon}{\lambda M^2}f\,, \tag{3.4}$$

where

$$f(r)=\frac{1-(4-6M/r)^2(2M/r)^6}{(1-2M/r)^2}\,, \tag{3.5}$$

$$h(r)=24\left[\frac{2M}{r}\right]^6\,. \tag{3.6}$$

The constants are defined by

$$\frac{\epsilon}{\lambda M^2}=\tfrac{1}{3}aT_H{}^4,\quad a=\frac{\pi^2}{30\hbar^3}\,,$$
$$\lambda=90(8^4)\pi^2\,. \tag{3.7}$$

The stress tensor is well behaved at $2M$, where $f(2M)=12$, $h(2M)=24$. Note that $T_t^t(2M)=T_r^r(2M)$, a necessary condition for regularity there, as pointed out by Page.[4] The trace anomaly[11] is the exact one,

$$\mathrm{tr}T=\frac{4\epsilon}{\lambda M^2}h(r)\,. \tag{3.8}$$

At large r, T_b^a approaches a flat-spacetime thermal stress tensor. One verifies that $\hat{\nabla}_b T_a^b=0$.

Physically, this tensor may be regarded as giving the stress-energy distribution of the heat bath that equilibrates the black hole, which would otherwise radiate into empty space. Thus, it should represent the gravitational effect of an infinite standing-wave pattern of scalar modes excited to the equilibrium temperature and therefore describes a stationary interference pattern of ingoing, outgoing, and circulating scalar waves. If the hole were radiating in vacuum, its dominant outgoing frequency[12] would be approximately that of its lowest, least-damped, quasinormal scalar mode[13] with angular index $l=0$. The real part of this frequency (measured at large r) is about $\omega=0.15M^{-1}$ with wavelength about $42M$.[14] At this frequency the modes have only a slight thermal excitation. This is illustrated by computing the mean quantum number

$$\langle n\rangle=(e^{h\omega/T_H}-1)^{-1}\approx 0.024 \tag{3.9}$$

and the thermal fluctuation ΔU of this mode. One finds

$$\frac{\Delta U}{U}\approx 6.29 \tag{3.10}$$

using standard results. Of course, $\langle n\rangle$ is smaller and $\Delta U/U$ larger for the dominant frequencies at $l=1,2,\ldots,$ which are higher than 0.15 M^{-1}. In this sense, T_b^a is almost all zero-temperature vacuum energy ("zero-point energy").

It is known that T_b^a is not expected to obey the classical "dominant energy condition" $T_{ab}t^au^b\geq 0$,[15] where t^a and u^b are any two future-directed timelike vectors. This condition would imply that the local four-momentum seen by any observer must be future-directed. However, as Page[4] has pointed out, the energy density $(-T_t^t)$ perceived by a static observer inside $r \simeq 2.3437M$ is negative. One can construct all radially infalling local orthonormal frames and further explore properties of T_b^a. An observer moving across $r=2M$ at nearly the speed of light perceives a four-momentum vector $\hat{P}^a = (\epsilon\lambda^{-1}M^{-2})$ [188, 260], which is spacelike. An observer crossing $2M$ at $v \simeq 0.4$ perceives a purely spacelike four-momentum, i.e., he sees $\hat{P}^0 = 0$. On the other hand, the classical null cone structure seen by all these observers is invariant. It is possible that these properties, which are unusual from a classical viewpoint, can ultimately be reconciled with local causality of the stress-energy-momentum by taking into account metric fluctuations.[7] In any case, the tensor T_b^a behaves "normally" at large r.

The trace anomaly[11] is also a feature of T^{ab} that affects the solution of the back-reaction problem in an important way. Its physical origins were discussed heuristically by Christensen and Fulling[16] in terms of the unstable circular photon orbit at $r=3M$.

IV. SOLUTION OF BACK-REACTION EQUATION

The solution of the back-reaction equation (2.3) can be reduced to two elementary quadratures as follows.

Working in Lorentzian signature, we transform the Schwarzschild metric and Page's stress tensor into advanced-time Eddington-Finkelstein coordinates:[17]

$$v = t + r + 2M \ln\left[\frac{r}{2M} - 1\right], \qquad \tilde{r} = r . \tag{4.1}$$

This yields

$$ds^2 = -\left[1 - \frac{2M}{r}\right] dv^2 + 2dv\, d\tilde{r} + r^2 d\omega^2 , \tag{4.2}$$

$$T_v^v = T_t^t = \frac{-3\epsilon}{\lambda M^2}(f-h) , \tag{4.3}$$

$$T_{\tilde{r}}^{v} = \frac{2\epsilon}{\lambda M^2}\left[1 - \frac{2M}{r}\right]^{-1}(2f-h) , \tag{4.4}$$

$$T_{\tilde{r}}^{\tilde{r}} = T_r^r = \frac{\epsilon}{\lambda M^2}(f+h) . \tag{4.5}$$

Of course, T_θ^θ and T_ϕ^ϕ are unchanged. Note that $T_{\tilde{r}}^{v}$ has the limit $224(\epsilon\lambda^{-1}M^{-2})$ at $r=2M$.

