Generic instabilities in first-order dissipative relativistic fluid theories

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We consider the stability of a general class of first-order dissipative relativistic fluid theories which includes the theories of Eckart and of Landau and Lifshitz as special cases. We show that all of these theories are unstable in the sense that small spatially bounded departures from equilibrium at one instant of time will diverge exponentially with time. The time scales for these instabilities are very short; for example, water at room temperature and pressure has an instability with a growth time scale of about 10^{-34} seconds in these theories. These results provide overwhelming motivation (we believe) for abandoning these theories in favor of the second-order (Israel) theories which are free of these difficulties.

I. INTRODUCTION

In this paper we investigate the stability of the equilibrium states in a class of first-order theories of relativistic dissipative fluid mechanics. These theories are referred to as "first order" because their entropy currents contain no terms higher than first order in deviations from equilibrium (heat flow, viscous stresses, etc.). The standard theories developed by Eckart[1] and by Landau and Lifshitz,[2] which are presented in most textbooks on gravitational physics,[3,4] are special cases of the class of theories considered here. We analyze the dynamics of small departures of these fluids from their equilibrium states and show that all of the theories predict rapid evolution away from equilibrium. We thus find these theories to be very unstable and consequently unacceptable as reasonable physical theories.

While these theories have existed in the literature for some time, to our knowledge no thorough analyses of their properties have ever been made. It has been known for some time[5] that under special restrictive conditions a parabolic equation may be obtained for the propagation of thermal fluctuations in these theories. This equation was considered problematic (because of its well-known noncausal properties) since one had hoped for a causal, truly relativistic theory. The question of the stability of the equilibrium states in these theories has also been raised.[6] It was shown that instabilities driven by thermal conductivity exist in the Eckart theory, but only for perturbations with length scales much smaller than is physically meaningful (i.e., smaller than the interparticle separation in the fluid). The instabilities which we discuss here occur at every length scale.

Since the theories predict nonsensical behavior for phenomena which should be well within their realm of applicability they must be abandoned. The second-order theories of Israel[7,8] have been shown to be causal, stable, and satisfy reasonable hyperbolic equations.[9] Thus they appear to be a most attractive replacement for the first-order theories.

To analyze the dynamics of the fluids in these theories we have derived the linearized equations which govern the evolution of small perturbations about equilibrium. We study the entire class of theories because the dynamics of these small perturbations vary widely from theory to theory; the theories are not equivalent in this sense.[7] The equations which govern the evolution of the perturbations can be shown to be nonhyperbolic; unfortunately, very little seems to be known about the properties of mixed hyperbolic-parabolic-elliptic systems of equations.[10] It is not clear, for example, to what extent these equations have a well-posed initial-value problem. Furthermore, the analysis of signal propagation to determine the causal properties of the equations is even more complicated than in the dispersive, dissipative electromagnetic case.[11] We have chosen to avoid these difficult problems in favor of one easier to analyze, the stability of the equilibrium states.

We are interested in determining the stability of these theories with respect to a class of "physically acceptable" perturbations away from equilibrium. If the theories had well-understood initial-value problems one would, for example, study the stability of the theory with respect to all initial data having compact spatial support, or perhaps all L^2 (square-integrable) initial data. This sort of initial-data choice embodies the intuitive idea of a local finite-energy change in the system and removes from consideration infinite-energy disturbances with sources at infinity. Since the initial-value problem is not well understood for these theories, an analogous criterion must be found to define the class of acceptable perturbations. We choose to examine only those solutions to the perturbation equations which have the property that on some spacelike surface the solutions are in L^2. One could equally well require them to be smooth functions with compact support or to be in Schwartz space. To determine whether the perturbations are stable, we examine these solutions on successive spacelike surfaces to determine whether they remain bounded.

We carry out this stability analysis for the case of a nongravitating special relativistic fluid in a spatially homogeneous equilibrium state. We consider the stability of these equilibrium states with respect to perturbations which are in L^2 on at least one of the t=constant hyperplanes in the Minkowski spacetime in which these fluids are found. These physically reasonable perturbations have a well-defined spatial Fourier transform since the solution is taken to be in L^2. This fact allows us to easily analyze the dynamics of the physically reasonable solutions. The evolution of each Fourier component is that of a simple exponential plane wave whose frequency is determined from the perturbation equations in the form of a dispersion relation. We show that the dispersion relations always contain exponentially growing modes. Finally, the Fourier components can be reassembled to form the real solutions of the perturbation equations. We show that any solution containing any unstable Fourier component will not remain bounded on successive spacelike surfaces. The L^2 norm of each such solution will in fact diverge exponentially in time. The unstable, exponentially growing Fourier components exist for all values of the wave number $\mathbf{k}$. Furthermore, the predicted exponential growth times are absurdly short; for example, water at room temperature and pressure is predicted to be unstable with a time scale of about 10^{-34} sec.

The details of the analysis described above will be found in the following sections of this paper. In Sec. II we develop a general covariant first-order theory of dissipative fluids which includes the theories of Eckart and of Landau and Lifshitz as special cases. To our knowledge most of the theories contained in this class have never been discussed previously in the literature. We display the general equations of motion for these theories and determine the equilibrium states in this section. In Sec. III we derive the linearized equations of motion for perturbations about an arbitrary equilibrium state. We then solve these equations for exponential plane waves on a homogeneous background state. The dispersion relations for these waves are analyzed and are found to contain growing modes for both transverse and longitudinal perturbations. In Sec. IV we show how the exponential plane waves of Sec. III can be assembled (via a Fourier transform) to form an acceptable physical perturbation. The existence of growing plane-wave perturbations is shown to imply the existence of unstable physical perturbations. Finally we evaluate the growth time scales for these unstable perturbations and find them to be unacceptably short.

