PHYSICAL REVIEW D VOLUME 31, NUMBER 4 15 FEBRUARY 1985

Feynman rules for finite-temperature Green's functions in an expanding universe

Gordon Semenoff and Nathan Weiss

Department of Physics, University of British Columbia, Vancouver, British Columbia, Canada V6T 2A6

(Received 18 July 1984)

We derive Feynman rules for evaluating the real-time Green's functions of a scalar field theory in a Robertson-Walker universe under the circumstance that at some initial time t_0 the system was in thermal equilibrium with density matrix $\exp[-\beta H(t_0)]$, where $H(t_0)$ is the Hamiltonian at time t_0.

I. INTRODUCTION

The subject of phase transitions in the very early universe has received much attention in recent years. Of interest are the confining-deconfining transition[1] of QCD at a temperature $T\sim 150$ MeV, the Weinberg-Salam $SU(2)\times U(1)$ symmetry restoration[2] at $T\sim 100$ GeV, and the grand-unified-theory (GUT) transition[3] at $T\sim 10^{16}$ GeV. The study of their physical effects on the universe requires not only a detailed knowledge of how field theories behave in a hot, dense environment but also a knowledge of the nonequilibrium properties of these systems as they pass through a phase transition. If we suppose that the system was in thermal equilibrium at some time *prior* to the phase transition, then the *deviation* from equilibrium as the universe expands is uniquely determined by the quantum (Heisenberg) evolution equations for the field in the curved background space-time of an expanding universe. If the initial state of the system is specified as a thermal density matrix, the density matrix at any later time is uniquely determined by these equations. In particular, it is possible (at least in principle) to follow the system through its critical point. In this paper, we show how to evaluate the real-time Green's functions for an interacting scalar field theory in a background Robertson-Walker (RW) universe subject to the condition of thermal equilibrium at some initial time t_0 (i.e., we assume that the density matrix at time t_0 is $\exp[-\beta H(t_0)]$ where $H(t_0)$ is the Hamiltonian at time t_0). These results are applied to the evolution equations for the Higgs field in a hot expanding universe in the following paper.[4]

The subject of field quantization in curved space-time has been studied extensively.[5] Both the choice of a vacuum state and the definition of a particle are ambiguous. In particular the vacuum state (and thus the vacuum Green's functions) depend crucially on the choice of coordinates. Even in flat space-time, the vacuum seen by quantizing in inertial coordinates appears as a thermal density matrix to a uniformly accelerating observer who quantized in Rindler coordinates.[6] Thus if a field is quantized in an expanding universe, the vacuum state will depend on the choice of coordinates. Furthermore, if the system is chosen to be in a vacuum state at some time t_0 [i.e., the ground state of the Hamiltonian $H(t_0)$], it will *not,* in general, be in an eigenstate of the Hamiltonian at a later time t. Our condition of thermal equilibrium will thus depend on the choice of coordinates as well as on the initial time t_0.

Several authors have previously considered thermal behavior in a background gravitational field. Drummond[7] has considered the case where the quantum Hamiltonian $H(\eta)$ is an analytic function of conformal time η. He postulates the initial density matrix $T\exp[+i\int_{t_0-i\beta/2}^{t_0+i\beta/2}H(\eta)d\eta]$ (where $T\exp$ is the time-ordered exponential) and derives Feynman rules for evaluating the Green's functions using complex-time contour methods. Hu[8] studied the evolution of the density matrix using a quasi-adiabatic expansion. In our work, we do not require H to be an analytic function of η. We deal with an initial condition in which the density matrix is $\exp[-\beta H(t_0)]$. We show that the complex contour methods discussed by Drummond[7] and by Niemi and Semenoff[9] can be generalized to this case.

Two coordinate systems are commonly used in the study of RW spaces.[6] The first is the Robertson-Walker coordinates described by the line element

$$ds^2=dt^2-a^2(t)dr^2\ , \tag{1.1}$$

where $a(t)$ is a real function of the time t. (We have specialized to spatially flat RW spaces.) Consider a scalar field with action

$$S=\int d^4x\sqrt{-g}\,[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-V(\phi)]\ , \tag{1.2}$$

where

$$V(\phi)=\frac{m^2\phi^2}{2}+\frac{\gamma\phi^3}{3!}+\frac{\lambda\phi^4}{4!}+\frac{\xi R\phi^2}{2}\ , \tag{1.3}$$

R is the scalar curvature and m^2, γ, λ, and ξ are real parameters. In the metric (1.1) the action becomes

$$S=\int dt\,d^3r\,a^3(t)\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2-\frac{1}{2}\frac{1}{a^2(t)}(\nabla\phi)^2-V(\phi)\right]\ . \tag{1.4}$$

Another common system uses conformal time,

$$\eta=\int^t\frac{d\tau}{a(\tau)}\ , \tag{1.5}$$

with line element

$$ds^2=C(\eta)(d\eta^2-dr^2)\ , \tag{1.6}$$

where $C(\eta)=a^2(t(\eta))$ is called the conformal factor. In these coordinates, it is clear that a spatially flat RW space is conformally related to flat space-time.

In conformal coordinates, the action (1.2) becomes

$$S=\int d\eta\, d^3r\, C(\eta)\left[\frac{1}{2}\left[\frac{\partial\phi}{\partial\eta}\right]^2-\frac{1}{2}(\nabla\phi)^2 -C(\eta)V(\phi)\right] . \quad (1.7)$$

We shall include both cases [(1.4) and (1.7)] by considering a generic action of the form

$$S=\int d^4x\left[\frac{\gamma_0(t)}{2}\left[\frac{\partial\phi}{\partial t}\right]^2-\frac{\gamma_1(t)}{2}(\nabla\phi)^2-V(\phi,t)\right] , \quad (1.8)$$

where

$$V(\phi,t)=\frac{m^2(t)\phi^2}{2}+\frac{\gamma(t)\phi^3}{3!}+\frac{\lambda(t)\phi^4}{4!} , \quad (1.9)$$

and $\gamma_i(t)$, $m^2(t)$, and $\lambda(t)$ are real functions of the time t.

In Sec. II we show how to evaluate the real-time Green's functions $\langle T(\phi(\mathbf{x}_1,t_1)\cdots\phi(\mathbf{x}_n,t_n))\rangle$ for the action (1.8) subject to the condition of thermal equilibrium at some initial time t_0. The *vacuum* Green's functions for the actions (1.4) and (1.7) have been studied for various definitions of the "vacuum."[5,7] We extend these results to the evaluation of the *thermal* Green's function for the general case (1.9) by finding a path-integral representation for them and deriving Feynman rules for their evaluation in perturbation theory. The Feynman rules for evaluating real-time thermal Green's functions in *flat* space-time have been developed recently[9,10] and our method is modeled after the work of Ref. 9. We conclude with two examples. First we derive the expression for the noninteracting Green's functions in de Sitter space. We then derive an expression for the expectation value of the stress-energy tensor in an arbitrary RW space (for the case $\xi=0$).