The perturbed metric is taken to have the form[18,19] (putting $\tilde{r}=r$)

$$ds^2 = -e^{2\psi}\left[1 - \frac{2m}{r}\right] dv^2 + 2e^{\psi} dv\, dr + r^2 d\omega^2 , \tag{4.6}$$

where $\psi = \psi(r)$ and $m = m(r)$ because the corrected metric should still be static and spherically symmetric. The field equations are

$$G_v^v = -\frac{2}{r^2}\frac{\partial m}{\partial r}, \quad G_r^v = \frac{2}{r}e^{-\psi}\frac{\partial \psi}{\partial r} , \tag{4.7}$$

$$G_r^r = \frac{2}{r^2}\left[r\left[1 - \frac{2m}{r}\right]\frac{\partial \psi}{\partial r} - \frac{\partial m}{\partial r}\right] . \tag{4.8}$$

All other components are zero except $G_\theta^\theta = G_\phi^\phi$, which follow from the Bianchi identity $\nabla_a G_r^a = 0$.

Linearization is achieved by setting

$$e^{\psi} \cong 1 + \epsilon\rho(r), \quad m \cong M[1 + \epsilon\mu(r)] . \tag{4.9}$$

This yields the linear back-reaction equations

$$\frac{\epsilon M}{4\pi r^2}\frac{\partial \mu}{\partial r} = \frac{3\epsilon}{\lambda M^2}(f-h) = T_v^v , \tag{4.10}$$

$$\frac{\epsilon}{4\pi r}\frac{\partial \rho}{\partial r} = \frac{2\epsilon}{\lambda M^2}\left[1 - \frac{2M}{r}\right]^{-1}(2f-h) = T_r^v , \tag{4.11}$$

$$\frac{\epsilon}{4\pi r^2}\left[(r-2M)\frac{\partial \rho}{\partial r} - M\frac{\partial \mu}{\partial r}\right] = \frac{\epsilon}{\lambda M^2}(f+h) = T_r^r . \tag{4.12}$$

We need solve only (4.10) and (4.11), as (4.12) then follows. Moreover the T_θ^θ and T_ϕ^ϕ equations hold automatically as a result of the linearized Bianchi identities (2.1). We find

$$\begin{aligned} K\mu(r) = {} & \left[1 - \frac{2M}{r}\right]^3 \left[\frac{r^3 - 72M^3}{24M^3}\right] \\ & + \left[1 - \frac{2M}{r}\right]^2 \left[\frac{r^2 + 12M^2}{2M^2}\right] \\ & + \left[1 - \frac{2M}{r}\right]\left[\frac{3r + 2M}{M}\right] \\ & + 64\frac{M^3}{r^3} - 4\ln\left[\frac{2M}{r}\right] - 8 + C_0 , \end{aligned}$$

or

$$K\mu(r) \equiv K\mu_0(r) + C_0 , \tag{4.13}$$

where $K\mu_0(2M) = 0$, and

$$\begin{aligned} K\rho(r) = {} & \frac{14}{3}\left[1 - \frac{2M}{r}\right]^3 + \left[1 - \frac{2M}{r}\right]^2\left[\frac{r^2 - 228M^2}{12M^2}\right] \\ & + \frac{1}{3}\left[1 - \frac{2M}{r}\right]\left[\frac{4r + 92M}{M}\right] - 4\ln\left[\frac{2M}{r}\right] + k_0 \\ \equiv {} & K\rho_0(r) + k_0 , \end{aligned} \tag{4.14}$$

where $\rho_0(2M) = 0$. The quantities C_0 and k_0 are integration constants to be determined and $K \equiv 3840\pi$.

By writing $g_{ab} = \hat{g}_{ab} + \Delta g_{ab}$, $\hat{g}_{ab}$ = Schwarzschild metric, we find

$$\Delta g_{vv} = -\left[1 - \frac{2M}{r}\right] 2\epsilon\rho(r) + \frac{2M\epsilon\mu(r)}{r} , \tag{4.15}$$

$$\Delta g_{vr} = \epsilon\rho(r) , \tag{4.16}$$

which are seen to be regular at $r=2M$. Note that the trace of this perturbation is

$$\mathrm{tr}\Delta g = \hat{g}^{ab}\Delta g_{ab} = 2\epsilon\rho(r) \neq 0 \ . \tag{4.17}$$

Hence, the constant k_0 in (4.14) has to do with a scale associated with the corrected geometry, as we shall see directly later. Clearly, the trace-free part of Δg_{ab} is not zero. Thus, the back-reaction changes the conformal structure of the spacetime, e.g., the Weyl tensor. This point is illustrated in Sec. X.

V. EVENT HORIZON AND ENERGY

The generator of the event horizon (EH) is an outgoing future-directed geodesic null vector field l^a which I take as[19,20]

$$l^a = \left[1, \frac{1}{2}e^{\psi}\left[1-\frac{2m}{r}\right], 0, 0\right] \ . \tag{5.1}$$