II. FIRST-ORDER DISSIPATIVE RELATIVISTIC FLUID MECHANICS

In this section we construct a general theory of dissipative relativistic fluids subject to the constraint that the expression for the entropy current contains no terms of higher than first order in deviations away from equilibrium. The theories of Eckart[1] and Landau and Lifshitz[2] will be seen to be special cases of this general first-order theory. In this section we first derive the equations of motion for the general theory, and then examine the equilibrium states of the theory.

The state of a relativistic dissipative fluid is described by a stress-energy tensor T^{ab} and a number current vector N^a, which obey the conservation laws

$$\nabla_a T^{ab}=0 \tag{1}$$

and

$$\nabla_a N^a=0 , \tag{2}$$

where ∇_a is the covariant derivative compatible with the spacetime metric g_{ab}.

The stress-energy tensor may be decomposed in the following manner:

$$T^{ab}=\rho u^a u^b+(p+\tau)q^{ab}+q^a u^b+q^b u^a+\tau^{ab} , \tag{3}$$

and the number current in a similar fashion,

$$N^a=nu^a+\nu^a . \tag{4}$$

In these expressions u^a is a unit timelike vector field which may be thought of as the four-velocity of the fluid; in the rest frame defined by u^a the energy density is ρ, and the particle number density is n. The tensor q^{ab} is a projection tensor formed from u^a and the metric g^{ab},

$$q^{ab}=g^{ab}+u^a u^b . \tag{5}$$

The additional four fields τ, q^a, ν^a, and τ^{ab} describe the deviations from local equilibrium in the fluid, and are defined to satisfy the following constraints:

$$0=u^a q_a=u^a \nu_a=u^a \tau_{ab}=\tau^a{}_a=\tau_{ab}-\tau_{ba} . \tag{6}$$

The vector field ν^a is a "particle diffusion" current, q^a is the heat flow (energy diffusion current), and τ and τ^{ab} are stresses caused by viscosity.

The energy density ρ and the particle number density n can in principle be measured by an observer moving along the vector field u^a. The entropy per particle s can then be defined by the equilibrium equation of state for the fluid:

$$s=s(\rho,n) . \tag{7}$$

All other thermodynamic variables are then defined by using the first law of thermodynamics. In particular, the temperature T and the pressure p are given by

$$T^{-1}=n\left[\frac{\partial s}{\partial \rho}\right]_n , \tag{8}$$

$$p=-\rho-n^2 T\left[\frac{\partial s}{\partial n}\right]_\rho . \tag{9}$$

To complete the theory, we must specify how τ, q^a, ν^a, and τ^{ab} are determined. Our choices for these variables are based on the need to satisfy the second law of thermodynamics. The total entropy associated with a spacelike surface Σ is obtained by integrating the entropy current vector field over the surface:

$$S(\Sigma)=\int_\Sigma s^a d^3x_a . \tag{10}$$

The second law of thermodynamics requires this total entropy to be a nondecreasing function of time for isolated systems; if Σ' is to the future of Σ, then we require

$$S(\Sigma')-S(\Sigma)=\int \nabla_a s^a d^4x \geq 0 , \tag{11}$$

where the two surface integrals have been converted into a volume integral by Gauss's theorem. If the inequality in Eq. (11) is to hold for all surfaces Σ' to the future of Σ, then the following inequality must also hold:

$$\nabla_a s^a \geq 0 . \tag{12}$$

The second law of thermodynamics is thus embodied locally in Eq. (12). In this paper we will consider a class of theories which includes the theories of Eckart and Landau and Lifshitz as special cases, and whose entropy current contains no terms of higher than first order in the deviations from equilibrium. We set

$$s^a = snu^a + \beta q^a - \Theta \nu^a , \tag{13}$$

where β and Θ are as yet undetermined zeroth-order thermodynamic functions. The divergence of this current can be evaluated using the equations of motion, Eqs. (1), (2), (8), and (9), to yield

$$\begin{aligned}\nabla_a s^a = & -T^{-1}\tau^{ab}\langle \nabla_a u_b \rangle - T^{-1}\tau \nabla_a u^a + (\beta - T^{-1})\nabla_a q^a \\ & + q^a(\nabla_a \beta - T^{-1} u^c \nabla_c u_a) \\ & - \left[\Theta - \frac{\rho+p}{nT} + s \right] \nabla_a \nu^a - \nu^a \nabla_a \Theta , \end{aligned} \tag{14}$$

where the brackets $\langle\ \rangle$ which appear in Eq. (14) have the meaning

$$\langle A_{ab} \rangle = \tfrac{1}{2} q_a^c q_b^d (A_{cd} + A_{dc} - \tfrac{2}{3} q_{cd} q^{ef} A_{ef}) \tag{15}$$

for any second-rank tensor. The simplest way to guarantee that Eq. (14) is consistent with the second law of thermodynamics, Eq. (12), is to require that

$$\beta = T^{-1} , \tag{16}$$

$$\Theta = \frac{\rho+p}{nT} - s , \tag{17}$$

$$\tau = -\zeta \nabla_a u^a , \tag{18}$$

$$q^a = -\kappa T q^{ab} \left[\frac{1}{T} \nabla_b T + u^c \nabla_c u_b \right] , \tag{19}$$

$$\nu^a = -\sigma T^2 q^{ab} \nabla_b \Theta , \tag{20}$$

$$\tau^{ab} = -2\eta \langle \nabla^a u^b \rangle . \tag{21}$$

Using these expressions the divergence of the entropy current may be written in the form

$$T\nabla_a s^a = \frac{\tau^2}{\zeta} + \frac{q^a q_a}{\kappa T} + \frac{\nu^a \nu_a}{\sigma T} + \frac{\tau^{ab}\tau_{ab}}{2\eta} \geq 0 , \tag{22}$$

which is manifestly positive if the four thermodynamic functions ζ, η, κ, and σ are required to be positive. These four functions may be identified as the bulk viscosity (ζ), the shear viscosity (η), the thermal conductivity (κ), and a particle-diffusion constant (σ) of the fluid.

