Particle and energy creation by moving mirrors

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We consider the creation of massless scalar particles by a moving mirror in two-dimensional space-time. The correct form for the Bogoliubov coefficients is given and their high-frequency behavior is investigated. We next consider the energy radiated by the mirror and show that this is related in the expected way to the number of particles produced only if a particular condition on the trajectory is fulfilled. The well-known moving-mirror formula of Fulling and Davies, which gives the radiated energy as a functional of the mirror trajectory, is here rederived from the Bogoliubov transformation, without recourse to regularization. Finally we analyze the response of a particle detector, and solve the paradox of how the detector can respond even when the mirror is radiating no energy.

I. INTRODUCTION

Quantum field theory in the presence of a moving boundary or "mirror" contains many of the special features of curved-space quantum field theory without the added complication of the background spacetime actually being curved.[1-3] Moreover moving-mirror systems turn out to be important in their own right, especially concerning the second law of thermodynamics, where they have been postulated both as a way of breaking it,[4] or, in the black-hole case, of saving it.[5]

However, even the simplest moving-mirror system—a massless scalar field in two dimensions—is not fully understood. The Bogoliubov coefficients, which describe the number of particles present, have been given only approximately and a folklore has arisen concerning their validity and high-frequency behavior. Few exactly soluble cases exist to test these properties.[6]

In addition, the relationship between the particles and energy radiated by the mirror is unclear.[7] For instance, it is known that the mirror can radiate zero, or even negative, energy while still producing particles. This gives rise to the following questions:

(i) Can we find the total energy released by adding the energy of each quantum, i.e., does

$$\int \langle T^{00}\rangle dx=\int \omega n(\omega)\,d\omega\ , \tag{1.1}$$

and if not, under what conditions does this cease to be true?

(ii) Will a particle detector respond to particles which appear to carry no energy?

(iii) If the detector does respond, where has the energy needed to excite it come from?

This paper is an attempt to put previous work on a firmer footing and to answer the above questions. The Bogoliubov coefficients are given in a form which is exact for most purposes, and definite statements concerning their validity and asymptotic behavior are made. Example trajectories are given. We next examine under what conditions Eq. (1.1) is true, and demonstrate that this equation is consistent with the known definitions of particles and energy. In so doing we rederive Fulling and Davies's[1] formula for the energy flux using the Bogoliubov coefficients. Finally we show that a detector will respond to particles which appear to carry no energy, and that this does not present a paradox. The solution of this problem contains a simple example of the particle detection mechanism given recently by Unruh and Wald.[8]

II. THE BOGOLIUBOV COEFFICIENTS

We will consider a massless scalar field Φ in two-dimensional space-time, in the presence of a moving boundary which follows a trajectory[1]

$$x=z(t),\quad |\dot z|<1\ . \tag{2.1}$$

We will impose perfectly reflecting boundary conditions on the field at the surface of the barrier. Solving the Klein-Gordon equation for this system enables us to define "in" and "out" modes, in the region to the right of the mirror:

$$\text{in:}\ \ \varphi_\omega=(4\pi\omega)^{-1/2}(e^{-i\omega v}-e^{-i\omega p(u)})\ , \tag{2.2}$$

$$\text{out:}\ \ \chi_\omega=(4\pi\omega)^{-1/2}(e^{-i\omega f(v)}-e^{-i\omega u})\ , \tag{2.3}$$

where $v=t+x$ and $u=t-x$ are advanced and retarded coordinates, and the mirror trajectory is related to the function p by

$$z(\tau_u)=\tfrac{1}{2}[p(u)-u]\ ,\qquad \tau_u=\tfrac{1}{2}[p(u)+u]\ , \tag{2.4}$$

and $f(p(u))=u$.

The mirror's velocity is given by

$$\frac{dz}{d\tau_u}=\frac{p'(u)-1}{p'(u)+1} \tag{2.5}$$

and the requirement that the trajectory be timelike imposes the restriction $p'(u)>0$ and $f'(v)>0$. We shall choose a coordinate system in which the mirror passes through the origin at time $t=0$. Hence $f(0)=p(0)=0$.

The modes (1.2) and (1.3) can be used to define in and out vacuum states, respectively. This is true even when the mirror is not asymptotically static in the past or future. Thus there always exists a well-defined in (out) vacuum state, despite the fact that there may never be a time in the past (future) when the field is empty of particles.[9] The in and out vacua will not, in general, be the same, and there exists a Bogoliubov transformation relating them[2] (see Ref. 3 for a review):

$$\varphi_\omega=\int d\omega'(\alpha_{\omega\omega'}\chi_{\omega'}+\beta_{\omega\omega'}\chi^*_{\omega'}) , \tag{2.6}$$

where

$$\alpha_{\omega\omega'}=(\varphi_\omega,\chi_{\omega'}) , \quad \beta_{\omega\omega'}=-(\varphi_\omega,\chi^*_{\omega'}) \tag{2.7}$$

and

$$(\varphi,\chi)=-i\int dx\,\varphi(x,t)\overleftrightarrow{\partial}_t\chi^*(x,t) \tag{2.8}$$

is the Klein-Gordon inner product.