II. GENERAL FORMALISM

Consider a scalar field theory with the action (1.8). The momentum π canonically conjugate to ϕ is given by

$$\pi=\gamma_0(t)\dot{\phi} , \quad (2.1)$$

and the Hamiltonian H is

$$H(t)=\int d^3x\left[\frac{\pi^2}{2\gamma_0(t)}+\gamma_1(t)\frac{(\nabla\phi)^2}{2}+V(\phi,t)\right] . \quad (2.2)$$

Let us suppose that the system is prepared at some initial time t_0 in thermal equilibrium at a temperature $T=\beta^{-1}$ [with respect to $H(t_0)$]. The density matrix at t_0 is thus chosen to be

$$\rho(t_0)=e^{-\beta H(t_0)}/\mathrm{Tr}(e^{-\beta H(t_0)}) . \quad (2.3)$$

The states of the system will evolve via the Hamiltonian $H(t)$, so that if we measure the expectation value of any operator Q in the evolved system, we find (in the Schrödinger representation)[11]

$$\langle Q\rangle(t)=\mathrm{Tr}[\rho(t_0)U(t,t_0)QU^\dagger(t,t_0)] , \quad (2.4)$$

where $U(t,t_0)=\exp_t[i\int_{t_0}^{t}H(\tau)d\tau]$ and t denotes time ordering of the exponential. The Green's functions for the system are given by

$$G(\mathbf{x}_1,t_1,\ldots,\mathbf{x}_n,t_n)=\mathrm{Tr}[\rho(t_0)(\phi(x_1)\cdots\phi(x_n))_t] , \quad (2.5)$$

with

$$\phi(x_i)\equiv U(t_i,t_0)\phi(\mathbf{x}_i,t_0)U^{-1}(t_i,t_0) ,$$

where $\phi(\mathbf{x}_i)$ is the Schrödinger-representation field operator.

To derive a path-integral representation for G, we suppose, without loss of generality, that $t_0<t_1<\cdots<t_n$ and rewrite (2.5) as

$$G(x_1,\ldots,x_n)=\mathrm{Tr}\left\{\rho(t_0)\left[\exp_t\left[i\int_{t_0}^{T}H(\tau)d\tau\right]\right]^{-1}\times\exp_t\left[i\int_{t_n}^{T}H(\tau)d\tau\right]\phi(\mathbf{x}_n)\exp_t\left[i\int_{t_{n-1}}^{t_n}H(\tau)d\tau\right]\phi(\mathbf{x}_{n-1})\cdots\phi(\mathbf{x}_1)\times\exp_t\left[i\int_{t_0}^{t_1}H(\tau)d\tau\right]\right\} , \quad (2.6)$$

where T is an arbitrary large time (eventually we shall let $T\to\infty$), and $\phi(\mathbf{x}_j)$ is understood to be $\phi(\mathbf{x}_j,t_0)$. Following the method of Ref. 9, we can write

$$G(x_1,\ldots,x_n)=N\int\prod_{\substack{\mathbf{x}\\ t\in P}}d\phi(\mathbf{x},t)\exp(iS_P)\phi(\mathbf{x}_1,t_1)\cdots\phi(\mathbf{x}_n,t_n) . \quad (2.7)$$

The integral is over fields $\phi(\mathbf{x},t)$ for t on the path P of Fig. 1 which are periodic; $\phi(t_0 \in P_1)=\phi(\beta \in P_3)$. P is divided into three parts. P_1 is the path from t_0 to $T-i\epsilon$, P_2 is the return path from $T-i\epsilon$ to $t_0-2i\epsilon$, and P_3 is a path from $t_0-2i\epsilon$ to $t_0-i\beta$ (see Fig. 1). The path action S_P is given by the sum

$$S_P=S_{P_1}+S_{P_2}+S_{P_3} ,$$

with $S_{P_1}\equiv S$ [Eq. (1.8)], $S_{P_2}=-S$, and

$$S_{P_3}=i\int_0^\beta d\tau \int d^3x \left[\frac{1}{2}\gamma_0(t_0)\left[\frac{\partial\phi}{\partial\tau}\right]^2 + \frac{1}{2}\gamma_1(t_0)(\nabla\phi)^2+V(\phi,t_0)\right] . \tag{2.8}$$

The integral (2.7) is over fields ϕ which satisfy the boundary condition (ϕ at t_0 on path P_1)=(ϕ at β on path P_3). The times $t_1,\ldots,t_n$ should be chosen on P_1 for evaluating the time-ordered product of the quantum fields and they should be chosen on P_2 for the anti-time-ordered operator product. N is a normalization factor

$$N^{-1}=\int \prod_{t\in P} d\phi(\mathbf{x},t)\exp(iS_P) . \tag{2.9}$$

Next, let us consider the two-point function $G(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2)$. By using (2.7), the definition of G can be generalized to include t_1 or t_2 in either P_1, P_2, or P_3. We define

$$G_{ab}(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2) = N\int D\phi \exp(iS_P)\phi(\mathbf{x}_1,t_1)\phi(\mathbf{x}_2,t_2)\Big|_{\substack{t_1\in P_a\\ t_2\in P_b}} . \tag{2.10}$$

It is straightforward to find the interpretation of G_{ab} in the field theory. G_{11} and G_{22} are the time-ordered and anti-time-ordered propagators, respectively. G_{33} is the Euclidean Green's function at t_0 and G_{12},G_{21} are the Wightman functions; $G_{12}=\langle\phi(\mathbf{x}_2,t_2)\phi(\mathbf{x}_1,t_1)\rangle$. The following continuity conditions now follow:

$$G_{a1}(\mathbf{x},t;\mathbf{y},T)=G_{a2}(\mathbf{x},t;\mathbf{y},T) ,$$

$$\frac{\partial}{\partial t_2}G_{a1}(\mathbf{x},t;\mathbf{y},t_2)\Big|_{t_2=T} = \frac{\partial}{\partial t_2}G_{a2}(\mathbf{x},t;\mathbf{y},t_2)\Big|_{t_2=T} , \tag{2.11a}$$

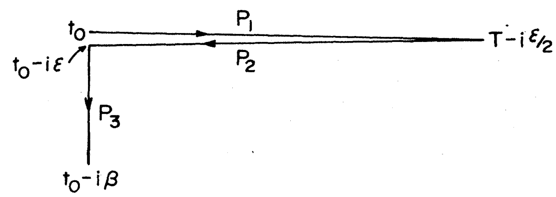


FIG. 1. The time path over which fields are defined in evaluating real-time thermal Green's functions.

