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Kaluza-Klein cosmologies and inflation. II

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The classical cosmology implied by Kaluza-Klein models is studied in some numerical detail. The constraints imposed on the parameters describing the early, n-dimensional epoch by the known properties of the current three-dimensional epoch are explored. The conclusion is that the present conditions are remarkably robust, depending essentially only on the total entropy within our horizon and requiring no fine tuning of the Kaluza-Klein parameters.

I. INTRODUCTION

There has recently been considerable interest in the study[1,2] of the early cosmology of Kaluza-Klein scenarios.[3] Several authors have noted that such models give rise to extremely rapid inflation of the scale factor describing the usual three dimensions at the time of the "collapse" of the extra, Kaluza-Klein dimensions. Such behavior can provide an explanation of the observed isotropy of the background radiation at the time it (re)enters the causal horizon, and thus a resolution of the well-known cosmological flatness problems. In a previous paper[1] we discussed this scenario (at least in classical terms) quite generally. The conclusion was that, for D, the number of extra dimensions, of order 40, the currently observed entropy ($\sim 10^{88}$) may be understood with only moderate ($\sim 10^2$) excitation[4] of each of the $n \equiv D+d = D+3$ dimensions. Here we consider the relevant equations in rather greater numerical detail. We integrate Einstein's equations numerically in order to investigate behavior near the collapse time. These numerical results are compared with the analytic "matching" scheme proposed in the previous paper and developed here. The agreement between the two approaches is surprisingly good. Thus the analytic technique is sufficiently reliable to afford a straightforward understanding of many of the important features of such scenarios. A point of particular interest is an appreciation of which parameters in the early universe are free and which are constrained by the physics, by reasonable assumptions, or by observations (necessarily) made in the later, three-dimensional universe. We find that present conditions are prescribed almost entirely by the magnitude of the entropy in the comoving volume of our present horizon and are insensitive to the detailed behavior at early times. Thus no fine tuning of the early parameters is necessary, a desirable situation, but also little can be inferred about the detailed behavior of the n-dimensional epoch from present observations. We close by elaborating on our earlier[1] discussion of the reliability of our general conclusions. In particular, we argue that, while quantum effects must surely play a sizeable dynamical role during the epoch of collapse, our conclusions will be insensitive to the inclusion of such effects.

II. THE MODEL

Our starting point[1] is a *classical* description of the universe which is assumed to be radiation dominated in the epoch of interest. The geometry is specified by a line element

$$ds^2 = -dt^2 + r^2(t) g_{ij} dx^i dx^j + R^2(t) g_{ab} dX^a dX^b \,. \tag{1}$$

The quantities r and R are the scale factors describing constant-curvature spaces of d ($=3$) ordinary and D extra dimensions, respectively. We can write Einstein's equations in the compact form

$$R^{\mu}{}_{\nu} = -8\pi \bar{G} S^{\mu}{}_{\nu} - \Lambda \delta^{\mu}{}_{\nu} \,. \tag{2a}$$

Here $R^{\mu}{}_{\nu}$ is the Ricci tensor where the indices μ and ν run over the values 0 to n. In terms of the stress-energy tensor $T^{\mu}{}_{\nu}$ the other tensor $S^{\mu}{}_{\nu}$ has the simple form

$$S^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} - \frac{1}{n-1} \delta^{\mu}{}_{\nu} T^{\lambda}{}_{\lambda} \,. \tag{2b}$$

Note that here we are allowing for the presence of a cosmological term and a more general stress-energy tensor than was used in Ref. 1. This is in order that the interval during which the extra dimensions are decoupling may be treated more adequately than was possible with the previous approach. As explained more fully below, the equations during the epoch from the big bang until R collapses to its final value R_{KK} are the same as those used in Ref. 1. The stress-energy tensor $T^{\mu}{}_{\nu}$ is assumed to exhibit the simple *diagonal* (perfect-fluid) form

$$T^{\mu}{}_{\nu}=\begin{bmatrix} -\rho & & & & & & \\ & p & & & & & \\ & & p & & & & \\ & & & p & & & \\ & & & & P & & \\ & & & & & \ddots & \\ & & & & & & P \end{bmatrix} . \tag{2c}$$