If the β coefficient is nonzero, then the vacua are inequivalent and particle production occurs. The number of particles produced in mode ω' is given by

$$n(\omega')=\int d\omega\,|\beta_{\omega\omega'}|^2 . \tag{2.9}$$

On substituting the modes (2.2) and (2.3) into (2.7), we can calculate the Bogoliubov coefficients. Since the inner product (2.8) is independent of time, we can choose to evaluate this at $t=0$ for simplicity. After an integration by parts, we find

$$\left.\begin{matrix}\alpha_{\omega\omega'}\\ \beta_{\omega\omega'}\end{matrix}\right\}=\pm(2\pi)^{-1}(\omega/\omega')^{1/2}\times\int_0^\infty dx[e^{-i\omega x\pm i\omega' f(x)}+p'(-x)e^{-i\omega p(-x)\mp i\omega' x}]+[(e^{-i\omega x}-e^{-i\omega p(-x)})\times(e^{\pm i\omega' f(x)}+e^{\mp i\omega' x})]_0^\infty . \tag{2.10}$$

We have temporarily assumed that the mirror is asymptotically static in the past and future. In general, because of the occurrence of δ-function-type terms, $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ are only well defined when integrated over ω or ω'. In this case the boundary term in (2.10) vanishes.

Fulling and Davies neglected the second term in the integral in (2.10) as being small compared to the first. However, we can simplify this term using the substitution $y=p(-x)$, $f(y)=-x$. We then obtain

$$\pm(2\pi)^{-1}(\omega/\omega')^{1/2}\int_{p(-\infty)}^{p(0)}dy\,e^{-i\omega y\pm i\omega' f(y)} .$$

We also know that $p(0)=0$, and since the mirror is asymptotically static in the past, $p(-\infty)=-\infty$. Combining this term with the first, we obtain

$$\left.\begin{matrix}\alpha_{\omega\omega'}\\ \beta_{\omega\omega'}\end{matrix}\right\}=\pm(2\pi)^{-1}(\omega/\omega')^{1/2}\int_{-\infty}^{\infty}dx\,e^{-i\omega x\pm i\omega' f(x)} . \tag{2.11}$$

If the mirror is not asymptotically static in the past and future, then this formula must be modified. Suppose the mirror becomes asymptotically null in the future, to the line $v=v_{\max}$, then incoming modes $e^{-i\omega v}$ with $v>v_{\max}$ will not intersect the mirror and so should be excluded from the integral (Ref. 2). Similarly if the mirror has a past asymptote $v=v_{\min}$ then the modes with $v<v_{\min}$ will not intersect it either (see Fig. 1). Thus the integral (2.11) should be replaced by

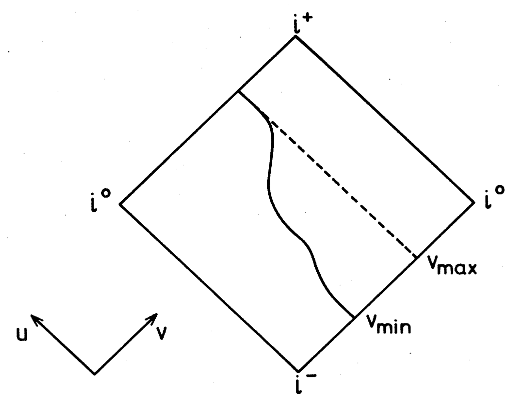

FIG. 1. Conformal diagram of a mirror trajectory with v asymptotes.

$$\left.\begin{matrix}\alpha_{\omega\omega'}\\ \beta_{\omega\omega'}\end{matrix}\right\}=\pm(2\pi)^{-1}(\omega/\omega')^{1/2}\int_{v_{\min}}^{v_{\max}}dv\,e^{-i\omega v\pm i\omega' f(v)} . \tag{2.12}$$

We have implied by replacing the x in (2.11) with v that we are really integrating along the v direction, over the region of overlap of the simple incoming modes $e^{-i\omega v}$ and the modes $e^{\pm i\omega' f(v)}$ that look simple after reflection from the mirror. Note that the boundary term no longer vanishes when either $v_{\min}$ or $v_{\max}$ are finite. When integrated over ω or ω', however, it will be small, and will vanish when integrated over both ω and ω', and thus it may be neglected. Alternatively, we may change variables and integrate by parts in (2.12) to obtain a formula in terms of the function $p(u)$:

$$\left.\begin{matrix}\alpha_{\omega\omega'}\\ \beta_{\omega\omega'}\end{matrix}\right\}=(2\pi)^{-1}(\omega'/\omega)^{1/2}\int_{u_{\min}}^{u_{\max}}du\,e^{-i\omega p(u)\pm i\omega' u} . \tag{2.13}$$

Here $u_{\min}$ and $u_{\max}$ are possible asymptotes of the trajectory in the u direction and may be taken to be infinite if no such asymptotes exist. The boundary term obtained in calculating (2.13) is similar to the one in (2.10) and may be disregarded for the same reasons.