Equations (18)—(21) and the conservation laws, Eqs. (1) and (2), form a complete system of equations for the dynamical variables $(n,\rho,u^a,\tau,q^a,\nu^a,\tau^{ab})$ of the general first-order theory of relativistic dissipative fluids. The gravitational interactions of the fluid may be included by adding Einstein's equation,

$$G_{ab} = 8\pi T_{ab} \tag{23}$$

to the system of equations. The general first-order theory includes as special cases the usual relativistic dissipative fluid theories of Eckart[1,3,4] and Landau and Lifshitz.[2] The theory of Eckart is obtained by setting $\sigma=0$ (and, hence, $\nu^a=0$). In this case the four-velocity of the fluid is equal to the four-velocity of the particles (u^a is parallel to N^a) and hence the particle number flux is zero in the rest frame of the fluid. The theory of Landau and Lifshitz is obtained by taking the opposite extreme limit (recalling that we require $\kappa,\sigma \geq 0$), by choosing $\kappa=0$. In this case $q^a=0$ always, the four-velocity of the fluid is a timelike eigenvector of T^{ab}, and hence the energy flux is always zero in the rest frame of the fluid. The Navier-Stokes-Fourier (NSF) theory of Newtonian dissipative fluids may be obtained by taking the Newtonian limit of our general theory and setting

$$\kappa_{\mathrm{NSF}} = \kappa + \left[\frac{\rho+p}{n} \right]^2 \sigma . \tag{24}$$

Determining the properties of the equilibrium configurations of the general first-order theory is a necessary first step in examining their stability, which we shall study in the following sections. In a state of equilibrium, the entropy of the fluid must not change with time. Equation (11) then implies that the divergence of the entropy current must be zero in this case. Since the divergence is a sum of positive terms [Eq. (22)], each term is required to vanish independently of the others. Thus, the viscous stresses and heat flows must vanish in equilibrium:

$$\tau = q^a = \nu^a = \tau^{ab} = 0 . \tag{25}$$

Equation (25), combined with the defining equations for the dissipative variables, Eqs. (18)—(21), imply the following conditions for equilibrium fluids:

$$\nabla_a u^a = 0 , \tag{26}$$

$$\langle \nabla_a u_b \rangle = 0 , \tag{27}$$

$$q^{ab}(\nabla_b T + T u^c \nabla_c u_b) = 0 , \tag{28}$$

$$q^{ab} \nabla_b \Theta = 0 . \tag{29}$$

When the above equilibrium conditions are imposed, the conservation laws Eqs. (1) and (2) yield the following:

$$u^a \nabla_a n = 0 , \tag{30}$$

$$u^a \nabla_a \rho = 0 , \tag{31}$$

$$q^{ab}[\nabla_b p + (\rho+p) u^c \nabla_c u_b] = 0 . \tag{32}$$

Equations (30) and (31) imply that all of the thermodynamic variables (s,T,p,Θ) must be constant along the integral curves of u^a, since all thermodynamic variables depend only on ρ and n through the equation of state. This in turn implies that the projection tensors appearing in Eqs. (28), (29), and (32) are superfluous, and, in particular, that the thermodynamic variable Θ [defined in Eq. (16)] has vanishing gradient in an equilibrium configuration, and is constant in value.

The conditions which follow from equilibrium [Eqs. (25)–(32)] are independent of the relative values of κ and σ. Thus, an Eckart equilibrium state is identical to a Landau-Lifshitz equilibrium state, or any other first-order theory's equilibrium state. Indeed, the equilibrium states found here are also identical to those of the general second-order (Israel) theory of dissipative fluids.[7]

III. LINEAR PERTURBATIONS

In this section we study fluid states which are nearly in equilibrium. Our particular goal is to determine whether the equilibrium states of the general first-order theory are stable or unstable. We first obtain the equations governing linear perturbations about an equilibrium state. Next, we wish to examine the properties of the physically acceptable solutions to these perturbation equations for the case of a homogeneous background equilibrium state. Since physically acceptable perturbations admit (by our definition) spatial Fourier transforms on at least one spacelike surface, it is most convenient to first study the properties of the exponential plane-wave solutions. The resulting dispersion relations are examined for growing, unstable, modes. We find that at least one transverse and one longitudinal mode are always unstable, except in the case when $\kappa=0$ (Landau-Lifshitz theory[2]). Finally, we examine plane waves on a homogeneous but moving background equilibrium state, and find that all first-order theories (including the Landau-Lifshitz theory) are unstable when examined on such spacelike surfaces. A more lengthy discussion of stability will follow in Sec. IV.

The perturbations about equilibrium will be analyzed in the Eulerian framework in order to avoid the gauge ambiguities inherent in the Lagrangian approach.[6,12] The difference between the actual nonequilibrium value of a field Q at a given spacetime point and the value of Q in the background, fiducial equilibrium state will be denoted by δQ. Fields which do not include the prefix δ (e.g., $n,\rho,u^a,\ldots$) will henceforth refer to the fiducial equilibrium state which is assumed to satisfy the conditions outlined in Sec. II. For the purposes of deriving the equations of motion for the perturbations, the fiducial equilibrium state considered here is not limited in any way, and in particular could include rapid rotation and/or strong gravitational fields. We will, however, consider only perturbations which leave the gravitational field fixed; i.e., $\delta g_{ab}=0$. This approximation is appropriate in any situation where gravity plays no role (e.g., special relativity), and also for short-wavelength perturbations of any equilibrium state.[6]