Here ρ is the energy density and p and P the pressures in the d- and D-dimensional spaces, respectively. The dots stand for $D-2$ diagonal P terms. The factor $\bar{G}$ in Eq. (2a) is the $(n+1)$-dimensional gravitational constant related to Newton's constant G_N by[3]

$$\bar{G}=V_D G_N = V_D \; , \tag{3}$$

where V_D is the volume of the compact D-dimensional manifold. For example, if this volume is a sphere (as will be assumed in what follows) with physical radius $\bar{R}_{\mathrm{KK}}$, then

$$V_D=\left[\frac{2\pi^{(D+1)/2}}{\Gamma\left[\dfrac{D+1}{2}\right]}\right]\bar{R}_{\mathrm{KK}}{}^{D} \; . \tag{4}$$

We proceed by assuming that at early times the universe is described by the "symmetrical" situation $p=P=\rho/n$ with $\Lambda=0$. In this case Einstein's equations resolve into three equations for $r(t)$ and $R(t)$ which are not independent due to the Bianchi identity and the conservation of energy-momentum. If the curvatures corresponding to the two scale factors are k_d and k_D, the three equations are

$$d\frac{\ddot{r}}{r}+D\frac{\ddot{R}}{R}=-8\pi\bar{G}\rho \; , \tag{5a}$$

$$\frac{k_d}{r^2}+\frac{d}{dt}\left[\frac{\dot{r}}{r}\right]+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\left[\frac{\dot{r}}{r}\right]=8\pi\bar{G}p=\frac{8\pi\bar{G}}{n}\rho \; , \tag{5b}$$

and

$$\frac{k_D}{R^2}+\frac{d}{dt}\left[\frac{\dot{R}}{R}\right]+\left[d\frac{\dot{r}}{r}+D\frac{\dot{R}}{R}\right]\left[\frac{\dot{R}}{R}\right]=8\pi\bar{G}P=\frac{8\pi\bar{G}}{n}\rho \; . \tag{5c}$$

The parameter k_D we scale to $+1$ so that R is not the physical radius but, for a sphere, $R=\bar{R}/\sqrt{D-1}$, where $\bar{R}$ is the physical radius [recall Eq. (4)]. The parameter k_d we take as zero in the present studies. (As discussed in Ref. 1, it is only necessary that k_d be small compared to k_D.)

To proceed in this classical framework we make the (more dubious) assumption that, during the epoch of interest, the interaction rates are such as to maintain thermal equilibrium. We shall return below to treat the issue of the reliability of this assumption. With this assumption entropy is conserved. Thus in a volume comoving with the expansion (or contraction) of the universe the entropy S is constant. In particular, in a comoving n-dimensional volume specified by r and R the entropy is given by[1]

$$S=(r^d R^D T^n) N_{\mathrm{pol}}^{(n)}\left[\frac{n+1}{n}\right]a_n\left[\frac{V_D}{R_{\mathrm{KK}}^{D}}\right]$$
$$=\text{constant} \; , \tag{6}$$

where $N_{\mathrm{pol}}^{(n)}$ is the number of participating polarizations (which we will take to be $n-1$ for the explicit calculations below) and a_n is the $(n+1)$-dimensional Stefan Boltzmann constant,

$$a_n=\frac{n\Gamma\left[\dfrac{n+1}{2}\right]\zeta(n+1)}{\pi^{(n+1)/2}} \tag{7}$$

with ζ the Riemann zeta function. In the special case $D=0$, the quantity in the square brackets in Eq. (6) is defined to be unity. If we set the constant on the right-hand side of Eq. (6) to $\sim 10^{88}$, we are implicitly defining r to correspond to the scale of the presently observable universe. The corresponding energy density ρ is specified by

$$\rho=N_{\mathrm{pol}}^{(n)} a_n T^{n+1} \; . \tag{8}$$

It is useful for the subsequent discussion to combine Eqs. (6) and (8) and define two constants of proportionality via

$$8\pi\bar{G}\rho \equiv f_n T^{n+1} \tag{9a}$$

and

$$T\equiv\frac{h_n}{(R^D r^d)^{1/n}} \; . \tag{9b}$$

Using Eqs. (5)–(8), the explicit form of the first of these constants can be deduced to be