The above discussion concerns the definition of particles. The energy density produced by the mirror is found by calculating the expectation value ${}_{\rm in}\langle 0|T_{\mu\nu}|0\rangle_{\rm in}$. This requires the use of some regularization technique, and was first given by Fulling and Davies[1] using point separation. They found

$$\langle T_{uu}\rangle=-(24\pi)^{-1}[p'''(u)/p'(u))-\tfrac{3}{2}(p''(u)/p'(u))^2] ,$$
$$\langle T_{uv}\rangle=\langle T_{vu}\rangle=\langle T_{vv}\rangle=0 . \tag{2.14}$$

The total energy released by the mirror is given by integrating (2.14) from $u_{\min}$ to $u_{\max}$:

$$E=\int_{z(t)}^{\infty}\langle T^{00}\rangle dx=\int_{u_{\min}}^{u_{\max}}\langle T_{uu}\rangle du \ . \tag{2.15}$$

As stated above neither the energy density, nor the total energy, is necessarily positive.

III. EXAMPLE TRAJECTORIES

The simplest example to consider is that of a mirror which remains static for all time. In this case $f(v)=v$, and we find from (2.10)

$$\alpha_{\omega\omega'}=\delta(\omega-\omega') \ ,$$
$$\beta_{\omega\omega'}=-(\omega/\omega')^{1/2}\delta(\omega+\omega')=0 \tag{3.1}$$

as expected, and, of course, no energy is radiated either.

A simple nontrivial trajectory is the hyperbolic trajectory[1,2] which corresponds to

$$F(v)=Bv/(B-v) \ . \tag{3.2}$$

This has $v_{\min}=-\infty$ and $v_{\max}=B$ (see Fig. 2). Substituting these values in (2.11), the integral may be evaluated exactly[10] to obtain[2]

$$\alpha_{\omega\omega'}=-\frac{B}{\pi}e^{-i(\omega+\omega')B}K_1(2iB(\omega\omega')^{1/2}) \ ,$$
$$\beta_{\omega\omega'}=\frac{iB}{\pi}e^{-i(\omega-\omega')B}K_1(2B(\omega\omega')^{1/2}) \ . \tag{3.3}$$

This trajectory is of particular interest because it is known that no energy is radiated. Thus we appear to have the paradoxical situation that particles are created without energy. This point will be investigated further in Secs. V and VI.

We may also construct a trajectory by joining together a static trajectory for $t\le 0$ with a hyperbolic trajectory for $t>0$, in a C^1 way (Fig. 3), giving

$$f(v)=\theta(v)Bv/(B-v)+v\theta(-v) \ . \tag{3.4}$$

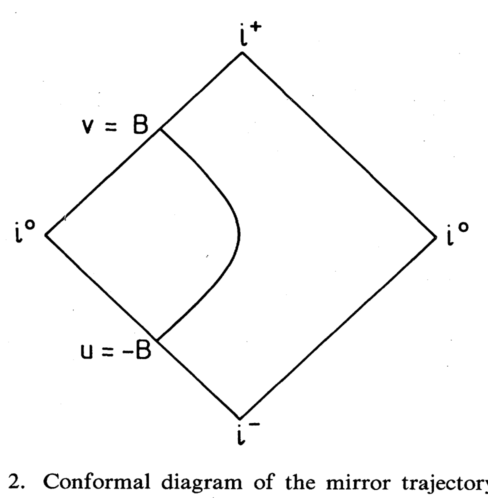

FIG. 2. Conformal diagram of the mirror trajectory corresponding to (3.2).

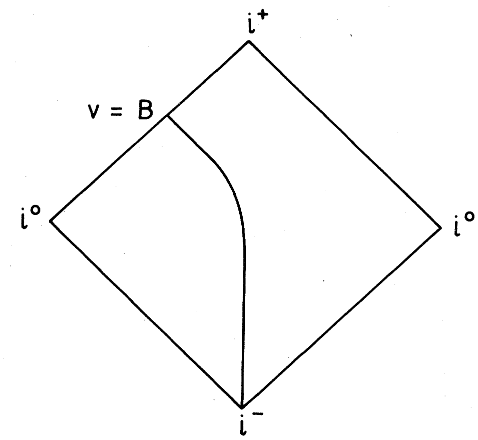

FIG. 3. Conformal diagram of the mirror trajectory corresponding to (3.4).

In this case the energy density is given by a δ function at the "join" and the total energy radiated is given by

$$E=\frac{1}{12\pi B} \ . \tag{3.5}$$

Unfortunately, α and β cannot be evaluated exactly, but it is easy to see that β will be nonzero. Moreover, the particles will be emitted continuously: they cannot be localized to the time when the energy was released.

Alternatively, we may join together two hyperbolic trajectories:

$$f(v)=\theta(v)Bv/(B-v)+\theta(-v)Av/(A-v) \tag{3.6}$$

which gives

$$E=(A-B)/12\pi AB \ . \tag{3.7}$$

This is an example of a trajectory where the total energy radiated need not even be positive. Again α and β cannot be evaluated exactly, but will be nonzero.

Another trajectory of interest corresponds to[2]

$$f(v)=-\kappa^{-1}\ln[(B-v)/A]-B \ , \tag{3.8}$$

where $A=B\exp(B\kappa)$ so that $f(0)=0$. This trajectory is well known to give a Planck spectrum of particles. However, previous calculations of $|\beta_{\omega\omega'}|^2$ have involved several approximations. When using the correct form (2.12) no approximations are necessary. The result is

$$|\beta_{\omega\omega'}|^2=(2\pi\kappa\omega)^{-1}(e^{2\pi\omega'/\kappa}-1)^{-1} \ . \tag{3.9}$$

The energy density is

$$T_{uu}=\kappa^2/48\pi \tag{3.10}$$

which also corresponds to thermal radiation.