$$G_{a2}(\mathbf{x},t;\mathbf{y},t_0)=G_{a3}(\mathbf{x},t;\mathbf{y},\tau=0) ,$$

$$\frac{\partial}{\partial t_2}G_{a2}(\mathbf{x},t;\mathbf{y},t_2)\Big|_{t_2=t_0} = i\frac{\partial}{\partial \tau}G_{a3}(\mathbf{x},t;\mathbf{y},\tau)\Big|_{\tau=0} , \tag{2.11b}$$

$$G_{a3}(\mathbf{x},t;\mathbf{y},\beta)=G_{a1}(\mathbf{x},t;\mathbf{y},t_0) ,$$

$$i\frac{\partial}{\partial \tau}G_{a3}(\mathbf{x},t,\mathbf{y},\tau)\Big|_{\tau=\beta} = \frac{\partial}{\partial t_2}G_{a1}(\mathbf{x},t;\mathbf{y},t_2)\Big|_{t_2=t_0} , \tag{2.11c}$$

where we have written $t=t_0-i\tau$ for $t\in P_3$. We shall need these results shortly.

The Green's functions can be evaluated in perturbation theory by separating S_P into its quadratic ($\lambda,\gamma_3\to 0$) and interaction parts and then expressing the interaction part of e^{iS_P} as a power series in λ and γ. The result will be a Feynman diagram series for the Green's functions.

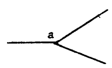
More precisely, let $S_P=S_P^{(2)}+S_P^I$ where $S_P^{(2)}$ and S_P^I are the quadratic and interaction parts of S_P, respectively. Then

$$G(x_1,\ldots,x_n) \propto \int D\phi \exp(iS_P^{(2)})\sum_{n=0}^{\infty}\frac{(iS_P^I)^n}{n!}\phi(x_1)\cdots\phi(x_n) . \tag{2.12}$$

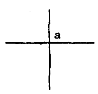
Now

$$S_P^I=-\int_{t\in P_1} d^4x\left[\tfrac{1}{3}\gamma_3(t)\phi^3(\mathbf{x},t)+\tfrac{1}{4}\lambda(t)\phi^4(\mathbf{x},t)\right]+\int_{t\in P_2} d^4x\left[\tfrac{1}{3}\gamma_3(t)\phi^3(\mathbf{x},t)+\tfrac{1}{4}\lambda(t)\phi^4(\mathbf{x},t)\right] +i\int_0^\beta d\tau\int d^3x\left[\tfrac{1}{3}\gamma^3(t_0)\phi^3(\mathbf{x},\tau)+\tfrac{1}{4}\lambda(t_0)\phi^4(\mathbf{x},\tau)\right] . \tag{2.13}$$

We thus have three types of ϕ^3 and ϕ^4 vertices which we call type 1, 2, and 3, corresponding to the various legs of the path P on which ϕ is defined. They are



$$-i\gamma_3(t),\ a=1\ ,$$
$$+i\gamma_3(t),\ a=2\ ,$$
$$-\gamma_3(t_0),\ a=3\ ,$$



$$-i\lambda(t),\ a=1\ ,$$
$$+i\lambda(t),\ a=2\ ,$$
$$-\lambda(t_0),\ a=3\ . \tag{2.14}$$

We next work out the propagator. To do this, we shall need to invert the operator

$$Q_1\equiv i\left[\gamma_0(t)\frac{\partial^2}{\partial t^2}+\dot\gamma_0(t)\frac{\partial}{\partial t}-\gamma_1(t)\nabla^2+m^2(t)\right]\ ,\quad t\in P_1\ ,$$
$$Q_2\equiv -Q_1,\quad t\in P_2\ , \tag{2.15}$$
$$Q_3\equiv -\gamma_0(t_0)\frac{\partial^2}{\partial\tau^2}-\gamma_1(t_0)\nabla^2+m^2(t_0)\ ,\quad \tau\in P_3\ ,$$

where an overdot denotes d/dt. This propagator, which we shall call K, will be

$$K\equiv K(\mathbf{x},t;\mathbf{y},t')\text{ with }(t,t')\in(P_1,P_2,P_3)\ .$$

K is, of course, just the free-field ($\gamma_3=\lambda=0$) two-point function, i.e., K coincides with G of Eq. (2.10) when $\gamma_3=\lambda=0$. We define $K_{ab}(\mathbf{x},t;\mathbf{y},t')$ to be the inverse of (2.15) when $t\in P_a$ and $t'\in P_b$. It satisfies the equations

$$Q_a(t)K_{ab}(\mathbf{x},t;\mathbf{y},t')=\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y})\delta(t-t')\ ,$$
$$K_{ab}(\mathbf{x},t,\mathbf{y},t')Q_b(t')=\delta_{ab}\delta^3(\mathbf{x}-\mathbf{y})\delta(t-t') \tag{2.16}$$

(with no summation on a,b). Furthermore K satisfies the continuity conditions of Eq. (2.11).

Spatial translation invariance can be used to write

$$K_{ab}(\mathbf{x},t;\mathbf{y},t')=\int\frac{d^3q}{(2\pi)^3}e^{-i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})}K^q_{ab}(t,t')\ , \tag{2.17}$$

and K^q_{ab} satisfies the equation

$$Q^q_a(t)K^q_{ab}(t,t')=\delta_{ab}\delta(t-t')\ , \tag{2.18}$$

where Q^q_a has the same form as Q_a (2.15) with ∇^2 replaced by $-q^2$.

A particularly useful form for K can be obtained by first finding *any* two linearly independent solutions to the homogeneous equations

$$Q^q_1(t)f^q_i(t)=0,\quad Q^q_3(\tau)h_i(\tau)=0\quad (i=1,2)\ , \tag{2.19}$$

satisfying the Wronskian conditions

$$\dot f_1(t)f_2(t)-f_1(t)\dot f_2(t)=-i/\gamma_0(t)\ ,$$
$$\dot h_1(\tau)h_2(\tau)-h_1(\tau)\dot h_2(\tau)=-1/\gamma_0(t_0)\ . \tag{2.20}$$