The radially ingoing geodesic null vector field is

$$\beta^a = [0, -e^{-\psi}, 0, 0] \ , \tag{5.2}$$

where the normalization $\beta_a l^a = -1$ is observed. The apparent horizon[15] is determined by the vanishing of the expansion Θ of l^a:

$$\Theta = \frac{1}{r}\left[1-\frac{2m}{r}\right]e^{\psi} \ . \tag{5.3}$$

In a static geometry, the apparent and event horizons coincide. Therefore $\Theta = 0$ yields in $O(\epsilon)$

$$r_{\mathrm{EH}} = 2m = 2M[1+\epsilon\mu(2M)] \equiv 2M_{\mathrm{BH}} \ , \tag{5.4}$$

which also defines the mass of the black hole M_{BH} in the corrected geometry. For future reference, we also exhibit the surface gravity

$$\begin{aligned}\kappa &= -\beta^a l^b \nabla_b l_a \\ &= \frac{1}{2}e^{\psi}\frac{\partial}{\partial r}\left[1-\frac{2m}{r}\right] + e^{\psi}\left[1-\frac{2m}{r}\right]\frac{\partial\psi}{\partial r} \ ,\end{aligned} \tag{5.5}$$

which yields at the event horizon in the first order

$$\kappa_{\mathrm{EH}} = \frac{1}{4M}\left[1+\epsilon\left[\rho-\mu-2M\frac{\partial\mu}{\partial r}\right]\right]_{r=2M} \ , \tag{5.6}$$

where

$$\rho(2M) = k_0 K^{-1}, \quad \mu(2M) = C_0 K^{-1} \ , \tag{5.7}$$

and

$$\left[2M\frac{\partial\mu}{\partial r}\right]_{r=2M} = -12K^{-1} \ . \tag{5.8}$$

The total effective mass energy inside a radius r for a static observer is given by

$$m(r) = M + \epsilon M\mu(r) \ . \tag{5.9}$$

The energy of the radiation is given in $O(\epsilon)$ by the familiar result for static spherical systems

$$E_{\mathrm{rad}}(r) = -\int_{2M}^{r} 4\pi r^2 T_t^t dr = \epsilon M\mu_0(r) \ . \tag{5.10}$$

(See, for example, Ref. 21.) Using (4.13) and (5.10), we write (5.9) as

$$\begin{aligned}m(r) &= M + \epsilon M\mu_0(r) + \epsilon M C_0 K^{-1} \\ &= M(1+\epsilon C_0 K^{-1}) + E_{\mathrm{rad}}(r) \\ &= M_{\mathrm{BH}} + E_{\mathrm{rad}}(r) \ ,\end{aligned} \tag{5.11}$$

where (5.4) and (5.7) were used in the last line. We see that the integration constant C_0 plays the physical role of locating the event horizon and thus defining the mass of the black hole by use of the well-known "irreducible mass" formula[17] $A_{\mathrm{EH}} = 4\pi r_{\mathrm{EH}}^2 = 16\pi M_{\mathrm{BH}}^2$ (as in 5.4).

We now note that $E_{\mathrm{rad}}(r)$ is determined by the behavior of $\mu_0(r)$, which is zero at $r = 2M$, becomes negative, and reaches its minimum $\mu_0 = -0.8896\ \mathrm{K}^{-1}$ at $r \cong 2.3437M$ (where $T_t^t = 0$). It then increases as r grows larger, crossing through zero at $r \cong 2.8018M$. For large r, we have

$$\mu_0(r) \underset{r\to\infty}{\sim} \frac{1}{24K}\left[\frac{r}{M}\right]^3 \ , \tag{5.12}$$

which gives the expected flat-space result

$$E_{\mathrm{rad}}(r) \underset{r\to\infty}{\sim} aT_H{}^4 V \ , \tag{5.13}$$

where V is the flat-space volume.

The unstable circular photon orbit $\bar{r} = 3M$ of the Schwarzschild solution is of decisive importance in many problems of black-hole physics,[17] for example, in perturbation theory, wave scattering and absorption, and estimation of resonant frequencies.[22] To find this orbit in the corrected geometry, one solves the appropriate null geodesic equations by standard methods. This yields in $O(\epsilon)$

$$\begin{aligned}\bar{r} &= 3M\left[1+\epsilon\left[\mu - M\frac{\partial\mu}{\partial r} + M\frac{\partial\rho}{\partial r}\right]_{r=3M}\right] \\ &= 3M_{\mathrm{BH}}\left[1+\frac{E_{\mathrm{rad}}(3M)}{M_{\mathrm{BH}}} + 36\pi M_{\mathrm{BH}}{}^2 T_r^r(3M)\right] \\ &\cong 3M_{\mathrm{BH}}[1+\epsilon(3.1\times10^{-4})] \ .\end{aligned} \tag{5.14}$$

The main point is that $\bar{r} > 3M_{\mathrm{BH}}$: the effective radiant energy inside the circular photon orbit is positive. This suggests, for example, that the back-reaction would slightly lower the resonant frequencies of the hole.[22] This could be significant for small holes, if the result holds at least qualitatively as $\epsilon \to 1$, because then at fixed temperature the thermal excitation of the modes will increase and the hole would be relatively hotter than when the back-reaction is ignored.

VI. SURFACE GRAVITY AND PERIOD OF THE REAL EUCLIDEAN SECTION

The surface gravity κ_{EH} is given by (5.6). We can express this result using (5.8) as

$$\kappa_{\mathrm{EH}} = \frac{1}{4M_{\mathrm{BH}}}\left[1+\epsilon\left[\frac{k_0+12}{3840\pi}\right]\right] \ , \tag{6.1}$$

which depends on the integration constant

$\rho(2M)=k_0K^{-1}$. The function $\rho(r)$ serves geometrically essentially to measure the departure of the coordinate r, which defines circumference and area in the usual way, from being an affine parameter for the ingoing radial null geodesics. Alternatively, it gives the normalization of $\partial/\partial v=\partial/\partial t$. Physically, it is related to the temperature and will be determined later. For the present, we note that its asymptotic behavior is

$$\rho(r) \underset{r\to\infty}{\sim} \frac{1}{12K}\left[\frac{r}{M}\right]^2 \tag{6.2}$$

so that k_0 cannot be fixed by asymptotic flatness.