All the above trajectories (except the trivial one) have at least one null asymptote. For a simple trajectory without this feature, consider a mirror which is initially static, accelerates for a short time, and moves with constant velocity thereafter. We can join these three parts together in a C^2 way, by choosing

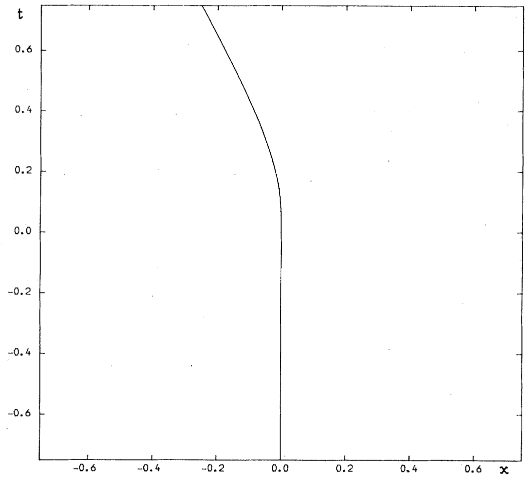

FIG. 4. The mirror trajectory corresponding to (3.8) ($A=1.0$, $B=0.5$).

$$f(v)=\begin{cases} v\ , & v\le 0 \\ av/b+(b-a)\pi^{-1}\sin(\pi/vb)\ , & 0<v<b \\ (2a/b-1)v+b-a\ , & v\ge b \end{cases} \tag{3.11}$$

(see Ref. 11 for a two-mirror analogy of this).

The trajectory is plotted in Fig. (4). We can evaluate the energy density for this motion from (2.14) and find

$$\langle T_{uu}\rangle=\tfrac{1}{12}\pi(b-a)[a\cos(\pi x/b)+\tfrac{1}{2}(b-a)\sin^2(\pi x/b)+b-a][a+(b-a)\cos(\pi x/b)]^{-4}\ . \tag{3.12}$$

This is plotted in Fig. 5. Note the discontinuity at the beginning and end of the acceleration period, which is a consequence of the fact that $f(x)$ is only a C^2 function. The total energy released from this motion is positive.

The Bogoliubov coefficients may be evaluated exactly.[10] We find

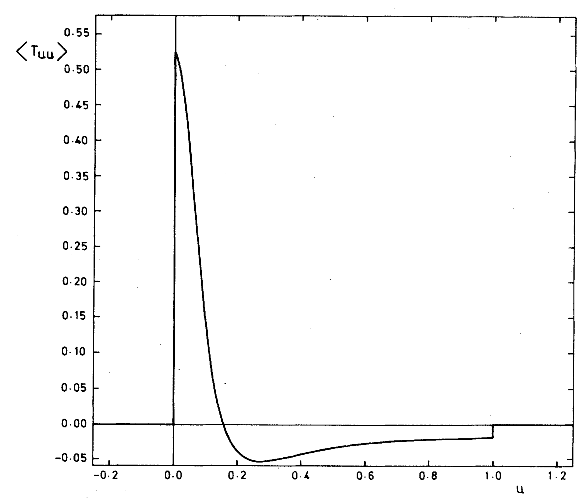

FIG. 5. The energy density for the trajectory in Fig. 4.

$$\left.\begin{matrix}\alpha_{\omega\omega'}\\ \beta_{\omega\omega'}\end{matrix}\right\}=b(2\pi)^{-1}(\omega/\omega')^{1/2}\times\left[\frac{i}{\pi}(\nu-z)^{-1}-\frac{i}{\pi}e^{-i\pi\nu}(\nu+z)^{-1}+\mathbf{J}_\nu(z)-i\mathbf{E}_\nu(z)\right] \tag{3.13}$$

where

$$\nu=(b\omega\mp a\omega')/\pi\ ,\quad z=\pm(b-a)\omega'/\pi$$

and $\mathbf{J}_\nu$ is an Anger function and $\mathbf{E}_\nu$ is a Weber function.

Smoother versions of this trajectory may easily be constructed, but in these cases the Bogoliubov coefficients cannot be constructed exactly. A trajectory which is C^∞, without null asymptotes, for which the Bogoliubov coefficients may be calculated exactly was given by Walker and Davies.[6]

IV. ASYMPTOTIC PROPERTIES OF $|\beta_{\omega\omega'}|^2$

Having obtained the formula for the Bogoliubov coefficients, we may now proceed to investigate their properties, in particular, the high-frequency behavior of $|\beta_{\omega\omega'}|^2$. It is generally believed that this quantity will fall off faster than any power of ω or ω' for large ω or ω', because we expect the production of particles of high energy to be suppressed. However, this statement does not seem to have been proved, so we shall investigate it here.[12]

Define

$$I_1(\alpha)=\int_0^\infty e^{-i\omega x-i\omega' f(x)-\alpha[x+f(x)]}dx\ ,$$
$$I_2(\alpha)=\int_{-\infty}^0 e^{-i\omega x-i\omega' f(x)+\alpha[x+f(x)]}dx\ . \tag{4.1}$$

Since $f(0)=0$ and $f'(x)>0$ for all x, we know that $f(x)>0$ for $x>0$ and $f(x)<0$ for $x<0$. Thus these integrals are convergent for $\alpha\ge 0$. We shall initially consider only asymptotically static trajectories, so that

$$\beta_{\omega\omega'}=-(2\pi)^{-1}(\omega/\omega')^{1/2}[I_1(0)+I_2(0)]\ . \tag{4.2}$$

We shall calculate the large-ω limit first.