[These conditions are preserved by Eqs. (2.19).] The most general solution to (2.18) can be written as a particular solution plus the general solution of the homogeneous equation:

$$K^q_{11}(t,t')=f_1(t)f_2(t')\theta(t-t')$$
$$+f_2(t)f_1(t')\theta(t'-t)+\sum_{i,j=1}^{2}\alpha^{(1,1)}_{ij}f_i(t)f_j(t')\ ,$$

$$K^q_{22}(t,t')=f_2(t)f_1(t')\theta(t-t')$$
$$+f_1(t)f_2(t')\theta(t'-t)+\sum_{i,j=1}^{2}\alpha^{(2,2)}_{ij}f_i(t)f_j(t')\ ,$$

$$K^q_{33}(\tau,\tau')=h_1(\tau)h_2(\tau')\theta(\tau-\tau')$$
$$+h_2(\tau)h_1(\tau')\theta(\tau'-\tau)+\sum_{i,j=1}^{2}\alpha^{(3,3)}_{ij}h_i(\tau)h_j(\tau')\ ,$$

$$K^q_{12}(t,t')=\sum_{i,j=1}^{2}\alpha^{(1,2)}_{ij}f_i(t)f_j(t')\ , \tag{2.21}$$

$$K^q_{21}(t,t')=\sum_{i,j=1}^{2}\alpha^{(2,1)}_{ij}f_i(t)f_j(t')\ ,$$

$$K^q_{a3}(t,\tau')=\sum_{i,j=1}^{2}\alpha^{(a,3)}_{ij}f_i(t)h_j(t')\quad (a=1,2)\ ,$$

Equation (2.21) will satisfy Eq. (2.18) for all values of $\alpha^{(a,b)}_{ij}$. Furthermore we shall see that the parameters $\alpha^{(a,b)}_{ij}$ are uniquely determined by the boundary conditions (2.11).

First consider (2.11a). We have

$$K^q_{a1}(t,T)=K^q_{a2}(t,T)\ ,$$
$$\frac{\partial}{\partial t'}K^q_{a1}(t,t')=\frac{\partial}{\partial t'}K^q_{a2}(t,t')\text{ at }t'=T\ . \tag{2.22}$$

We define

$$\tilde\alpha^{(a,a)}_{ij}=\alpha^{(a,a)}_{ij}+\delta_{i2}\delta_{j1}\ ,$$
$$\tilde\gamma^{(a,a)}_{ij}=\alpha^{(a,a)}_{ij}+\delta_{i1}\delta_{j2}\ . \tag{2.23}$$

From (2.21), (2.23), and (2.22) with $a=1$, we have, for all $t\in P_1$,

$$\sum_{i,j}\tilde\alpha^{(1,1)}_{ij}f_i(t)f_j(T)=\sum_{i,j}\alpha^{(1,2)}_{ij}f_i(t)f_j(T)\ , \tag{2.24}$$

$$\sum_{ij} \tilde{\alpha}_{ij}^{(1,1)} f_i(t) \dot{f}_j(T) = \sum_{ij} \alpha_{ij}^{(1,2)} f_i(t) \dot{f}_j(T) ,$$

or

$$\tilde{\alpha}^{(1,1)} \begin{bmatrix} f_1 & \dot{f}_1 \\ f_2 & \dot{f}_2 \end{bmatrix}_T = \alpha^{(1,2)} \begin{bmatrix} f_1 & \dot{f}_1 \\ f_2 & \dot{f}_2 \end{bmatrix}_T . \tag{2.25}$$

The matrix

$$\begin{bmatrix} f_1 & \dot{f}_1 \\ f_2 & \dot{f}_2 \end{bmatrix}_T$$

is invertible since its determinant $f_1\dot{f}_2 - \dot{f}_1 f_2 \neq 0$ [see Eq. (2.20)]. We thus conclude

$$\tilde{\alpha}^{(1,1)} = \alpha^{(1,2)} . \tag{2.26}$$

(The matching matrix at T is the identity.) Similarly one finds

$$\alpha^{(1,2)} = \tilde{\gamma}^{(2,2)}, \quad \alpha^{(3,1)} = \alpha^{(3,2)} . \tag{2.27}$$

We can now consider the matching conditions at the point P_{23} where the paths P_2 and P_3 meet. From (2.11b), we have, for example,

$$\begin{aligned} &K_{12}^q(t,t_0) = K_{13}^q(t,\tau=0) , \\ &\frac{\partial}{\partial t'} K_{12}^q(t,t') \Big|_{t'=t_0} = i \frac{\partial}{\partial \tau} K_{13}^q(t,\tau) \Big|_{\tau=0} . \end{aligned} \tag{2.28}$$

Using (2.21), we find

$$\alpha^{(1,2)} \begin{bmatrix} f_1(t_0) & \dot{f}_1(t_0) \\ f_2(t_0) & \dot{f}_2(t_0) \end{bmatrix} = \alpha^{(1,3)} \begin{bmatrix} h_1(0) & i\dot{h}_1(0) \\ h_2(0) & i\dot{h}_2(0) \end{bmatrix} , \tag{2.29}$$

which can be written

$$\alpha^{(1,3)} = \alpha^{(1,2)} M_{(23)} ,$$

where

$$M_{(23)} = \begin{bmatrix} f_1(t_0) & \dot{f}_1(t_0) \\ f_2(t_0) & \dot{f}_2(t_0) \end{bmatrix} \begin{bmatrix} h_1(0) & i\dot{h}_1(0) \\ h_2(0) & i\dot{h}_2(0) \end{bmatrix}^{-1} , \tag{2.30}$$

is the matching matrix at P_{23}.

The functions $h_i(\tau)$ can be evaluated explicitly since they satisfy the equation

$$-\gamma_0(t_0)\ddot{h}(\tau) + [\gamma_1(t_0)q^2 + m^2(t_0)]h(\tau) = 0 . \tag{2.31}$$

Let us define

$$\mu^2(t) = [\gamma_1(t)q^2 + m^2(t)]/\gamma_0(t) . \tag{2.32}$$

Then

$$\ddot{h}(\tau) = \mu^2(t_0) h(\tau) . \tag{2.33}$$

We can choose, as solutions to this equation,

$$\begin{aligned} h_1(\tau) &= \frac{1}{[2\gamma_0(t_0)\mu(t_0)]^{1/2}} \exp[-\mu(t_0)\tau] , \\ h_2(\tau) &= \frac{1}{[2\gamma_0(t_0)\mu(t_0)]^{1/2}} \exp[\mu(t_0)\tau] . \end{aligned} \tag{2.34}$$

Here h_1 satisfy both the equation (2.33) and Wronskian condition (2.20). $M_{(23)}$ [Eq. (2.30)] can now be evaluated:

$$M_{(23)} = -i \left[\frac{\gamma_0(t_0)}{2\mu(t_0)} \right]^{1/2} \begin{bmatrix} -\dot{f}_1 + i\mu f_1 & \dot{f}_1 + i\mu f_1 \\ -\dot{f}_2 + i\mu f_2 & \dot{f}_2 + i\mu f_2 \end{bmatrix}_{t_0} . \tag{2.35}$$