We will assume that the perturbation variables (δQ) are small enough so that their evolution is adequately described by the equations of motion [Eqs. (1), (2), and (18)–(21)] linearized about the background equilibrium state. After linearization in the perturbation variables, these equations become

$$\nabla_a \delta T^{ab}=0\ , \tag{33}$$

$$\nabla_a \delta N^a=0\ , \tag{34}$$

$$\delta\tau=-\zeta\nabla_a\delta u^a\ , \tag{35}$$

$$\delta q^a=-\kappa T q^{ab}\left[\nabla_b\left[\frac{\delta T}{T}\right]+u^c\nabla_c\delta u_b+\delta u^c\nabla_c u_b\right]\ , \tag{36}$$

$$\delta\nu^a=-\sigma T^2 q^{ab}\nabla_b\delta\Theta\ , \tag{37}$$

$$\delta\tau^{ab}=-2\eta\langle\nabla^a\delta u^b+\delta u^a u^c\nabla_c u^b\rangle\ , \tag{38}$$

where the perturbed stress-energy tensor and perturbed particle number current are given by

$$\delta T^{ab}=(\rho+p)(\delta u^a u^b+u^a\delta u^b)+\delta\rho u^a u^b+(\delta p+\delta\tau)q^{ab} + u^a\delta q^b+u^b\delta q^a+\delta\tau^{ab}\ , \tag{39}$$

$$\delta N^a=\delta n u^a+n\delta u^a+\delta\nu^a\ . \tag{40}$$

The derivative which appears in these equations is the covariant derivative compatible with the background spacetime metric tensor g_{ab}; spacetime indices are raised and lowered using g_{ab}, e.g., $\delta u_a=g_{ab}\delta u^b$. The perturbation variables satisfy a number of constraints which follow from linearizing Eq. (6):

$$0=u^a\delta q_a=u^a\delta\nu_a=u^a\delta\tau_{ab}=u^a\delta u_a=\delta\tau_{ab}-\langle\delta\tau_{ab}\rangle\ . \tag{41}$$

We now consider solutions of the perturbation equations for a general first-order theory subject to the following restrictions.

(1) The background, fiducial, equilibrium state is assumed to be homogeneous in space and the background spacetime is assumed to be flat Minkowski space, so that all background field variables have vanishing gradients.

(2) We look only for exponential plane-wave solutions to the perturbation equations,

$$\delta Q=\delta Q_0\exp(ikx+\Gamma t)\ , \tag{42}$$

where δQ_0 is constant and t and x are two of the orthonormal coordinates on Minkowski space. We will, at first, consider an equilibrium background state in which the fluid is at rest, so that

$$u^a\partial_a=\partial_t\ . \tag{43}$$

Given these restrictions, the set of perturbation equations takes the form

$$M^A_B\delta Y^B=0\ , \tag{44}$$

where δY^B represents the list of fields which describe the perturbation of the fluid, and M^A_B is the 17×17 complex-valued matrix which describes the linearized equations of motion. The index B runs over the 17 perturbation variable fields, while the index A runs over the 17 equations of motion. The system matrix M^A_B takes on a particularly simple form when one chooses the following set of perturbation variables:

$$\delta Y^B=\{\delta\rho,\delta n,\delta u^x,\delta\tau,\delta q^x,\delta v^x,\delta\tau^{xx},\delta u^y,\delta q^y,\delta\tau^{xy},\delta u^z,\delta q^z,\delta\tau^{xz},\delta v^y,\delta v^z,\delta\tau^{yz},\delta\tau^{yy}-\delta\tau^{zz}\}\ . \tag{45}$$

With this set of variables, the matrix $\mathbf{M}$ takes a block-diagonal form as follows:

$$\mathbf{M}=\begin{bmatrix}\mathbf{Q} & 0 & 0 & 0\\ 0 & \mathbf{R} & 0 & 0\\ 0 & 0 & \mathbf{R} & 0\\ 0 & 0 & 0 & \mathbf{I}\end{bmatrix}\ . \tag{46}$$

The submatrices $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{I}$ are defined as follows:

$$\mathbf{Q}=\begin{bmatrix} 0 & \Gamma & ink & 0 & 0 & ik & 0\\ \Gamma & 0 & i(\rho+p)k & 0 & ik & 0 & 0\\ i\left[\frac{\partial p}{\partial\rho}\right]_n k & i\left[\frac{\partial p}{\partial n}\right]_\rho k & (\rho+p)\Gamma & ik & \Gamma & 0 & ik\\ 0 & 0 & ik & \frac{1}{\zeta} & 0 & 0 & 0\\ \frac{i}{T}\left[\frac{\partial T}{\partial\rho}\right]_n k & \frac{i}{T}\left[\frac{\partial T}{\partial n}\right]_\rho k & \Gamma & 0 & \frac{1}{\kappa T} & 0 & 0\\ iT\left[\frac{\partial\Theta}{\partial\rho}\right]_n k & iT\left[\frac{\partial\Theta}{\partial n}\right]_\rho k & 0 & 0 & 0 & \frac{1}{\sigma T} & 0\\ 0 & 0 & ik & 0 & 0 & 0 & \frac{3}{4\eta}\end{bmatrix}, \tag{47}$$

$$\mathbf{R}=\begin{bmatrix}(\rho+p)\Gamma & \Gamma & ik\\ \Gamma & \frac{1}{\kappa T} & 0\\ ik & 0 & \frac{1}{\eta}\end{bmatrix}, \tag{48}$$

and $\mathbf{I}$ is the 4×4 unit matrix. These equations immediately imply that the four variables δv^y, δv^z, $\delta\tau^{yz}$, and $\delta\tau^{yy}-\delta\tau^{zz}$ vanish identically.