Assuming that $e^{-(i\omega'+\alpha)f(x)}$ admits a Taylor series expansion about $x=0$, then

$$I_1(\alpha)=\sum_{r=0}^\infty\left[\frac{\partial^r}{\partial x^r}e^{-(i\omega'+\alpha)f(x)}\right]_{x=0^+}\times\int_0^\infty x^r e^{-(i\omega+\alpha)x}dx\ . \tag{4.3}$$

Now write

$$\int_0^\infty dx\, e^{-(i\omega+\alpha)x}=\sum_{r=0}^\infty A_r\alpha^r \tag{4.4}$$

so that

$$I_1(0)=\sum_{r=0}^\infty\left[\frac{\partial^r}{\partial x^r}e^{-i\omega' f(x)}\right]_{x=0^+}(-1)^r A_r\ . \tag{4.5}$$

And, evaluating the integral in (4.4), $A_r=-(i/\omega)^{r+1}$.

Expanding I_2 in a similar way, we find

$$\beta_{\omega\omega'} = -(2\pi)^{-1}(\omega/\omega')^{1/2}\left[\frac{\partial^r}{\partial x^r}(e^{-i\omega' f(x)})_{x=0^+} - \frac{\partial^r}{\partial x^r}(e^{-i\omega' f(x)})_{x=0^-}\right]\left[\frac{-i}{\omega}\right]^{r+1} . \tag{4.6}$$

Now if $f(x)$ is C^p at $x=0$, then $f^{(r)}(0^+)=f^{(r)}(0^-)$ if and only if $r \le p$, so that the leading term in (4.5) gives

$$\lim_{\omega\to\infty} |\beta_{\omega\omega'}|^2 = \omega'(2\pi)^{-2}\omega^{-(2p+3)}\,|f^{(p+1)}(0^+) - f^{(p+1)}(0^-)|^2 . \tag{4.7}$$

Similarly, we find for the ω' limit

$$\lim_{\omega'\to\infty} |\beta_{\omega\omega'}|^2 = \omega(2\pi)^{-2}\omega'^{-(2p+3)}[f'(0)]^{-(2p+4)}\,|f^{(p+1)}(0^+) - f^{(p+1)}(0^-)|^2 . \tag{4.8}$$

Thus if $f(x)$ is C^∞, we do indeed find that $|\beta_{\omega\omega'}|^2$ falls off faster than any power of ω or ω'. However, the only physical limitation on the continuity of $f(x)$ is that $p \ge 1$ (so that the velocity of the mirror is continuous). Therefore, in general, $|\beta_{\omega\omega'}|^2$ will fall off like $\omega^{-(2p+3)}$ or $\omega'^{-(2p+3)}$.

We can check this limit using $f(x)$ as given by (3.11) with the corresponding $\beta_{\omega\omega'}$ given by (3.13). The asymptotic properties of $\mathbf{J}_\nu - i\mathbf{E}_\nu$ are given in Watson,[13] or may be calculated direct from the integral representation by a saddle-point method. The result is

$$\lim_{\omega\to\infty} |\beta_{\omega\omega'}|^2 = \frac{\pi^2}{4} b^{-6}|a-b|^2\omega'\omega^{-7} ,$$
$$\lim_{\omega'\to\infty} |\beta_{\omega\omega'}|^2 = \frac{\pi^2}{4} b^{-6}|a-b|^2\omega\omega'^{-7} . \tag{4.9}$$

This is in agreement with the results calculated from (4.7) and (4.8) with $p=2$.

If the trajectory has a v asymptote then the above method breaks down as the limits on the integrals in (4.1) become $v_{\min}$ and $v_{\max}$. However, if we define

$$f(v) = \begin{cases} f(v_{\min}) , & v \le v_{\min} \\ f(v) , & v_{\min} < v < v_{\max} \\ f(v_{\max}) , & v \ge v_{\max} \end{cases} \tag{4.10}$$

and note that $f(v_{\min}) = -\infty$, and $f(v_{\max}) = +\infty$, then we may extend the limits to $\pm\infty$ without changing the value of the integral. This procedure does not affect the continuity of $f(x)$ at the origin so the results (4.7) and (4.8) go through unchanged.