$M_{(23)}$ is the matching matrix at P_{23} and we find

$$\begin{aligned} &\alpha^{(1,3)} = \alpha^{(1,2)} M_{(23)} , \\ &\alpha^{(2,3)} = \tilde{\alpha}^{(2,2)} M_{(23)} , \\ &\tilde{\gamma}^{(3,3)} = \alpha^{(3,2)} M_{(23)} . \end{aligned} \tag{2.36}$$

Finally, we must evaluate the matching matrix at the boundaries; $t_0 \in P_1$ and $\beta \in P_3$. Let us call this point "β." From (2.11c), we have, for example,

$$\begin{aligned} &K_{13}^q(t,\beta) = K_{11}^q(t,t_0) , \\ &i \frac{\partial}{\partial \tau} K_{13}^q(t,\tau) \Big|_{\tau=\beta} = \frac{\partial}{\partial t'} K_{11}^q(t,t') \Big|_{t'=t_0} . \end{aligned} \tag{2.37}$$

Using (2.23), we find

$$\alpha^{(1,3)} \begin{bmatrix} h_1(\beta) & i\dot{h}_1(\beta) \\ h_2(\beta) & i\dot{h}_2(\beta) \end{bmatrix} = \tilde{\gamma}^{(1,1)} \begin{bmatrix} f_1(t_0) & \dot{f}_1(t_0) \\ f_2(t_0) & \dot{f}_2(t_0) \end{bmatrix} . \tag{2.38}$$

We can now define the matching matrix M_β by

$$\tilde{\gamma}^{(1,1)} = \alpha^{(1,3)} M_\beta . \tag{2.39}$$

M_β can be evaluated using Eq. (2.34),

$$M_\beta = -i \left[\frac{\gamma_0(t_0)}{2\mu(t_0)} \right]^{1/2} \begin{bmatrix} e^{-\beta\mu}(\dot{f}_2 + i\mu f_2) & -e^{-\beta\mu}(\dot{f}_1 + i\mu f_1) \\ e^{\beta\mu}(\dot{f}_2 - i\mu f_2) & -e^{\beta\mu}(\dot{f}_1 - i\mu f_1) \end{bmatrix}_{t_0} . \tag{2.40}$$

We then have

$$\tilde{\gamma}^{(1,1)}=\alpha^{(1,3)}M_\beta\ ,\quad \alpha^{(2,1)}=\alpha^{(2,3)}M_\beta\ ,\quad \alpha^{(3,1)}=\tilde{\alpha}^{(3,3)}M_\beta\ . \tag{2.41}$$

We now use (2.26), (2.27), (2.36), and (2.41) to obtain

$$\tilde{\gamma}^{(1,1)}=\tilde{\alpha}^{(1,1)}M_{(23)}M_\beta\ ,\quad \tilde{\gamma}^{(2,2)}=\tilde{\alpha}^{(2,2)}M_{(23)}M_\beta\ ,\quad \tilde{\gamma}^{(3,3)}=\tilde{\alpha}^{(3,3)}M_\beta M_{(23)}\ . \tag{2.42}$$

Recall the definition of $\tilde{\alpha}$ and $\tilde{\gamma}$ [Eq. (2.23)]. The solution to the equation $\tilde{\gamma}=\tilde{\alpha}M$ for any matrix M is simply

$$\tilde{\alpha}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}(1-M)^{-1} \tag{2.43}$$

[provided, of course $\det(1-M)\neq 0$]. First consider $\tilde{\alpha}^{(3,3)}$ which is the simplest to evaluate. From (2.35) and (2.40), we find, using (2.20),

$$M_\beta M_{(23)}=\begin{bmatrix}e^{-\beta\mu(t_0)} & 0\\ 0 & e^{\beta\mu(t_0)}\end{bmatrix}\ . \tag{2.44}$$

Using (2.42) and (2.43), we have

$$\tilde{\alpha}^{(3,3)}=\frac{1}{e^{\beta\mu(t_0)}-1}\begin{bmatrix}0 & 1\\ e^{\beta\mu(t_0)} & 0\end{bmatrix}\ . \tag{2.45}$$

When combined with K^q_{33} in Eq. (2.21), this leads to the usual Euclidean Green's function at finite temperature.

The expression for $\tilde{\alpha}^{(1,1)}$ and $\tilde{\alpha}^{(2,2)}$ is more complicated. From (2.35) and (2.40),

$$M_{(23)}M_\beta=\frac{-\gamma_0(t_0)}{2\mu(t_0)}\begin{bmatrix}2Q(t_0)\sinh\beta\mu-\frac{2\mu}{\gamma_0}\cosh\beta\mu & -2Q_1(t_0)\sinh\beta\mu\\ 2Q_2(t_0)\sinh\beta\mu & -2Q(t_0)\sinh\beta\mu-\frac{2\mu}{\gamma_0}\cosh\beta\mu\end{bmatrix}_{t_0} \tag{2.46}$$

with

$$Q_i(t)=[\dot{f}_i^{\,2}(t)+\mu^2(t)f_i^{\,2}(t)]\quad (i=1,2)\ ,\qquad Q(t)=[\dot{f}_1(t)\dot{f}_2(t)+\mu^2(t)f_1(t)f_2(t)]\ . \tag{2.47}$$

A simple calculation yields $\det(M_{(23)}M_\beta)=1$, $\det(1-M_{(23)}M_\beta)=-4\sinh^2[\beta\mu(t_0)/2]$. Thus

$$\tilde{\alpha}^{(1,1)}=\tilde{\alpha}^{(2,2)}=\frac{\gamma_0(t_0)}{2\mu(t_0)}\begin{bmatrix}-Q_2\coth\left[\beta\frac{\mu}{2}\right] & Q\coth\left[\beta\frac{\mu}{2}\right]-\frac{\mu}{\gamma_0}\\ Q\coth\left[\beta\frac{\mu}{2}\right]+\frac{\mu}{\gamma_0} & -Q_1\coth\left[\beta\frac{\mu}{2}\right]\end{bmatrix}_{t_0}\ . \tag{2.48}$$

Equation (2.43) gives us $\alpha^{(3,1)}=\tilde{\alpha}^{(3,3)}M_\beta$ which yields

$$\alpha^{(3,1)}=-i[\gamma_0(t_0)/2\mu(t_0)]^{1/2}(1-e^{-\beta\mu(t_0)})^{-1}\begin{bmatrix}\dot{f}_2-i\mu f_2 & -(\dot{f}_1-i\mu f_1)\\ e^{\beta\mu}(\dot{f}_2+i\mu f_2) & -e^{\beta\mu}(\dot{f}_1+i\mu f_1)\end{bmatrix}\Bigg|_{t_0}\ . \tag{2.49}$$

Finally, one can also show that

$$\alpha^{(1,3)}=(\alpha^{(3,1)})^T\ , \tag{2.50}$$

where the superscript T denotes the matrix transpose.