$$f_n=\frac{16n(n-1)\Gamma\left[\dfrac{n+1}{2}\right]\zeta(n+1)(D-1)^{D/2}}{\pi^{(d-2)/2}\Gamma\left[\dfrac{D+1}{2}\right]}R_{\mathrm{KK}}{}^{D} \tag{10a}$$

for $D>1$ which, for $D=40$, has magnitude

$$f_{43}=1.021\times 10^{38} R_{\mathrm{KK}}{}^{40} \; , \tag{10b}$$

while for the special case $D=0$ (with $\bar{G}=G=1$)

$$f_3=16.5 \; . \tag{10c}$$

The second coefficient is given by

$$h_n = \left[\frac{\Gamma\left[\frac{D+1}{2}\right]\pi^{d/2}S}{2(n^2-1)\Gamma\left[\frac{n+1}{2}\right]\zeta(n+1)(D-1)^{D/2}} \right]^{1/n} , \quad (11a)$$

again for $D > 1$, with magnitude

$$h_{43} = 15.66 \quad (11b)$$

for $D = 40$, while for the special case $D = 0$ (with the ratio $V_D/R_{KK}{}^D$ set to unity)

$$h_3 = 2.25 \times 10^{29} . \quad (11c)$$

To obtain the latter numerical results the specific value $S = 10^{88}$ was used. This same value will be used in the numerical analysis discussed below.

It is straightforward to evaluate the time dependence of the scale factors predicted by the classical equations [Eqs. (5)]. As noted in Ref. 1, the behavior near the big bang, $t = 0$, is easily found to be

$$R(t) \underset{t \to 0}{\sim} At^{2/(n+1)} \quad (12a)$$

and

$$r(t) \underset{t \to 0}{\sim} at^{2/(n+1)} . \quad (12b)$$

Here the coefficients a and A specify the initial conditions. Having specified the total entropy via Eqs. (9)—(11), these constants are, in fact, not independent. Using Eqs. (9)—(12) in Eqs. (5) yields the relation

$$a = A^{-D/d} h_n{}^{n/d} \left[f_n \frac{(n+1)^2}{2n(n-1)} \right]^{n/d(n+1)} . \quad (13)$$

Thus we see that there are basically five input parameters to the classical numerical problem, three of which are either fixed or at least constrained by observations. In the latter category is the total entropy S, which appears in h_n, the final size of the compact dimensions R_{KK}, which appears in f_n, and the number of ordinary dimensions $d = 3$. The other two parameters are the number of compact dimensions, which we will take here as $D = 40$, as an example, and the initial condition on the Kaluza-Klein scale factor, input here via the coefficient A [which then specifies the corresponding coefficient a for the ordinary dimensions via Eq. (13)]. As a specific example of the implied relationships consider the further constraint of full (but unjustified, as discussed below) symmetry, $a = A$. In this case both a and A are specified to have the value (indicating the extra symmetry constraint by a tilde)

$$\tilde{a} = \tilde{A} = 112.9 R_{KK}{}^{10/11} . \quad (14)$$

Here the numerical coefficient depends on the values of S and D while the general dimensional scale is carried essentially (for D large) by R_{KK}. This result exhibits a general feature of the solutions of the classical equations for D large and for comparable values of the coefficients A and a. Most dimensionful quantities, for example, the values of r and R when R is maximum, and the times when R is maximum and when it collapses, scale essentially as R_{KK} to the first power. An important exception is the length of the interval between the time that the collapse stops (t_c) and the time by when the extra dimensions have decoupled (t_d) which scales essentially as $R_{KK}{}^2$. More specifically it follows from Eqs. (10a) and (13) that the characteristic length scale $(r^d R^D)^{1/n}$ varies as $R_{KK}{}^{D/(n+1)} = R_{KK}{}^{10/11}$. This will be illustrated by the numerical results discussed below.