V. RELATIONSHIP BETWEEN PARTICLES AND ENERGY

The relationship between particles and energy in quantum field theory is far from clear,[7] and moving-mirror systems provide a good illustration of the problems involved. One would expect to be able to calculate the total energy by summing the energy of each particle in the obvious way (1.1), but in general this naive prescription will fail. For example, the hyperbolic trajectory given in (3.2) radiates particles but no energy, and it is even possible to construct trajectories that radiate negative energy [e.g., (3.6)—see also Ref. 4]. Even in the more physical case of a mirror which is static for $t \le 0$ (3.4), we know that the energy need not be where the particles are. In this section we shall examine the conditions required so that the total energy is given by (1.1). The problem of whether we can detect particles which carry no energy is dealt with in Sec. VI.

Starting with the definition of $T_{\mu\nu}$, and expanding the in modes in terms of the out modes using (2.6), Davies and Fulling[2] obtained

$$\langle T_{uu}\rangle = \int_0^\infty dp\,dp'\left[2\rho_{pp'}\frac{\partial\chi_p}{\partial u}\frac{\partial\chi^*_{p'}}{\partial u} + 2\,\mathrm{Re}\left[\mu_{pp'}\frac{\partial\chi^*_p}{\partial u}\frac{\partial\chi^*_{p'}}{\partial u}\right]\right] , \tag{5.1}$$

where

$$\rho_{pp'} = \int_0^\infty \beta_{\omega p}\beta^*_{\omega p'}d\omega ,$$
$$\mu_{pp'} = \int_0^\infty \alpha^*_{\omega p}\beta_{\omega p'}d\omega . \tag{5.2}$$

We have disregarded a divergent term which is present even when the mirror is always static.

Substituting for the modes from (2.3), and integrating over all u, we obtain

$$E = (2\pi)^{-1}\int_{u_{\min}}^{u_{\max}} du \int_0^\infty dp\,dp'(pp')^{1/2}[\rho_{pp'}e^{-i(p-p')u} + \mathrm{Re}(\mu_{pp'}e^{i(p+p')u})] . \tag{5.3}$$

If the mirror has no u asymptotes, $u_{\min} = -\infty$, $u_{\max} = +\infty$, and the u integration can be performed to give two δ functions. This gives

$$E = \int dp\,p \int d\omega\,|\beta_{\omega p}|^2 \tag{5.4}$$

as stated in Eq. (1.1)

Thus it seems the simple particle-counting prescription for finding the energy will be true in general only if the mirror does not have a u asymptote. If it does, other terms arise in (5.3), and (5.4) will no longer be valid.

We can put Eq. (5.4) on a firmer footing by substituting for $\beta_{\omega p}$ from (2.12):

$$E=(2\pi)^{-2}\int_0^{\infty}dp\,p\int_0^{\infty}d\omega\int_{v_{\min}}^{v_{\max}}dx\,dx'e^{-i\omega(x-x'-i\epsilon)-ip[f(x)-f(x'+i\epsilon)]}\ , \tag{5.5}$$

where we have replaced x' by $x'+i\epsilon$ for convergence, noting that $\mathrm{Im}[f(x'+i\epsilon)]>0$ as $f'(x')>0$.

Performing the p and ω integrations first, we obtain

$$E=i(2\pi)^{-2}\int_{v_{\min}}^{v_{\max}}dx\,dx'(x-x'-i\epsilon)^{-2}[f(x)-f(x'+i\epsilon)]^{-1}\ . \tag{5.6}$$

We next change variables to $x-x'$ and $x+x'$, and use

$$\int d(x-x')=P\int d(x-x')-\tfrac{1}{2}\int_C d(x-x')\ , \tag{5.7}$$

where the contour in (5.7) is taken clockwise around the pole at $x-x'=i\epsilon$, and P denotes the Cauchy principal value. However, the principal value will be zero, as we are integrating an odd function of $x-x'$ between symmetric limits. Evaluating the contour integral in (5.7) gives

$$(24\pi)^{-1}\int_{v_{\min}}^{v_{\max}}dy[f'(y)]^{-3}\{f'''(y)f'(y)-\tfrac{3}{2}[f''(y)]^2\}\ , \tag{5.8}$$

where $y=x+x'$.

Substituting $y=p(u)$ gives

$$E=-(24\pi)^{-1}\int_{-\infty}^{\infty}du[p'''(u)/p'(u)-\tfrac{3}{2}(p''(u)/p'(u))^2]\ . \tag{5.9}$$

Thus by using the Bogoliubov transformation to evaluate the energy of created particles, we have recovered the familiar result (2.14) of Fulling and Davies, which was originally obtained using a regularization procedure (point-separation) on the formally divergent quantity $\langle T_{\mu\nu}\rangle$.

As a final check on consistency (5.9) may be integrated by parts to give[1]

$$-(24\pi)^{-1}p''(u)/p'(u)\,|_{-\infty}^{\infty}+(48\pi)^{-1}\int_{-\infty}^{\infty}du(p''(u)/p'(u))^2\ . \tag{5.10}$$

We can rewrite the boundary term in (5.10) using (2.4) as

$$-(12\pi)^{-1}(1+\dot z)^{1/2}(1-\dot z)^{-1/2}\alpha\,|_{\tau_u=-\infty}^{\tau_u=+\infty}\ , \tag{5.11}$$

where $\alpha=(1-\dot z^2)^{-3/2}\ddot z$ is the proper acceleration. If $\alpha(\pm\infty)\neq 0$ there will either be a u or a v asymptote, with $\dot z\to+1$ or -1, respectively. Hence the boundary term will vanish if there is no u asymptote, and the overall result will be positive as required. If the mirror has a u asymptote, then (5.4) will no longer be true, but (5.9) will still hold (with appropriate limits). Then the boundary term in (5.10) will no longer vanish, and the integrated energy need no longer be positive.