In summary, to evaluate a Green's function of type $(a,b)[G^q_{ab}(t,t')]$, write down all Feynman graphs with two external legs labeled by $\mathbf{q},a,t$ and $\mathbf{q},b,t'$ (see Fig. 2). There are ϕ^3 and ϕ^4 vertices of types 1, 2, and 3 given by Eq. (2.11). For each line of momentum q, the propagator is given by $K^q_{ab}(t,t')$ of Eq. (2.23) with $\tilde{\alpha}^{(3,3)}$, $\tilde{\alpha}^{(1,1)}$, and $\tilde{\alpha}^{(2,2)}$ given by Eqs. (2.47), (2.25), and (2.48). $\alpha^{(1,3)}$ and $\alpha^{(3,1)}$ are given by Eqs. (2.49) and (2.50). $\alpha^{(1,3)}=\alpha^{(2,3)}$ and $\alpha^{(3,1)}=\alpha^{(3,2)}$. $\alpha^{(1,2)}=\tilde{\alpha}^{(1,1)}$ [see Eqs. (2.23) and (2.48)] and $\alpha^{(2,1)}=\tilde{\gamma}^{(2,2)}$.

As an illustration of the Feynman rules, let us look at the form of the two-point function in the pure ϕ^4 theory ($\gamma=0$) to lowest order in $\lambda(t)$. Using the Feynman rules above, we evaluate the graphs of Fig. 2 and obtain

$$G^q_{ab}(t,t')=K^q_{ab}(t,t')+\sum_{c=1}^{3}\epsilon_c\int dt''\int\frac{d^3k}{(2\pi)^3}\lambda(t'')K^q_{ac}(t,t'')K^q_{cb}(t'',t')K^k_{cc}(t'',t'')\ , \tag{2.51}$$

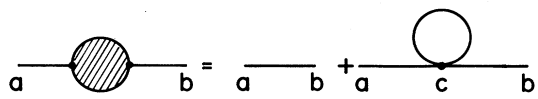


FIG. 2. One-loop correction to the propagator in ϕ^4 field theory.

with $\epsilon_1=i$, $\epsilon_2=-i$, and $\epsilon_3=-1$. The integral over t'' runs from t_0 to ∞ for $c=1,2$ and from 0 to β for $c=3$.

III. PARAMETRIZATION INDEPENDENCE OF THE GREEN'S FUNCTION

Recall that $f_a(t)$ of (2.21) were *any* two linearly independent solutions to Eq. (2.19) which satisfied the Wronskian conditions (2.20). The Green's functions (2.21) must be totally independent of the choice of f's. We can check this explicitly by considering another set $g_a^q(t)$ of solutions to (2.19) satisfying (2.20). Suppose

$$f_a^q(t)=U_{ab}g_b^q(t) \ . \tag{3.1}$$

Let us keep the solutions $h_a(\tau)$ on the path P_3 [Eq. (2.36)] unchanged. Equation (2.20) requires

$$\dot{f}_a f_b - f_a \dot{f}_b = \dot{g}_a g_b - g_a \dot{g}_b = \epsilon_{ab}[-i/\gamma_0(t)] \ . \tag{3.2}$$

This implies

$$U\epsilon U^T=\epsilon \ ,$$

i.e.,

$$\det U = 1 \ . \tag{3.3}$$

The Green's function K_{33}^q is clearly unchanged. Consider first

$$K_{a3}^q(t,\tau')=\sum_{i,j}\alpha_{ij}^{(a,3)}(f)f_i(t)h_j(\tau') \quad (a=1,2) \ , \tag{3.4}$$

where the f dependence of α_{ij} [as seen in Eq. (2.49)] is explicitly shown. Under the transformation (3.1),

$$K_{a3}^q \to \sum_{i,j,k}\alpha_{ij}^{(a,3)}(f)U_{ik}g_k(t)h_j(\tau') \ . \tag{3.5}$$

We thus require

$$\sum_i \alpha_{ij}^{(a,3)}(f)U_{ik}=\alpha_{kj}^{(a,3)}(g) \ . \tag{3.6}$$

This can easily be checked from (2.49) and (2.50) using the transformation properties of f.

For $a,b=1,2$ we write

$$K_{ab}^q(t,t')=N_{ij}^{(a,b)}(f)f_i(t)f_j(t') \ , \tag{3.7}$$

where $N^{(a,b)}$ is either $\tilde{\alpha}^{(1,1)}$, $\tilde{\alpha}^{(2,2)}$, $\tilde{\gamma}^{(1,1)}$, or $\tilde{\gamma}^{(2,2)}$ depending on a,b and the sign of $t-t'$. We need to show

$$U^T N(f) U = N(g) \ . \tag{3.8}$$

Since

$$U^T\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}U=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} , \tag{3.9}$$

it is sufficient to show

$$U^T\begin{bmatrix}Q_2(f) & -Q(f)\\ -Q(f) & Q_1(f)\end{bmatrix}U=\begin{bmatrix}Q_2(g) & -Q(g)\\ -Q(g) & Q_1(g)\end{bmatrix} . \tag{3.10}$$

To see this, define

$$\bar{Q}_{ab}=\dot{f}_a\dot{f}_b+\mu^2(t)f_af_b \ . \tag{3.11}$$

Then (3.10) requires

$$U^T\bar{Q}(f)^{-1}U=Q(g)^{-1} \ . \tag{3.12}$$

But under (3.1)

$$\bar{Q}(f)=U\bar{Q}(g)U^{-1}, \quad \bar{Q}(f)^{-1}=U\bar{Q}(g)^{-1}U^{-1} \ . \tag{3.13}$$

This establishes (3.12) and thus the parametrization independence of the free Green's functions. The Feynman rules then clearly imply that this result holds for the fully interacting Green's functions.

IV. DE SITTER SPACE—AN EXAMPLE

As an example of the formalism of this section, let us consider quantizing a scalar field in de Sitter space given by the line element (1.1) with

$$a(t)=e^{Ht} \ . \tag{4.1}$$

In conformal coordinates, $\eta=-H^{-1}\exp(-Ht)$,

$$C(\eta)=(H\eta)^{-2} \ . \tag{4.2}$$

We shall consider quantization in both of these coordinate systems. In RW coordinates, the action (1.8) of the scalar field has

$$\begin{gathered}\gamma_0(t)=a^3(t), \quad \gamma_1(t)=a(t) \ ,\\ m^2(t)=a^3(t)(m^2+\xi R) \ .\end{gathered} \tag{4.3}$$

In conformal coordinates,

$$\begin{gathered}\gamma_0(t)=C(\eta), \quad \gamma_1(t)=C(\eta) \ ,\\ m^2(t)=C^2(\eta)(m^2+\xi R) \ ,\end{gathered} \tag{4.4}$$

where $R=12H^2$ is the curvature scalar in de Sitter space. (We shall not consider the interaction terms in this discussion.)