Since it will be instructive to connect the currently discussed Kaluza-Klein epoch to the subsequent Robertson-Walker regime, we shall make one further assumption before proceeding to integrate numerically. We shall make the (highly dubious) approximation that the classical equations of *motion* describing the universe hold right through the period of collapse, while there will be discontinuous changes in the equations of *state* when the physics is thought to change. In particular, when R collapses to the value R_{KK} we will simply set $R \equiv R_{KK}$ for all subsequent times (with all time derivatives of R set to zero in Einstein's equations). This is specifically realized in Eq. (2) by setting the pressure for the D dimensions to $P = 1/(8\pi\bar{G}R_{KK}{}^2)$ and taking $p = \rho/d$. At the same time, in order to ensure that the large time equations match the usual Robertson-Walker forms, we introduce a cosmological constant

$$\Lambda = 8\pi\bar{G}DP/(n-1) = D/R_{KK}{}^2(n-1) .$$

Thus $T^\mu{}_\nu$ is henceforth traceless in only $d+1$ dimensions. The same is true of $R^\mu{}_\nu$ which, due to our explicit assumptions, is proportional to $T^\mu{}_\nu$ during all epochs. These specific classical definitions are intended only as (crude) approximations to whatever microphysics serves to stabilize R at R_{KK}. Thus P and Λ are *not* to be considered as distinct, new parameters but rather as simply a naive classical characterization of this microphysics such that the parameter

$$R_{KK}{}^2 = 1/(8\pi\bar{G}P) = D/\Lambda(n-1) .$$

With these choices Eqs. (5) are replaced by

$$d\frac{\ddot{r}}{r} = -8\pi\bar{G}\rho , \quad (15a)$$

$$\frac{\ddot{r}}{r} + (d-1)\left[\frac{\dot{r}}{r}\right]^2 = 8\pi\bar{G}\frac{\rho}{d} , \quad (15b)$$

and

$$R = R_{KK} = \text{constant} . \quad (15c)$$

Likewise, instead of treating in detail the physics of the decoupling from the energy density ρ of the extra degrees of freedom as the temperature decreases,[5] we will simply discontinuously change from full excitation of $D+d$ dimensions to excitation of only d dimensions. While this is a fairly gross approximation, it will have little effect on our detailed results and *no* effect on our general conclusions. Note that in any case these simplifications are relevant only during the tiny time interval of collapse when the most serious approximations are the restriction to equilibrium and classical behavior. The time of "decoupling" of the extra dimensions is then specified by the requirement that r, R, and the temperature be continuous while the physics (i.e., the equations) are not. Thus if $r_\pm$, $R_\pm$, $T_\pm$ refer to the values of parameters just before and just after decoupling, the requirement of continuity is that

$$\begin{aligned} r_- &= r_+ , \\ R_- &= R_+ , \\ T_- &= T_+ . \end{aligned} \tag{16a}$$

From Eq. (9b), however, we have that

$$\left[\frac{T_-}{h_n}\right]^n r_-{}^d R_-{}^D = 1 = \left[\frac{T_+ r_+}{h_d}\right]^d . \tag{16b}$$

These imply a relation which can be rewritten in two particularly useful forms. At decoupling R and T satisfy

$$RT\,|_{\text{decoupling}} = \left[\frac{(h_n)^n}{(h_d)^d}\right]^{1/D} \tag{17a}$$

and

$$\left.\frac{R}{r}\right|_{\text{decoupling}} = \left[\frac{h_n}{h_d}\right]^{n/D} . \tag{17b}$$

Note, in particular, that Eq. (11) guarantees that the right-hand side of Eq. (17a) is independent of S and depends only on D and d. For $d=3$ and $D=40$ this constant is 0.12 which guarantees that, for $R_{\rm KK} \geq R_{\rm Pl}$, T at the time of decoupling is bounded above by $0.12 M_{\rm Pl}$. Thus the collapse of the Kaluza-Klein dimensions does *not* necessarily lead to a reheating to temperatures above the Planck temperature. We shall return to this point below and explicitly show that in general the reheating leaves the temperature below $M_{\rm Pl}$.

As discussed in Ref. 1, there is the question at this point as to which occurs first: the "stabilization" of R at $R_{\rm KK}$ or the decoupling of the D dimensions? This issue can be simply restated in terms of whether the quantity RT, at the time R reaches $R_{\rm KK}$ assuming no decoupling, is greater than or less than the bound in Eq. (17a). The answer is greater for any sizable entropy value as will be demonstrated shortly by our numerical results and using the analytic approximations of Ref. 1. Thus the collapse stops before decoupling occurs for "realistic" parameter values.