To further interpret the new metric, we transform back to Schwarzschild-type coordinates by a transformation very similar to the Eddington-Finkelstein transformation (4.1). We have (restoring $\tilde{r}=r$)

$$ds^2=-e^{2\psi}\left[1-\frac{2m}{r}\right]+2e^{\psi}dv\,d\tilde{r}+r^2d\omega^2 . \tag{6.3}$$

We define the transformation to (t,r,θ,ϕ) by

$$\frac{\partial v}{\partial t}=1,\quad \frac{\partial v}{\partial r}=e^{-\psi}\left[1-\frac{2m}{r}\right]^{-1} \tag{6.4}$$

and $\tilde{r}=r$. This yields the Lorentzian metric

$$ds^2=-e^{2\psi}\left[1-\frac{2m}{r}\right]dt^2 + \left[1-\frac{2m}{r}\right]^{-1}dr^2+r^2d\omega^2 . \tag{6.5}$$

In order ϵ, this metric can be written as

$$ds^2=-\left[1-\frac{2m(r)}{r}\right](1+2\epsilon\rho)dt^2+\left[1-\frac{2m(r)}{r}\right]^{-1}dr^2+r^2d\omega^2 \tag{6.6}$$

$$=-\left[1-\frac{2M_{\rm BH}}{r}\right]\left[1+2\epsilon\rho-2\epsilon\frac{\mu_0M_{\rm BH}}{r}\left[1-\frac{2M_{\rm BH}}{r}\right]^{-1}\right]dt^2 + \left[1-\frac{2M_{\rm BH}}{r}\right]^{-1}\left[1+2\epsilon\frac{\mu_0M_{\rm BH}}{r}\left[1-\frac{2M_{\rm BH}}{r}\right]^{-1}\right]dr^2+r^2d\omega^2 , \tag{6.7}$$

which is regular at the event horizon. Note that k_0 appears explicitly in ρ but that C_0 has been absorbed into $M_{\rm BH}$. Hence, C_0 plays no further role and without loss of generality, *we henceforth replace* $M_{\rm BH}$ *by* M, with the understanding that M denotes the mass of the "dressed" black hole through $O(\epsilon)$.

The real Euclidean section of this geometry is given by the Wick rotation $t\to -it$, and we find a metric of Euclidean signature for $r\geq 2M$, which is just (6.6) or (6.7) with the sign of g_{tt} changed. This geometry is regular ("nonconical") at the axis $r=2M$ provided that t is identified with the period β given by (3.2). We find

$$\beta=\frac{2\pi}{\kappa_{\rm EH}}=8\pi M\left[1-\epsilon\left[\frac{k_0+12}{3840\pi}\right]\right] . \tag{6.8}$$

Note that β and $\kappa_{\rm EH}$ are related as in the Schwarzschild metric but that β has the Schwarzschild value $8\pi M$ if and only if we choose $k_0=-12$. However, k_0 has many other allowed values that are determined by the boundary conditions.

Next, we establish the domain of validity implicit in regarding the effect of T^a_b as a perturbation of the Schwarzschild geometry. As there are two relevant physical scales in the problem, defined by $\epsilon=\hbar M^{-2}$ and by r_0M^{-1}, $r_0=$ radius of cavity, it is helpful to introduce another small dimensionless parameter ϵ_0 (a pure number) such that over the domain $r=r_{\rm EH}=2M$ to $r=r_{\rm max}$, we have uniformly

$$\frac{|\Delta g_{ab}|}{|\hat{g}_{ab}|}\lesssim\epsilon_0\ll 1,\quad 0\leq\epsilon\leq\epsilon_0 . \tag{6.9}$$

The metric corrections Δg_{tt} and Δg_{rr} both grow as r^2 at large distances and have, asymptotically, equal magnitudes for all finite values of k_0. Examination of the functions μ_0 and ρ shows that (6.9) holds if

$$4\leq\left[\frac{r}{M}\right]^2\ll(12K)\left[\frac{\epsilon_0}{\epsilon}\right] \tag{6.10}$$

or, in terms of areas, with $A=4\pi r^2$,

$$1\leq AA_{\rm EH}^{-1}\ll 3K\left[\frac{\epsilon_0}{\epsilon}\right] . \tag{6.11}$$

If we choose as a reasonable upper limit in (6.9) $\epsilon_0=0.1$, then choosing also $\epsilon=0.1$ ($M=\sqrt{10}\times$ Planck mass), we find from (6.10) that $r_{\rm max}\lesssim 380M$; if $\epsilon=0.01$, we find $r_{\rm max}\lesssim 1200M$, etc. Thus, the cavity can always be chosen to be relatively large. For instance, if $\epsilon_0=\epsilon=0.1$ and $r=350$ M, we have $E_{\rm rad}(r)M^{-1}\approx 15$, so that most of the energy is in the radiation. Though the fractional metric perturbations resulting from the choice $r=r_{\rm max}$ can be of order $\epsilon_0\geq\epsilon$, it is reasonable to assume that higher-order corrections (in ϵ) would yield fractional metric changes of order $\epsilon\epsilon_0\ll\epsilon$ so that linearization of the Einstein operator in terms of ϵ is justified even for a relatively large amount of radiant energy.