In conformal coordinates, the functions $f_a(\eta)$ satisfy Eq. (2.19),

$$\ddot{f}_a(\eta)-\frac{2}{\eta}\dot{f}_a(\eta)+[k^2+(12\xi+m^2/H^2)\eta^{-2}]f_a(\eta)=0 \ . \tag{4.5}$$

A set of normalized [via (2.20)] solutions to (4.5) is given by

$$\begin{aligned}f_1^k(\eta)&=H\eta\left[\frac{\pi\eta}{4}\right]^{1/2}H_\nu^{(2)}(k\eta) \ ,\\ f_2^k(\eta)&=H\eta\left[\frac{\pi\eta}{4}\right]^{1/2}H_\nu^{(1)}(k\eta) \ ,\end{aligned} \tag{4.6}$$

with $\nu_2 = \frac{9}{4} - 12\xi - m^2/H^2$. Let us now suppose that at η_0, the system is in thermal equilibrium at a temperature β^{-1}. The Green's functions are then given by (2.21), (2.48), and (2.49). Let us now take the limit as $\eta_0 \to -\infty$. In this case,

$$f_{1(2)}^{(k)}(\eta_0) \underset{\eta_0 \to -\infty}{\sim} \frac{H\eta_0}{\sqrt{2k}} \exp\left[\mp i \left[k\eta_0 - \frac{\pi}{2}\nu - \frac{\pi}{4}\right]\right] . \tag{4.7}$$

For fixed k,

$$\mu^2(\eta_0) = k^2 + (12\xi + m^2/H^2)\eta_0^{-2} \underset{\eta_0 \to -\infty}{\to} k^2 ,$$

so that from Eq. (2.47), to leading order,

$$Q(t) \to kH^2\eta_0^2, \quad Q_{1,2}^{(t)} \to O((\eta_0)^0) . \tag{4.8}$$

From (2.48),

$$\tilde{\alpha}^{(1,1)} = \tilde{\alpha}^{(2,2)} = \alpha^{(1,2)} = \begin{bmatrix} 0 & \frac{1}{2}\left[\coth\left[\frac{\beta k}{2}\right] - 1\right] \\ \frac{1}{2}\left[\coth\left[\frac{\beta k}{2}\right] + 1\right] & 0 \end{bmatrix} . \tag{4.9}$$

Thus, for example, the Wightman function

$$K_{12}^k(\eta,\eta') = \frac{1}{2}\left[\coth\left[\frac{\beta k}{2}\right] - 1\right] f_1^k(\eta) f_2^k(\eta') + \frac{1}{2}\left[\coth\left[\frac{\beta k}{2}\right] + 1\right] f_1^k(\eta') f_2^k(\eta)$$

$$= \frac{\pi H^2}{4}(\eta\eta')^{3/2} H_\nu^{(2)}(k\eta') H_\nu^{(1)}(k\eta) + (e^{\beta k} - 1)^{-1}[H_\nu^{(1)}(k\eta) H_\nu^{(2)}(k\eta') + H_\nu^{(1)}(k\eta') H_\nu^{(2)}(k\eta)] . \tag{4.10}$$

The first term corresponds to the "vacuum" Green's function with respect to the $\eta_0 \to -\infty$ vacuum[12] and the other term is the thermal correction.

Note that the thermal effects at $\eta_0 \to -\infty$ are *not* red-shifted away in conformal coordinates. This is *not* true for quantization in RW coordinates.

To see this note that the functions $f_a^k(t)$ in RW coordinates are obtained from (4.6) by substituting $\eta = -(Ha)^{-1}$. We find

$$f_1^k(t) = (-a)^{-3/2}\left[\frac{\pi}{4H}\right]^{1/2} H_\nu^{(2)}\left[-\frac{k}{Ha}\right], \quad f_2^k(t) = (-a)^{-3/2}\left[\frac{\pi}{4H}\right]^{1/2} H_\nu^{(1)}\left[-\frac{k}{Ha}\right], \tag{4.11a}$$

$$\mu^2(t) = k^2/a^2(t) + m^2 + \xi R . \tag{4.11b}$$

Furthermore

$$\alpha^{(1,2)} = \tilde{\alpha}^{(1,1)} = \left.\begin{bmatrix} 0 & \frac{1}{2}\left[\coth\left[\frac{\beta\mu}{2}\right] - 1\right] \\ \frac{1}{2}\left[\coth\left[\frac{\beta\mu}{2}\right] + 1\right] & 0 \end{bmatrix}\right|_{t=-\infty} . \tag{4.12}$$

Now $a \to 0$ as $t \to -\infty$. So for fixed β, k

$$\alpha^{(1,2)} = \tilde{\alpha}^{(1,1)} \underset{t_0 \to -\infty}{\to} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} . \tag{4.13}$$

Thermal effects at $t_0 \to -\infty$ do not affect the Green's functions since the wavelengths of all the particles present at $t_0 = -\infty$ have been red-shifted to infinity.

As discussed in the Introduction, it is difficult to decide which coordinate system is more relevant for specifying the initial state. We do not propose a solution to this problem—we simply show how to calculate the time evolution of any of the states (or density matrices) in question.

V. THE STRESS-ENERGY TENSOR

As a further example, we derive the expression for the expectation value of the stress-energy tensor, $T^{\mu\nu}$ for the action (1.4) with $\xi = 0$ in the lowest-order (noninteracting) approximation. The definition[5]

$$T_{\mu\nu}=\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\ , \tag{5.1}$$

yields a conserved energy-momentum tensor. For the free-field version of the action (1.4) with $\xi=0$, we have

$$T^{\mu\nu}=D^{\mu}\phi D^{\nu}\phi-g^{\mu\nu}(\tfrac{1}{2}D_{\alpha}\phi D^{\alpha}\phi-\tfrac{1}{2}m^2\phi^2)\ . \tag{5.2}$$

The energy density and pressure are defined as

$$\rho=\langle T^0_0\rangle\ ,\qquad -p=\langle T^1{}_1\rangle=\langle T^2{}_2\rangle=\langle T^3{}_3\rangle\ . \tag{5.3}$$

We then have

$$\rho=\left\langle-\frac{\dot\phi^2}{2}+\frac{(\nabla\phi)^2}{2a^2}+\frac{m^2\phi^2}{2}\right\rangle\ ,\qquad -p=\left\langle-\frac{\dot\phi^2}{2}+\frac{(\nabla\phi)^2}{3a^2}+\frac{m^2\phi^2}{2}\right\rangle\ . \tag{5.4}$$