This, then demonstrates that there is complete consistency between the predictions of particle and energy production by the mirror. If the mirror does not have a u asymptote, then the energy radiated will be positive and may be found by adding up the energy of each particle as in (5.4). If the mirror does have a u asymptote, however, then the energy radiated may be negative (or zero) and (5.4) is no longer valid. The examples given in Sec. III all agree with this.

The physical significance of the u asymptote is that the mirror is traveling towards the region under consideration at the speed of light, even if only in the infinite past or future. Thus the mirror is moving at the same velocity as the particles it emits. It is no surprise, therefore, that this is a special case.

Note that (5.4) only says that the total energy will be equal to the sum of the energies of each particle. Indeed, only if $\rho_{pp'}$ is diagonal and $\mu_{pp'}=0$ can we localize the particles to the same region of spacetime as the energy, for in that case (5.1) gives[7]

$$\langle T_{uu}\rangle=\int n(\omega)\omega\,d\omega\ , \tag{5.12}$$

and we must construct wave packets to convert the particle number per mode to a number rate. A trajectory with this property is the thermal one (3.8).

VI. PARTICLE DETECTORS

We have shown that the theory of particle and energy production is entirely self-consistent, and that it is quite possible for particles to be produced even though the mirror is emitting zero or negative energy or that the particles and energy need not be "in the same place." To make sense of this we must answer the questions raised in Sec. I concerning the response of a particle detector.

The treatment given below is based on the detector model of Unruh and Wald.[8] The free Hamiltonian of the detector is

$$H_D=\Omega A^{\dagger}A \tag{6.1}$$

and the interaction Hamiltonian is

$$H_I=\epsilon\int_{\Sigma}\Phi(x)(\psi A+\psi^* A^{\dagger})dx\ , \tag{6.2}$$

where ψ represents the internal field of the detector which can occupy two states denoted by $|\downarrow\rangle$ and $|\uparrow\rangle$. A and $A^{\dagger}$ are the raising and lowering operators for these states, and Ω is the energy difference between them. ϵ is a small coupling constant which we have taken to be time independent. We shall restrict the detector to be a pointlike object, and so ψ and ψ^* will become δ functions of position.

We shall assume the field is initially in the in vacuum state and the detector unexcited so the initial state is $|0\rangle_{\rm in}\otimes|\downarrow\rangle$. Then the final state will be, to first order in perturbation theory,

$$|s_{\infty}\rangle=|0\downarrow\rangle-i\epsilon\int e^{i\Omega t}\psi^*(x)\Phi(x)dx\,dt\,|0\uparrow\rangle\ . \tag{6.3}$$

We are working in the interaction picture, and have included the time dependence of the operator $A^\dagger$. Substituting the decomposition for Φ in terms of the in modes

$$\Phi=\int d\omega(a_\omega\varphi_\omega+a_\omega^\dagger\varphi_\omega^*)\ , \tag{6.4}$$

we obtain

$$|s_\infty\rangle=|0\downarrow\rangle \\ -i\epsilon\int dx\,dt\,e^{+i\Omega t}\psi^*(x,t)\int d\omega\,\varphi_\omega^*(x,t)\,|1_\omega\uparrow\rangle\ . \tag{6.5}$$

Assuming for simplicity the detector to be stationary at $x=0$,[14] we replace $\psi^*(x)$ with $\delta(x)$. The result is

$$|s_\infty\rangle=|0\downarrow\rangle-i\epsilon\int_{u_{\min}}^{u_{\max}}dt\int d\omega\,e^{i\Omega t+i\omega p(t)}\,|1_\omega\uparrow\rangle \\ =|0\downarrow\rangle-i\epsilon(\pi/\Omega)^{1/2}\int d\omega\,\beta_{\omega\Omega}^*\,|1_\omega\uparrow\rangle\ , \tag{6.6}$$

where we have used (2.13). The limits on the t integral will be $u_{\min}$ and $u_{\max}$ as no interference between the modes can occur except between these times.

Thus the probability for excitation is given by

$$P_\uparrow=\pi\epsilon^2\Omega^{-1}\int d\omega\,|\beta_{\omega\Omega}|^2\ . \tag{6.7}$$

This shows that the detector response is indeed a measure of the number of out particles present, whether they appear to carry energy or not.

The answer to the next question, of where does the energy needed for the response come from, is similar to that given by Unruh and Wald[8] for the Rindler case, and the result given here is a good illustration of their mechanism.