There will exist exponential plane-wave solutions of Eq. (44) whenever Γ and k have values which satisfy the dispersion relation

$$\det\mathbf{M}=0\ . \tag{49}$$

The determinant of $\mathbf{M}$ is simply the product of the determinants of its diagonal blocks,

$$\det\mathbf{M}=(\det\mathbf{Q})(\det\mathbf{R})^2\ . \tag{50}$$

The roots of Eq. (50) are simply the collection of roots obtained by setting the determinants of $\mathbf{Q}$ and $\mathbf{R}$ separately to zero.

The determinant of the matrix $\mathbf{R}$ is given by

$$-\eta\kappa T\det\mathbf{R}=\kappa T\Gamma^2-(\rho+p)\Gamma-\eta k^2=0\ . \tag{51}$$

This can be solved for Γ to yield

$$\Gamma_\pm=\frac{1}{2\kappa T}\{(\rho+p)\pm[(\rho+p)^2+4\eta\kappa Tk^2]^{1/2}\}\ . \tag{52}$$

These roots are referred to as transverse modes, since the matrix $\mathbf{R}$ involves the components of the perturbation variables which are orthogonal to the direction of spatial variation (x) of a disturbance. The frequencies of these transverse modes [Eq. (52)] are purely real for real wave numbers k, and hence the modes do not actually propagate. An observer at a fixed coordinate x would observe only a monotonically decaying or growing perturbation. The existence of the positive real root (Γ_+) implies the existence of a growing mode, and hence an instability in the fluid (except in the case $\kappa=0$). Inspection of Eq. (52) shows that the root Γ_+ is in fact positive for all real wave numbers k, and hence the fluid is unstable to a growing transverse mode for perturbations of all wavelengths. In the exceptional case $\kappa=0$ (which corresponds to the Landau-Lifshitz theory) the dispersion relation for transverse modes reduces to

$$\Gamma=-\eta k^2/(\rho+p)\ . \tag{53}$$

Since δq^y (and δq^z) are identically zero if $\kappa=0$, there are fewer transverse modes in this exceptional case, and hence only a single root for Γ. Note that Γ is again purely real, so that the transverse modes are nonpropagating, but now the single root for Γ is negative, and hence represents a decaying mode. At first sight it thus appears that the theory of Landau and Lifshitz escapes the instability which occurs in all other first-order theories; however, one should be suspicious of the way the stability appears. If one takes a generic first-order theory ($\kappa\neq 0$) and then approaches the Landau-Lifshitz theory by reducing $\kappa\to 0$, the positive real part of the frequency in Eq. (52) diverges rather than going smoothly to zero. Thus, as one approaches the Landau-Lifshitz theory as a limit, the characteristic growth time of the unstable mode, Γ^{-1}, approaches zero; the instability becomes stronger and faster. The theory of Landau and Lifshitz is thus a rather peculiar singular limit. We will investigate its stability more fully when we examine a nonrest background state at the end of this section.

The determinant of the matrix $\mathbf{Q}$ is a fourth-order real polynomial $F(\Gamma)$ in the frequency Γ:

$$
\begin{aligned}
F(\Gamma)&=\tfrac{4}{3}\eta\zeta\kappa\sigma T^2\det\mathbf{Q}\\
&=\kappa T\Gamma^4+\left[\kappa\sigma T^3\left[\frac{\partial\Theta}{\partial n}\right]_\rho k^2-(\rho+p)\right]\Gamma^3\\
&\quad+\left\{\kappa T\left[\left[\frac{\partial p}{\partial\rho}\right]_s+\left[\frac{\partial\Theta}{\partial s}\right]_p\right]-\sigma T^2(\rho+p)\left[\frac{\partial\Theta}{\partial n}\right]_\rho-(\zeta+\tfrac{4}{3}\eta)\right\}k^2\Gamma^2\\
&\quad+\left\{\kappa\sigma T^3\left[\frac{\partial\Theta}{\partial n}\right]_\rho\left[\frac{\partial p}{\partial\rho}\right]_\Theta k^2-(\rho+p)\left[\frac{\partial p}{\partial\rho}\right]_s-(\zeta+\tfrac{4}{3}\eta)k^2\left[\kappa\left[\frac{\partial T}{\partial\rho}\right]_n+\sigma T^2\left[\frac{\partial\Theta}{\partial n}\right]_\rho\right]\right\}k^2\Gamma\\
&\quad+\left\{-(\tfrac{4}{3}\eta+\zeta)\kappa\sigma T^2\left[\frac{\partial\Theta}{\partial n}\right]_\rho\left[\frac{\partial T}{\partial\rho}\right]_\Theta k^2+\left[\kappa+\left[\frac{\rho+p}{n}\right]^2\sigma\right]T\left[\frac{\partial p}{\partial\rho}\right]_s\left[\frac{\partial\Theta}{\partial s}\right]_p\right\}k^4=0\ .
\end{aligned}
\tag{54}
$$

The four complex roots of Eq. (54) are dispersion relations for the longitudinal modes of the fluid, since they involve the components of the perturbation variables parallel to the direction of propagation (or the direction of spatial variation in the case of nonpropagating modes). Purely real solutions of Eq. (54) correspond to growing or decaying nonpropagating modes, while complex roots are propagating modes which grow or decay when the real part of the root is, respectively, positive or negative.