VII. MICROCANONICAL BOUNDARY CONDITIONS

We consider here as boundary an ideal massless perfectly reflecting spherical wall of area $4\pi r_0{}^2$, which corresponds to a closed cavity as envisioned in defining a microcanonical ensemble. We specify as boundary conditions the total effective energy at r_0:

$$m(r_0)=M+E_{\rm rad}(r_0)\ . \tag{7.1}$$

To stay within the domain of validity of the perturbation theory previously defined, $m(r_0)$ is chosen to be close to the value (Schwarzschild mass) plus $E_{\rm rad}(r_0)$, the latter being given by (5.10). Then we may regard the M in the formulas for $E_{\rm rad}$, μ, and ρ as being the physical black-hole mass. Hence, (7.1) is to be regarded as giving the relation between the physical black-hole mass and the total energy.

The constant k_0 is determined by noting that outside the reflecting wall, we have an asymptotically flat Schwarzschild geometry of mass-energy $m(r_0)$ if we ignore the mass of the wall. In this case

$$-g_{tt}(r>r_0)=1-\frac{2m(r_0)}{r}\ . \tag{7.2}$$

For a very thin wall we determine k_0 by requiring g_{tt} to be continuous across $r=r_0$. From (4.14) and (7.2) it follows that

$$k_0K^{-1}=-\rho_0(r_0)\ . \tag{7.3}$$

Observe that g_{rr} is also continuous across $r=r_0$. There are finite discontinuities in the radial derivatives of the metric that yield the surface stresses of the massless wall. Following, for example, Ref. 17, p. 552, we find the proper surface pressure (units: cm^{-1}) in $O(\epsilon)$:

$$\begin{aligned} S^{\phi}_{\phi}&=\frac{\epsilon}{16\pi}\left[1-\frac{2M}{r}\right]^{-3/2}\left[\frac{2M}{r}\frac{\partial\mu}{\partial r}-2\left[1-\frac{2M}{r}\right]\frac{\partial\rho}{\partial r}\right]\\ &=-\frac{1}{2}r\left[1-\frac{2M}{r}\right]^{-3/2}T^{r}_{r}\ . \end{aligned} \tag{7.4}$$

Hence, if $T^r_r>0$, as holds for the scalar field, there is in the reflecting wall surface tension that maintains the mechanical equilibrium of the wall as it is bombarded by radiation from inside. If we consider a small, slow, spherical expansion of the system, for r_0 large in accordance with (6.10), we find that the work associated with the restraining surface tension is

$$|S^{\phi}_{\phi}|\,dA_0\rightarrow\tfrac{1}{2}\epsilon_0 dr_0\ . \tag{7.5}$$

In the case of large r_0, the volume pressure of the radiation is $p\rightarrow 3^{-1}aT_H{}^4$ so that

$$p\,dV_0\rightarrow\tfrac{1}{2}\epsilon_0 dr_0=|S^{\phi}_{\phi}|\,dA_0\ , \tag{7.6}$$

which indicates the mechanical equilibrium of the reflecting wall.

The spacetime geometry is now uniquely determined. For $2M\le r\le r_0$, we have

$$\begin{aligned} ds^2=&-\left[1-\frac{2m(r)}{r}\right]\left[1+2\epsilon\rho_0(r)-2\epsilon\rho_0(r_0)\right]dt^2\\ &+\left[1-\frac{2m(r)}{r}\right]^{-1}dr^2+r^2d\omega^2\ . \end{aligned} \tag{7.7}$$

For $r\ge r_0$,

$$\begin{aligned} ds^2=&-\left[1-\frac{2m(r_0)}{r}\right]dt^2\\ &+\left[1-\frac{2m(r_0)}{r}\right]^{-1}dr^2+r^2d\omega^2\ . \end{aligned} \tag{7.8}$$

VIII. EQUILIBRIUM TEMPERATURE DISTRIBUTION

The equilibrium temperature of a static, self-gravitating system is not constant but rather is given by the "red-shifted" temperature distribution

$$T_{\rm loc}(r)\,|g_{tt}(r)|^{1/2}=\text{constant}\equiv T_*\ , \tag{8.1}$$

$$\frac{\partial T_*}{\partial r}=0\ . \tag{8.2}$$

(See, for example, Ref. 17, p. 568.) In the Schwarzschild geometry, $T_*=T_H=\hbar(8\pi M)^{-1}$. We know $g_{tt}(r)$ through $O(\epsilon)$; the question is whether or not $T_*=\hbar(8\pi M)^{-1}$ when the back-reaction is included. In other words, are all $O(\epsilon)$ corrections incorporated in g_{tt}? The answer cannot be affirmative with this choice of T_*, as I shall now show.