The key point to notice is that the detector excitation is accompanied by the emission of an in particle. Thus the energy of the field will go up when detection occurs. This can be demonstrated by considering the expectation value $\langle T_{uu}\rangle_\uparrow$ in the excited state:

$$\langle T_{uu}\rangle_\uparrow=\langle s_\infty|\tfrac{1}{2}(I+C_0)T_{uu}\tfrac{1}{2}(I+C_0)|s_\infty\rangle/P_\uparrow\ , \tag{6.8}$$

where $\frac{1}{2}(I+C_0)$ is the projection operator onto the $|\uparrow\rangle$ state. We have $T_{uu}=[\partial\Phi/\partial u]^2$, where Φ is given by (6.4). This gives

$$\delta\langle T_{uu}\rangle_\uparrow=\langle T_{uu}\rangle_\uparrow-{}_{\rm in}\langle 0|T_{uu}|0\rangle_{\rm in} \\ =2\left|\int d\omega\beta_{\omega\Omega}\partial\varphi_\omega^*/\partial u\right|^2\Big/\int d\omega\,|\beta_{\omega\Omega}|^2\ , \tag{6.9}$$

and so the energy of the field increases when the particle is detected. This can occur because the field was not initially in an eigenstate of energy, and the detection comprises a partial measurement of the field.

It may appear from the above that we are getting energy from nothing. To investigate this, we must expand $|s_\infty\rangle$ to second order in ϵ. There will be two extra terms, both corresponding to no excitation of the detector. These terms are

$$-\tfrac{1}{2}\pi\epsilon^2\Omega^{-1}\int d\omega\,|\beta_{\Omega\omega}|^2|0\downarrow\rangle \\ +\int d\omega\,d\omega' P_{\omega\omega'}|1_\omega 1_{\omega'}\downarrow\rangle\ , \tag{6.10a}$$

where

$$P_{\omega\omega'}=-\tfrac{1}{2}\pi\epsilon^2\Omega^{-1}\alpha_{\omega\Omega}^*\beta_{\omega'\Omega}^*\ . \tag{6.10b}$$

We can calculate the probability that the detector is not excited, $P_\downarrow$. As expected we find that $P_\downarrow=1-P_\uparrow$, where the $P_\uparrow$ came from the cross terms in $\langle s_\infty|s_\infty\rangle$.

Now we can evaluate the expected energy density when no detection has occurred. We obtain

$$\delta\langle T_{uu}\rangle_\downarrow=\int d\omega\,d\omega'(P_{\omega\omega'}\langle 1_\omega 1_{\omega'}|T_{uu}|0\rangle+\text{c.c.})/P_\downarrow\ . \tag{6.11}$$

Thus

$$\delta\langle T_{uu}\rangle_\downarrow=-\epsilon^2\pi\Omega^{-1}\int d\omega\,d\omega'\left[\alpha_{\omega\Omega}^*\beta_{\omega'\Omega}^*\frac{\partial\varphi_\omega}{\partial u}\frac{\partial\varphi_{\omega'}}{\partial u}+\text{c.c.}\right]/P_\downarrow\ . \tag{6.12}$$

We next expand the in modes φ_ω in terms of the out modes using (2.6) and integrate over all u, giving a δ function in frequency. We then find

$$\int du(\langle s_\infty|T_{uu}|s_\infty\rangle-{}_{\rm in}\langle 0|T_{uu}|0\rangle_{\rm in}) \\ =\int du(P_\uparrow\delta\langle T_{uu}\rangle_\uparrow+P_\downarrow\delta\langle T_{uu}\rangle_\downarrow) \\ =\tfrac{1}{2}\epsilon^2\pi\Omega^{-1}\int dp\,p\int d\omega\,d\omega'[2\beta_{\omega\Omega}\beta_{\omega'\Omega}^*(\alpha_{\omega p}^*\alpha_{\omega' p}+\beta_{\omega p}^*\beta_{\omega' p})-\alpha_{\omega\Omega}^*\beta_{\omega'\Omega}^*(\alpha_{\omega' p}\beta_{\omega p}+\alpha_{\omega p}\beta_{\omega' p}) \\ -\alpha_{\omega\Omega}\beta_{\omega'\Omega}(\alpha_{\omega' p}^*\beta_{\omega p}^*+\alpha_{\omega p}^*\beta_{\omega' p}^*)]=-\epsilon^2\pi\int d\omega\,|\beta_{\omega\Omega}|^2\ , \tag{6.13}$$

where we have used the orthogonality properties of the Bogoliubov coefficients.[15]

This term is just the energy absorbed by the detector Ω multiplied by the probability $P_\uparrow$ of its being excited. So the expected field energy in the final state is just the energy contained in the in state, minus the expected energy stored in the detector.

This, then, explains how particles which appear to carry no energy can "exist." The motion of the mirror disturbs the vacuum fluctuations of the field and causes the detector to respond with probability $P_\uparrow$. The energy required for this response is independent of the energy radiated by the mirror and the field energy goes up on detection. One might hope to extract infinite energy by continually repeating the process. This is not so, because the detector will sometimes not respond, with probability $1-P_\uparrow$, and when this occurs the field energy will decrease by just the right amount to ensure energy conservation in the mean.

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[14]Equation (6.5) is invariant under spatial translations $x \rightarrow x + \text{const}$, just as the inner product (2.8) is invariant under time translations.
[15]In the discussion of the second-order terms it was necessary to integrate over u from $-\infty$ to $+\infty$. Thus the result (6.13) will no longer be true when the mirror has a u asymptote, precisely the case where (5.4) also breaks down. Even so, it is still true that the energy of the field goes up when a particle is detected.