The qualitative locations of these roots may be determined for fluids which satisfy the perfect-fluid stability conditions:

$$\left[\frac{dp}{d\rho}\right]_s\geq 0\ , \tag{55}$$

$$\left[\frac{\partial\rho}{\partial s}\right]_p\left[\frac{\partial p}{\partial s}\right]_\Theta\geq 0\ . \tag{56}$$

These conditions and the thermodynamic identities developed in Ref. 5, Sec. III(c), imply that the value of the polynomial $F(\Gamma)$ is negative at $\Gamma=0$:

$$F(0)\leq 0\ . \tag{57}$$

Since the coefficient of the Γ^4 term in F is non-negative, this implies the existence of at least two real roots of Eq. (54), one positive and one negative (except in the special cases $\kappa=0$ and $k=0$). The positive real root, which exists for all $k\neq 0$ and all $\kappa\neq 0$, represents a growing but nonpropagating longitudinal mode in the fluid. The special case where $k=0$ (a spatially homogeneous perturbation) is also unstable for all $\kappa\neq 0$; in this case the only nonzero root of Eq. (54) is

$$\Gamma=(\rho+p)/\kappa T\ . \tag{58}$$

Note that, as in the transverse case, as the limit $\kappa\to 0$ is taken, the positive root diverges, and the growth time of the unstable mode (Γ^{-1}) approaches zero.

Now consider the special case when $\kappa=0$ (the theory of Landau and Lifshitz). The longitudinal dispersion relations in this instance are the roots of a cubic equation for Γ, which can be obtained by setting $\kappa=0$ in Eq. (54). This equation has the form

$$G(\Gamma)=A_3\Gamma^3+A_2\Gamma^2+A_1\Gamma+A_0=0\ ; \tag{59}$$

using the perfect-fluid stability conditions [Eqs. (55) and (56)] and the thermodynamic identities given in Ref. 5, it follows that all the A_i are negative. Consequently there are no positive real roots of G. The complex roots may be analyzed as follows. Divide Γ into its real and imaginary parts, $\Gamma=\Gamma_R+i\Gamma_I$, and write out the real and imaginary parts of Eq. (59). These may be manipulated to yield the following equation for Γ_R:

$$8A_3{}^2\Gamma_R{}^3+8A_2A_3\Gamma_R{}^2+2(A_1A_3+A_2{}^2)\Gamma_R+A_1A_2-A_0A_3=0\ . \tag{60}$$

The combination $A_1A_2-A_0A_3$ may be shown, using Eqs. (55) and (56), to be positive. The coefficients of each power of Γ_R in Eq. (60) are thus all positive, and consequently there exist no positive roots for Γ_R. The longitudinal modes in the theory of Landau and Lifshitz are thus all damped; there is no instability for exponential plane waves in the rest frame of the fluid.

In a generic situation, there will not exist a spacelike surface which is everywhere orthogonal to the four-velocity of the fluid, u^a. This is the case whenever there is any rotational motion within the fluid. In order to determine the stability of the fluid in a general equilibrium state, therefore, it is necessary to consider the solutions to the linear perturbation equations which are physically acceptable with respect to surfaces which are not comoving with the fluid. In the comoving case we limited our attention to exponential plane-wave solutions, and by extension any solution having a well-defined Fourier transform on some comoving $t=\text{const}$ surface. In the non-comoving case, by analogy, we consider exponential plane-wave solutions defined on the noncomoving surfaces. We do this by Lorentz-transforming to a frame moving along the x axis (in the direction of propagation of the linear waves) with velocity v. Since the general theory with $\kappa\neq 0$ is unstable even in the fluid rest frame, we restrict the discussion here to the special case $\kappa=0$, the theory of Landau and Lifshitz. Similar conclusions can be reached for the general theory with $\kappa\neq 0$, and for Lorentz boosts in other directions. The form of the exponential plane-wave solution will now be [cf. Eq. (42)]

$$\delta Q=\delta Q_0 \exp(i\tilde{k}\tilde{x}+\tilde{\Gamma}\tilde{t}) , \tag{61}$$

where $\tilde{t}$ and $\tilde{x}$ are related to t and x by

$$\tilde{t}=\gamma t-v\gamma x , \tag{62}$$

$$\tilde{x}=-v\gamma t+\gamma x , \tag{63}$$

where

$$\gamma=(1-v^2)^{-1/2} . \tag{64}$$

The corresponding transformation on the wave four-covector $(-i\Gamma,k)$ is

$$k=\gamma\tilde{k}+iv\gamma\tilde{\Gamma} , \tag{65}$$

$$\Gamma=\gamma\tilde{\Gamma}-iv\gamma\tilde{k} . \tag{66}$$

The dispersion relations for the exponential plane waves on the boosted surface may be obtained by substituting these transformations [Eqs. (65) and (66)] into the comoving dispersion relations, Eqs. (51) and (54), with κ set equal to zero for the Landau-Lifshitz theory.

The boosted dispersion relation for the transverse modes is

$$\gamma v^2\eta\tilde{\Gamma}^2-[(\rho+p)+2i\gamma v\eta\tilde{k}]\tilde{\Gamma}-\gamma\eta\tilde{k}^2+i(\rho+p)v\tilde{k}=0 . \tag{67}$$

The roots of Eq. (67) are always complex (except when $\tilde{k}=0$); the real parts of the roots satisfy the following as a consequence of Eq. (67):

$$\tilde{\Gamma}_{R1}+\tilde{\Gamma}_{R2}=\frac{\rho+p}{\gamma v^2\eta}>0 , \tag{68}$$

$$\tilde{\Gamma}_{R1}\tilde{\Gamma}_{R2}=-\left[\Gamma_{I1}-\frac{\tilde{k}}{v}\right]^2\leq 0 . \tag{69}$$

Equations (68) and (69) imply that exactly one of the two transverse modes is unstable, except in the even-more-special case when $\eta=0$ as well as κ. In this case ($\eta=0$) there is only a simple undamped mode ($\tilde{\Gamma}_R=0$), and so we must examine the longitudinal modes in the boosted frame to determine the stability of the theory.