Using (2.21)

$$\rho(t)=\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\sum_{i,j=1}^{2}\alpha_{ij}^{(1,2)}\left[\dot f_i^k(t)\dot f_j^k(t)+\left[\frac{k^2}{a^2}+m^2\right]f_i^k(t)f_j^k(t)\right]\ , \tag{5.5}$$

$$p(t)=\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\sum_{i,j=1}^{2}\alpha_{ij}^{(1,2)}\left[\dot f_i^k(t)\dot f_j^k(t)-\left[\frac{2k^2}{3a^2}+m^2\right]f_i^k(t)f_j^k(t)\right]\ . \tag{5.6}$$

Now use (2.48) to obtain

$$\rho(t)=a^3(t_0)\int\frac{d^3k}{(2\pi)^3}\frac{\coth[\beta\mu(t_0)/2]}{2\mu(t_0)}\{Q(t_0)Q(t)-\tfrac{1}{2}[Q_1(t_0)Q_2(t)+Q_2(t_0)Q_1(t)]\}\ , \tag{5.7a}$$

$$p(t)=a^3(t_0)\int\frac{d^3k}{(2\pi)^3}\frac{\coth[\beta\mu(t_0)/2]}{2\mu(t_0)}\{Q(t_0)S(t)-\tfrac{1}{2}[Q_2(t_0)S_1(t)+Q_1(t_0)S_2(t)]\}\ , \tag{5.7b}$$

with

$$S(t)=-\dot f_1\dot f_2+(\mu^2-\tfrac{1}{3}k^2)f_1f_2,\quad S_i(t)=-\dot f_i^{\,2}+(\mu^2-\tfrac{1}{3}k^2)f_i^{\,2}\ . \tag{5.7c}$$

The dependence on β is in the factor $\coth[\beta\mu(t_0)/2]$. Despite the divergences in $T^{\mu\nu}$ at $\beta\to\infty$, the difference $T^{\mu\nu}(\beta)-T^{\mu\nu}(\infty)$ is finite. The question of renormalization of $T^{\mu\nu}$ is discussed in detail in Ref. 5.

VI. SUMMARY

Using complex-time functional integrals, we have derived Feynman rules for evaluating real-time thermal Green's functions for a scalar field theory in a spatially flat Robertson-Walker background. We believe that the method can be extended in a straightforward manner to include fermions and gauge fields as well as open and closed universes. We start with a system which is in thermal equilibrium at some initial time t_0. The system is then allowed to evolve as dictated by its (time dependent) Hamiltonian. The n-point Green's functions at times $t_1,\ldots,t_n$ are then given by drawing all of the usual Feynman graphs with n external legs but with each vertex having a label $i=1,2,3$ and each propagator having a corresponding label (i,j). The vertex factors are given by Eq. (2.14). The propagator is given by Eq. (2.21). It depends on the functions $h(\tau)(0<\tau<\beta)$ [Eq. (2.34)] and on the mode functions $f(t)$ which are solutions to Eq. (2.19) [see also (2.15)], satisfying the Wronskian condition (2.20). The matrices α are given by Eqs. (2.48), (2.49), and (2.50). Some illustration of the use of these rules is also given in this paper.

In the free-field case, although the system starts out in thermal equilibrium at t_0, the expansion of the universe will, in general, drive the system out of equilibrium due to the red-shifting of the wavelengths. A sufficient condition for equilibrium to be maintained is that the various matrices $\alpha(k,t_0)$ [see, for example, Eq. (2.48)] are independent of t_0 for a suitable (k independent) choice of a (t_0 dependent) temperature, $\beta(t_0)$. A sufficient condition for this to occur is that $\beta(t_0)\mu(t_0)$ is t_0 independent for all k and that Q, Q_1, and Q_2 are t_0 independent. The first condition can only be satisfied if $m^2(t)=0$ in which case $\beta\mu(t_0)=\beta\gamma_1(t_0)k/\gamma_0(t_0)$. If $\beta(t)$ is chosen equal to $\gamma_0(t)/\gamma_1(t)$, then $\beta(t_0)\mu(t_0)$ is constant. Now $m^2(t)=0$

requires a massless, *minimally* coupled field ($m^2=0$, $\xi=0$). The second condition can be satisfied either in flat space-time or for a conformally coupled field ($m^2=0$, $\xi=\frac{1}{6}$). We thus see that unless the scalar curvature $R=0$, equilibrium is never precisely maintained for free fields. This result contrasts with other choices of the density matrix[7,8] for which conformally coupled fields remain in equilibrium. Even for our choice of density matrix, if R is small or if the expansion is slow, then Q, Q_1, and Q_2 are nearly constant and if m is small, the system will remain nearly in equilibrium. In such a case, we would expect the interactions to reequilibrate the system.

In the following paper, these methods are applied to the derivation of evolution equations for the Higgs field in a hot, expanding universe. Further applications to cosmology as well as questions of renormalization are presently under investigation.

ACKNOWLEDGMENT

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada.

[1]L. Susskind, Phys. Rev. D **20**, 2610 (1979); A. M. Polyakov, Phys. Lett. **72B**, 477 (1978); J. Kuti, J. Polonyi, and K. Szlachanyi, *ibid.* **98B**, 199 (1981); L. McLerran and B. Svetitsky, *ibid.* **98B**, 195 (1981).
[2]A. Kirzhnits and A. Linde, Phys. Lett. **42B**, 471 (1972); S. Weinberg, Phys. Rev. D **9**, 3357 (1974).
[3]A. Guth, Phys. Rev. D **23**, 347 (1981); A. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).
[4]G. Semenoff and N. Weiss, following paper, Phys. Rev. D **31**, 699 (1985).
[5]N. Birrell and P. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).
[6]W. Unruh, Phys. Rev. D **14**, 870 (1976).
[7]I. T. Drummond, Nucl. Phys. **B190** [FS3], 93 (1981).
[8]B. L. Hu, Phys. Lett. **108B**, 19 (1982).
[9]A. Niemi and G. Semenoff, Ann. Phys. (N.Y.) **152**, 105 (1984); Nucl. Phys. **B230** [FS10], 181 (1984); R. L. Kobes, G. Semenoff, and N. Weiss, UBC Report, 1984 (unpublished).
[10]H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermofield Dynamics and Condensed States* (North-Holland, Amsterdam, 1982).
[11]L. V. Keldysh, Zh. Eksp. Teor. Fiz. **47**, 1515 (1964) [Sov. Phys. JETP **20**, 1018 (1965)].
[12]T. Bunch and P. Davies, Proc. R. Soc. London **A360**, 117 (1978).