We recall that in the Schwarzschild geometry, we can write T_* in an equivalent but more general form as

$$T_*=(\kappa_{\rm EH}\hbar)(2\pi)^{-1}\ . \tag{8.3}$$

This form is also valid for all other time-independent black-hole geometries. I therefore adopt it here and show it implies the essential physical property that $T_{\rm loc}(r)$ *is independent of* k_0. Thus, recall from (6.5) that

$$\begin{aligned} |g_{tt}(r)|^{-1/2}&=e^{-\psi}\left[1-\frac{2m}{r}\right]^{-1/2}\\ &\cong[1-\epsilon\rho_0(r)-\epsilon k_0K^{-1}]\left[1-\frac{2m}{r}\right]^{-1/2}\ . \end{aligned} \tag{8.4}$$

From (6.1) and (8.3) we find that T_* depends on k_0,

$$T_*=\frac{\hbar}{8\pi M}(1+\epsilon k_0K^{-1}+12\epsilon K^{-1})\ , \tag{8.5}$$

but then $T_{\rm loc}(r)$ is independent of k_0:

$$\begin{aligned} T_{\rm loc}(r)=&\frac{\hbar}{8\pi M}[1-\epsilon\rho_0(r)\\ &+12\epsilon K^{-1}]\left[1-\frac{2m(r)}{r}\right]^{-1/2}\ . \end{aligned} \tag{8.6}$$

It is vital that $T_{\rm loc}(r)$ be independent of k_0 because, as is easily shown, the motion of test particles and photons at $r<r_0$ is independent of k_0 and the locally measured temperature should be as well. For example, if r_0 is very large, the nature of the boundary at r_0 should be unimportant for the mode functions and for measurements of local thermal properties at $r\ll r_0$. If we did not choose T_* using (8.3), we would find local thermal properties at

$r \ll r_0$ would be, for example, significantly different in $O(\epsilon)$ for a black hole of a given mass if we used canonical (next section) rather than microcanonical boundary conditions. This would not be physically reasonable.

Another argument concerning the important question of the local temperature distribution can be applied: In the Euclidean description, the local temperature is determined by the proper length of the periodically identified "time" axis for fixed r, θ, and ϕ:[23]

$$\hbar[T_{\rm loc}(r)]^{-1} = \int_0^\beta |g_{tt}(r)|^{1/2} dt \ , \tag{8.7}$$

which gives precisely the previous result (8.6).

We can apply these results to a black hole in a closed cavity. Imagine attaching a small tube from r_0 to infinity or opening a small shutter on the surface of the reflecting wall. For $r > r_0$, we have

$$|g_{tt}(r>r_0)|^{-1/2} = \left[1 - \frac{2m(r_0)}{r}\right]^{-1/2} \tag{8.8}$$

so that from (8.1), we find that an observer at asymptotically flat infinity perceives thermal radiation characterized by a temperature

$$T_\infty = T_* = (\kappa_{\rm EH}\hbar)(2\pi)^{-1} \tag{8.9}$$

in accord with Hawking's results for all other time-independent asymptotically flat black-hole geometries. We see that *both* T_∞ and the "gravitational potential" $|g_{tt}|$ acquire corrections in $O(\epsilon)$, but that these corrections compensate each other to the extent that $T_{\rm loc}$ inside the cavity does not depend explicitly on the constant k_0 determined by the boundary conditions.

IX. CANONICAL BOUNDARY CONDITIONS

With canonical boundary conditions, the system is thermally coupled to a large heat reservoir and we fix temperature rather than energy. Do we fix $T_{\rm loc}(r_0)$ or T_*? (Note that here T_* cannot be identified with T_∞ because space outside r_0 is no longer asymptotically flat.) The natural choice is to fix $T_{\rm loc}(r_0)$ and then to solve (8.6) for the "irreducible" black-hole mass $M = M[T_{\rm loc}(r_0), r_0]$. By examining the Schwarzschild formula for $T_{\rm loc}(r_0)$, one can see that if $T_{\rm loc}(r_0)$ and r_0 admit a positive solution M_1, then there is also always another positive solution M_2. One finds $M_1 \neq M_2$ if $r_0 \neq 3M_1$; $r_0 = 3M_1$ implies $M_1 = M_2$. Thus, in general there will be two possibilities for the value of M if a black hole is present. [This is analogous to fixing $m(r_0)$ and r_0 in the microcanonical ensemble, where these quantities allow two possibilities for M. See, for example, Hawking (Ref. 20).] This procedure does not determine k_0. However, in the canonical ensemble spacetime is not asymptotically flat and neither t nor k_0 has an invariant meaning. One can use a constant rescaling of the coordinate t without changing the geometry or the physics. Formula (3.2) shows that such a scaling alters β (and therefore κ and T_*) and can be used to arrange that $\beta = 8\pi M[T_{\rm loc}(r_0), r_0]$ so that the period of the Euclidean time coordinate is determined by the mass of the black hole just as in the unperturbed Schwarzschild geometry. (This is analogous to fixing the period of all azimuthal angles in geometry as 2π.) The metric inside the cavity is then given by (7.7) with M replaced by $M[T_{\rm loc}(r_0), r_0]$, $\rho_0(r_0)$ replaced by $12K^{-1}$, and t understood as the rescaled time.