The dispersion relation for the longitudinal modes in a Lorentz boosted frame is obtained from Eq. (54) by substituting in Eqs. (65) and (66). The result is a complex fifth-order polynomial in $\tilde{\Gamma}$, which we do not display in its full generality. If one sets $\tilde{k}=0$ (spatially homogeneous perturbation in the boosted frame) then the dispersion relation reduces to a real quadratic equation for $\tilde{\Gamma}$:

$$\sigma T(\zeta+\tfrac{4}{3}\eta)\left[\frac{\rho+p}{n^2}\right]\left[\frac{\partial\Theta}{\partial s}\right]_\rho\gamma^2v^4\tilde{\Gamma}^2+\left[\zeta+\tfrac{4}{3}\eta-\sigma T\left[\frac{\rho+p}{n}\right]^2\left\{\left[\frac{\partial\Theta}{\partial s}\right]_p\left[1-\left[\frac{\partial p}{\partial\rho}\right]_s v^2\right]-\frac{1}{n^2T^2}\left[\frac{\partial\rho}{\partial p}\right]_s\left[\frac{\partial p}{\partial s}\right]_\rho^2\right\}\right]\gamma v^2\tilde{\Gamma}-(\rho+p)\left[1-\left[\frac{\partial p}{\partial\rho}\right]_s v^2\right]=0 . \tag{70}$$

When the conditions for the stability of a perfect fluid, Eqs. (55) and (56), are satisfied, then there are two positive real roots to Eq. (70). Thus both nontrivial longitudinal modes in the $\tilde{k}=0$ limit are growing, unstable modes. These modes are unstable as long as any one of the dissipation coefficients (ζ,η,σ) is nonzero. A tedious argument can be made to demonstrate that there exists at least one growing, unstable longitudinal mode in the general case when $\tilde{k}\neq 0$, provided only that one of (ζ,η,σ) is nonzero.

In summary, we have examined exponential plane-wave perturbations within our general theory of first-order relativistic dissipative fluid mechanics, and found that there exist growing transverse and longitudinal modes whenever any one of the dissipation coefficients $(\zeta,\eta,\kappa,\sigma)$ is nonzero. If $\kappa\neq 0$, then there are growing modes in a frame in which the background equilibrium state is at rest, while the theory of Landau and Lifshitz ($\kappa=0$) contains only decaying modes in such a frame. When analyzed in a frame in which the background equilibrium state is in motion, all of the theories examined contain growing modes.

IV. DISCUSSION

In this section we show how a physically acceptable solution to the perturbation equations may be assembled from the plane-wave normal mode solutions discussed in Sec. III. The existence of exponentially growing plane-wave modes is shown to imply that the physically acceptable solutions grow without bound as well. Finally, we evaluate the timescales on which these instabilities grow in these theories, and find them to be unacceptably short.

Let (t,x) be Cartesian coordinates on Minkowski spacetime, chosen so that the surface $t=0$ is the initial surface on which a physically acceptable solution to the perturbation equations, δY^A, is square integrable, i.e., is in L^2. This solution admits a spatial Fourier transform on this surface which we denote as $\delta\hat{Y}^A$, defined as

$$\delta\hat{Y}^A(\mathbf{k})=(2\pi)^{-3/2}\int \delta Y^a(0,\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x \ , \tag{71}$$

where the index A runs over the 17 perturbation variables and the integration is performed over all space. The Fourier transform may be further decomposed into the normal-mode vectors which satisfy Eq. (44) as follows:

$$\delta\hat{Y}^A(\mathbf{k})=\sum_\alpha c^\alpha(\mathbf{k})\delta\hat{Y}^A_\alpha(\mathbf{k}) \ , \tag{72}$$

where $\delta\hat{Y}^A_\alpha(\mathbf{k})$ are the components of the αth normal mode, and the $c^\alpha(\mathbf{k})$ are coefficients which may depend on $\mathbf{k}$. The time evolution of each normal mode is a simple exponential function whose frequency $[\Gamma_\alpha(\mathbf{k})]$ is given by the dispersion relations derived in Sec. III. Consequently the time evolution of the physically acceptable perturbations is given by

$$\delta Y^A(t,\mathbf{x})=(2\pi)^{-3/2}\int e^{i\mathbf{k}\cdot\mathbf{x}}\sum_\alpha c^\alpha(\mathbf{k})\delta\hat{Y}^A_\alpha(\mathbf{k}) \times\exp[\Gamma_\alpha(\mathbf{k})t]d^3k \ , \tag{73}$$

for as long as the perturbation remains physically acceptable.

Now consider the L^2 norm of the perturbation functions on successive spacelike surfaces:

$$||\delta\mathbf{Y}||^2(t)\equiv\sum_A\int |\delta Y^A(t,\mathbf{x})|^2d^3x \ . \tag{74}$$

Using Parseval's theorem, this can be converted to an expression involving the components of the Fourier transform:

$$||\delta\mathbf{Y}||^2(t)=\sum_A\int\left|\sum_\alpha c^\alpha(\mathbf{k})\delta Y^A_\alpha(\mathbf{k})\exp[\Gamma_\alpha(\mathbf{k})t]\right|^2 d^3k \ . \tag{75}$$

Next, consider a subset Ω of $\mathbf{k}$ space where at least one of the normal-mode frequencies, say, $\Gamma_0(\mathbf{k})$, has positive real part, i.e., $\mathrm{Re}[\Gamma_0(\mathbf{k})]\geq\Gamma_{\min}>0$. Let us assume as well that Γ_0 is the frequency of the mode having the largest $\mathrm{Re}[\Gamma_0(\mathbf{k})]$ on Ω. Let Ω' be a subset of Ω where $\mathrm{Re}[\Gamma_0(k)]$ is at least ϵ (for some positive ϵ) greater than the real part of the frequency of any other mode: i.e., $\mathrm{Re}[\Gamma_0(\mathbf{k})]-\mathrm{Re}[\Gamma_{\alpha\neq 0}(\mathbf{k})]\geq\epsilon>0$. We restrict the domain of integration in Eq. (75) to this subset Ω' to obtain the inequality

$$||\delta\mathbf{Y}||^2(t)\geq\sum_A\int_{\Omega'}|c_0(\mathbf{k})\delta Y^A_0(\mathbf{k})+f^A(\mathbf{k},t)|^2 \times\exp\{2\,\mathrm{Re}[\Gamma_0(\mathbf{k})]t\}d^3k \ , \tag{76}$$

where we have set

$$f^A(\mathbf{k},t)=\sum_{\alpha\neq 0}c^\alpha(\mathbf{k})\delta Y^A_\alpha(\mathbf{k})\exp\{[\Gamma_\alpha(\mathbf{k})-\Gamma_0(\mathbf{k})]t\} \ . \tag{77}$$