There is an alternative procedure that allows a direct comparison of the metrics inside the cavity in the two ensembles. We regard M as fixed and specify T_*, which will determine k_0. Here also there is a natural choice of T_*. Let r_0 and M take the same values in the canonical ensemble as in the microcanonical ensemble, where the meaning of t is unambiguous in the latter. Then choose T_* such that the time is normalized at r_0 in the same way in both cases. This means we choose

$$T_* = \frac{\hbar}{8\pi M}[1 - \epsilon\rho_0(r_0) + 12\epsilon K^{-1}] \ . \tag{9.1}$$

In this case, the microcanonical and canonical metrics inside r_0 have identically the same form (7.7). Any physical differences could only result from explicit effects of finite size on the mode functions and from explicit quantum effects at the wall, just as in any thermal system. Of course, this is not to say that global stability properties of the two ensembles are equivalent.[8]

X. CURVATURE INVARIANTS

Curvature invariants of dimension (length)$^{-4}$ are well known to be of fundamental importance for quantum field theory in curved spacetime and in quantum gravity.[2] It is therefore of some interest to investigate the changes brought about in these quantities by the back-reaction. For present purposes, I list these invariants as $({\rm Weyl})^2 = C_{abcd}C^{abcd}$, $({\rm Ricc})^2 = R_{ab}R^{ab}$, R^2, and $\Box R = g^{ab}\nabla_a\nabla_b R$.

The invariants (Ricc)^2 and R^2 can be computed directly from the stress-energy tensor T^a_b. They are of $O(\epsilon^2)$ and in this order the correction Δg_{ab} to the metric is unimportant. However, the invariants (Weyl)^2 and $\Box R = \hat{\Box} R + O(\epsilon^2)$ do acquire changes in $O(\epsilon)$. Such changes would also be caused by a one-loop graviton stress-energy tensor were it present in the calculations. In $O(\epsilon)$, we have $({\rm Riem})^2 = R_{abcd}R^{abcd} = ({\rm Weyl})^2$, where, for a static metric of the form (4.6),

$$\begin{aligned}({\rm Riem})^2 = {} & 4e^{-2\psi}\left[\frac{\partial\kappa}{\partial r}\right]^2 + 8r^{-2}\left[1-\frac{2m}{r}\right]^2\left[\frac{\partial\psi}{\partial r}\right]^2 \\ & + 8r^{-2}\left[1-\frac{2m}{r}\right]\left[\frac{\partial\psi}{\partial r}\right]\frac{\partial}{\partial r}\left[1-\frac{2m}{r}\right] \\ & + 4r^{-2}\left[\frac{\partial}{\partial r}\left[1-\frac{2m}{r}\right]\right]^2 + 16m^2r^{-6} \ . \end{aligned} \tag{10.1}$$

For the Schwarzschild metric, we have the familiar result $({\rm Riem})^2 = 48M^2r^{-6}$, which yields $3(4M^4)^{-1}$ at the event horizon. It is straightforward to evaluate (10.1) in the corrected geometry. I shall exhibit the result in $O(\epsilon)$ only at the event horizon:

$$({\rm Riem})^2(2M) = ({\rm Weyl})^2(2M) = \frac{3}{4M^4}\left[1 + \frac{40\epsilon}{3K}\right] \ . \tag{10.2}$$

In the Schwarzschild geometry, we have of course $\Box R = \hat{\Box}\hat{R} = 0$. In the corrected geometry, we have in $O(\epsilon)$ that $\Box R = \hat{\Box} R$, which gives at the event horizon

$$\hat{\Box} R(2M) = \frac{\epsilon}{M^4}\left[\frac{12}{K}\right]. \tag{10.3}$$

Thus the corrections to $\Box R$ and $(\mathrm{Weyl})^2$ are comparable near the event horizon.

The invariant $\Box R$ is often neglected in considering the one-loop effective action for quantum gravity, on the grounds that it is a pure divergence and therefore has a vanishing functional derivative with respect to the metric.[24] However, as is now well known in the case of self-gravitating systems,[25] pure divergences in the action cannot be in general ignored; they can produce nontrivial variations of the action, even when the field equations hold. The result will depend on the boundary conditions. Another way of saying this is that such terms can affect the value of the action, and, hence, the results of a sum-over-histories. In the semiclassical solutions, we have seen the necessity of introducing a boundary at a finite radius. Thus, it is of some interest to see that in $O(\epsilon)$

$$\int_\Omega \Box R\sqrt{g}\, d^4x = \epsilon\frac{128\pi}{5}\left[\frac{M}{r_0}\right]^5\left[1-\frac{2M}{r_0}\right], \tag{10.4}$$

where the integral has been evaluated in Euclidean space. The value of this integral need not be negligible.

I shall now consider corrections to the conformal scalar-field trace anomaly originating in the back-reaction caused by this same scalar field. For a massless free conformal scalar field, it is well known that the trace anomaly is given by[11] (in Page's[4] notation)

$$\mathrm{tr}T = \alpha\mathscr{H} + \beta\mathscr{G} = \gamma\Box R\ , \tag{10.5}$$

where $\alpha = 2\hbar(3840\pi^2)^{-1}$, $\beta = -\frac{1}{3}\alpha$, $\gamma = \frac{2}{3}\alpha$, and

$$\mathscr{H} = (\mathrm{Weyl})^2\ , \tag{10.6}$$

$$\mathscr{G} = ({}^*\mathrm{Riem})^2 = (\mathrm{Riem})^2 - 4(\mathrm{Ricc})^2 + R^2\ . \tag{10.7}$$

The correction to $\mathrm{tr}T$ will be of $O(\epsilon^2)$. This will come about because the scalar-field calculation will still be only a one-loop calculation,[2] but it would now presumably be carried out on the corrected real Euclidean black-hole thermal geometry $g_{ab} = g_{ab}(\mathrm{Schwarzschild}) + \Delta g_{ab}$, $\Delta g_{ab} = O(\epsilon)$. (Page's techniques[4] can still be used in this case.) It is then easy to see that to obtain $\mathrm{tr}T$ in $O(\epsilon^2)$, we need only know g_{ab} in $O(\epsilon)$. Denoting the altered value by $\mathrm{tr}T'$, we have from (10.5) and the previous results

$$\mathrm{tr}T' = \frac{4\hbar}{3(3840\pi^2)}[(\mathrm{Riem})^2 + \Box R] \tag{10.8}$$

$$= \mathrm{tr}T + \frac{4\hbar}{3(3840\pi^2)}[\Delta(\mathrm{Riem})^2 + \hat{\Box} R]$$

$$= \mathrm{tr}T + \Delta(\mathrm{tr}T)\ , \tag{10.9}$$

where $\mathrm{tr}T$ is given by (3.8). It follows that the *fractional* correction to $\mathrm{tr}T$ is of $O(\epsilon)$. Evaluated at the event horizon, the fractional correction is

$$\frac{\Delta(\mathrm{tr}T)}{\mathrm{tr}T}(2M) = \frac{88}{3(3840\pi)}\epsilon\ . \tag{10.10}$$

XI. CONCLUSION

The semiclassical back-reaction program as far as it has been carried out here seems to make sense. It was necessary to solve the problem using linearization in a finite region. On the mathematical side, there is no reason to worry about linearization because uniform approximations can be made and "linearization instability" in the sense defined by Fischer and Marsden[26] is not an issue here. Physically, the finiteness and boundary conditions seem natural from either the point of view of statistical mechanics or of regarding them as a device that would ultimately allow embedding the equilibrium system into the "universe" using matching conditions.[27] This "matching" was carried out in an asymptotically flat universe in the microcanonical case.

The most important limitation of the results is that a one-loop graviton tensor τ^{ab} has not been included. Once we have this quantity, especially if an excellent closed-form approximation is found, it will be possible to repeat easily the calculations of this work. We would then know more about the role of metric fluctuations near $r = 2M$, where they appear to play a decisive role in the dynamical origin of the temperature of black holes.[7] This direction appears to be the most fruitful. The way will then be open for obtaining more precise statistical-mechanical understanding of black holes in and near equilibrium. This might also illuminate the behavior of other multiphase systems in which one phase involves the presence of a horizon.

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[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[3] J. B. Hartle and S. W. Hawking, Phys. Rev. D **13**, 2188 (1976); W. Israel, Phys. Lett. **57A**, 107 (1976); G. W. Gibbons and M. J. Perry, Phys. Rev. Lett. **36**, 985 (1976).
[4] D. N. Page, Phys. Rev. D **25**, 1499 (1982).
[5] K. W. Howard and P. Candelas, Phys. Rev. Lett. **53**, 403

(1984).
[6]K. W. Howard, University of Texas at Austin report, 1984 (unpublished).
[7]J. W. York, Jr., Phys. Rev. D **28**, 2929 (1983).
[8]S. W. Hawking, Phys. Rev. D **13**, 191 (1976).
[9]Y. Nambu, Phys. Lett. **26B**, 626 (1966).
[10]G. W. Gibbons and M. J. Perry, Proc. R. Soc. London **A358**, 467 (1978).
[11]D. M. Capper and M. J. Duff, Nuovo Cimento **23A**, 173 (1974); Phys. Lett. **53A**, 361 (1975); S. Deser, M. J. Duff, and C. J. Isham, Nucl. Phys. **B111**, 45 (1976); M. J. Duff, *ibid.* **B125**, 334 (1978), and references therein.
[12]D. N. Page, Phys. Rev. D **13**, 198 (1976).
[13]See, for example, S. Detweiler, in *Sources of Gravitational Radiation,* edited by L. Smarr (Cambridge University Press, Cambridge, 1979).
[14]S. Detweiler (private communication).
[15]S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, 1973).
[16]S. M. Christensen and S. A. Fulling, Phys. Rev. D **15**, 2088 (1977).
[17]See, for example, C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[18]J. M. Bardeen, Phys. Rev. Lett. **46**, 382 (1981).
[19]J. W. York, Jr., in *Quantum Theory of Gravity: Essays in Honor of the Sixtieth Birthday of Bryce S. DeWitt,* edited by S. Christensen (Adam Hilger, Bristol, 1984).
[20]See, for example, B. Carter, in *General Relativity,* edited by S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
[21]D. Pavón and W. Israel, Gen. Relativ. Gravit. **16**, 563 (1984).
[22]W. Press, Astrophys. J. **170L**, 105 (1971); C. J. Goebel, *ibid.* **172L**, 95 (1972).
[23]Boundary conditions of both microcanonical and canonical types for perturbations of black holes were discussed recently in a different context by B. Allen, Phys. Rev. D **30**, 1153 (1984). Allen refers to S. W. Hawking in *General Relativity* (Ref. 20).
[24]See, however, the remarks by N. H. Barth and S. M. Christensen, Phys. Rev. D **28**, 1876 (1983).
[25]J. W. York, Jr., Phys. Rev. Lett. **28**, 1082 (1972); G. W. Gibbons and S. W. Hawking, Commun. Math. Phys. **66**, 291 (1979).
[26]A. E. Fischer and J. E. Marsden, in *General Relativity* (Ref. 20).
[27]J. A. Wheeler, in *Relativity, Groups, and Topology,* edited by C. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964).