It follows that the magnitude of $f^A(\mathbf{k},t)$ on the domain Ω' decreases exponentially with time at least as rapidly as $\exp(-\epsilon t)$. Since the normal-mode vectors δY^A_α are linearly independent it follows that the norm of the vector $c^0(\mathbf{k})\delta Y^A_0(\mathbf{k})+f^A(\mathbf{k},t)$ will never approach zero for $\infty\geq t\geq 0$ unless $c^0(\mathbf{k})=0$; i.e.,

$$\sum_A |c^0(\mathbf{k})\delta Y^A_0(\mathbf{k})+f^A(\mathbf{k},t)|^2\geq g^2(\mathbf{k})>0 \ .$$

By restricting attention to sufficiently late times we can clearly make $g^2(\mathbf{k})$ as close to $\sum_A |c^0(\mathbf{k})\delta Y^A_0(\mathbf{k})|^2$ as we wish. Finally, it follows that the norm of δY^A must be bounded below by

$$||\delta\mathbf{Y}||^2(t)\geq[\exp(2\Gamma_{\min}t)]\int_{\Omega'}g^2(\mathbf{k})d^3k \ . \tag{78}$$

Thus we see that the existence of an unstable plane-wave mode implies that real physically acceptable perturbations will grow exponentially with time as well. The only perturbations which will not grow exponentially in time are those which are chosen to contain no growing Fourier components. These special perturbations are a set of measure zero in the space of all physically acceptable perturbations.

It might be argued that these instabilities are not sufficient grounds on which to condemn the first-order theories; perhaps one should use the first-order theory, but simply discard runaway, growing solutions. The classical theory of electromagnetic radiation reaction, using the Abraham-Lorentz[13] equation of motion for the electron, suffers from runaway solutions, and yet the equation is still considered to be a useful and accurate approximation. We do not believe that this is a reasonable or compelling analogy for the following reasons. First, in the fluid case, the Newtonian theory (Navier-Stokes-Fourier) of dissipative fluids is stable; it seems reasonable to expect that a relativistic theory could be found which is stable. Second, in the fluid case, it is known that the reasonably simple extension of the theory to second order (i.e., to the Israel theory of fluids[7,8]) will yield a stable, causal theory.[9] Since the Israel theory possesses the desired properties, it does not seem worthwhile to expend much effort in attempting to save the first-order theory by some *ad hoc* restriction on solutions of the equations of motion, as one does in the case of the Abraham-Lorentz equation.[14]

Finally, it is important to determine the time scales associated with these instabilities. If, for all astrophysically imaginable conditions, the e-folding time for growth of the instability were much longer than the age of the universe, then the instabilities would be of only pedagogic interest; the first-order theory would be a reasonable approximation to use on shorter time scales. It is easiest to determine the time scales for the transverse modes. Equation (52) shows that Γ_+, the frequency of the growing transverse mode, is bounded below by

$$\Gamma_+\geq\frac{(\rho c^2+p)c^2}{\kappa T} \ , \tag{79}$$

where the speed of light c has been explicitly reinserted into the expression. The characteristic time scale is given by $\tau=\Gamma_+{}^{-1}$, hence

$$\tau \le \frac{\kappa T}{(\rho c^2+p)c^2} . \tag{80}$$

This is an extremely short time scale; as an example, for water at a temperature of 293 K and pressure of 1 bar, one finds

$$\tau \lesssim 2\times 10^{-34}\ \text{sec} . \tag{81}$$

This time scale is ridiculously short; it clearly indicates that the first-order theory can never be used as a reasonable approximation. Another interesting point concerning the time scale is that in a more relativistic fluid (higher temperature), the unstable modes grow more slowly. Notice also that in the Newtonian limit ($c \to \infty$), the growth time scale goes to zero rather than infinity; the Navier-Stokes-Fourier theory is thus a singular limit of the relativistic first-order theory. In the special case of the Landau-Lifshitz theory, $\kappa=0$, and the time-scale estimates of Eqs. (79)—(81) are thus inapplicable. Equations (67)—(69) imply that in this case

$$\tau \lesssim \frac{\gamma v^2 \eta}{(\rho c^2+p)c^4} . \tag{82}$$

This is again an exceedingly short time scale for nearly Newtonian fluids; again the Navier-Stokes-Fourier theory is a singular limit of the relativistic first-order theory.

In conclusion, we have shown that the "standard," first-order theories of dissipative relativistic fluids are unstable; and further, that they are unstable on microphysical time scales. On this basis, we feel that these theories should be discarded in favor of the second-order theory developed by Israel.[7–9] Israel's second-order theory contains as special cases the unstable first-order theories; however, it also contains new parameters which kinetic theory calculations have shown to be nonzero.[8,15,16] If these new parameters are chosen to lie within the range which results in stable equilibria, then Israel's theory has been shown[9] to possess the following additional desirable properties: (i) the equations of motion for linear perturbations form a symmetric hyperbolic system,[10] and (ii) the characteristic velocities are subluminal. Israel's theory thus seems to be an acceptable relativistic theory, while the first-order theories of Eckart and Landau and Lifshitz are highly pathological.

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