III. NUMERICAL INTEGRATION

We turn now to the results of numerical integration of Eqs. (5) for $d=3$, $D=40$, and $S=10^{88}$ for various values of $R_{\rm KK}$ and A. For small times we simply numerically integrate Eqs. (5) using Eqs. (9)–(11) to write the right-hand sides in terms of R and r with $n=43$. This yields R and r as functions of time, while Eq. (9b) gives the corresponding temperature. According to the discussion of the last paragraph the first change in the equations occurs at t_c, when R collapses to $R_{\rm KK}$. For subsequent times $R=R_{\rm KK}$ so that Eqs. (15a) and (15b) are relevant, and as noted earlier the form of $T^\mu{}_\nu$ changes. The right-hand sides are still given by Eqs. (9)–(11) with $n=43$ (the D dimensions have not yet decoupled), but with $R=R_{\rm KK}$.

Finally, when the decoupling relation of Eq. (17) is satisfied (at t_d), n is set to 3 and the special values of Eqs. (10c) and (11c) are used in the integration for subsequent times. This defines the calculations performed. With $R_{\rm KK}=1$ (in Planck units) in the symmetric limit of Eq. (14), the results for R, r, and T are illustrated by the solid curves in Figs. 1(a)–1(d), where these quantities are plotted versus time (all in Planck units). These figures illustrate several important features. The general behavior is as expected. The scale R first increases *very* rapidly with time and is then essentially constant [recall Eq. (12) and the fact that $t^{1/22}$ is a good approximation to a Θ function] until it collapses *very* rapidly to $R_{\rm KK}$. Note that the bulk of the time variation occurs, in fact, within time intervals of length less than one Planck time. Similarly r rises rapidly and essentially equally with R, is constant and then inflates *very* rapidly until the collapse of R is complete. In subsequent times it continues to grow but in a different fashion. In the intervals $t_c < t < t_d$ and $t_d < t$ the equations may be integrated analytically to give the following behavior for r:

$$\begin{aligned} r &= \left[\frac{h_n}{R_{\rm KK}^{D/n}}\right]^{n/d} \left[\frac{d(n+1)^2}{2(d-1)n^2} f_n\right]^{n/d(n+1)} \\ &\quad \times (t-\hat{t})^{2n/d(n+1)} \text{ for } t_c < t < t_d , \\ r &= h_3(\tfrac{4}{3} f_3)^{1/4} \sqrt{t-\bar{t}} \text{ for } t_d < t , \end{aligned} \tag{18}$$

where $\hat{t}$ and $\bar{t}$ are constants of integration. Note that for $t \gg \bar{t}$, this is the standard Robertson-Walker behavior (with $S=10^{88}$). The integration constant $\bar{t}$ depends on the

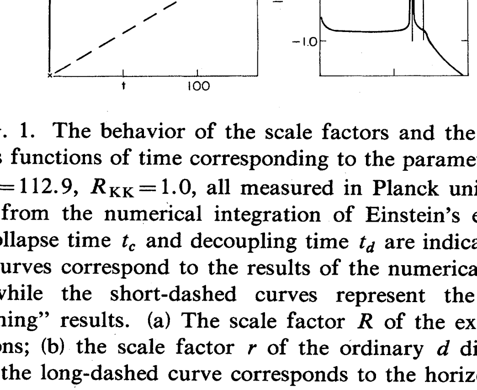

FIG. 1. The behavior of the scale factors and the temperature as functions of time corresponding to the parameter choice $\tilde{A}=\tilde{a}=112.9$, $R_{\rm KK}=1.0$, all measured in Planck units, which result from the numerical integration of Einstein's equations. The collapse time t_c and decoupling time t_d are indicated. The solid curves correspond to the results of the numerical integration while the short-dashed curves represent the analytic "matching" results. (a) The scale factor R of the extra D dimensions; (b) the scale factor r of the ordinary d dimensions where the long-dashed curve corresponds to the horizon "size" ct; (c) the logarithm of the scale factor r; (d) the logarithm of the temperature T.

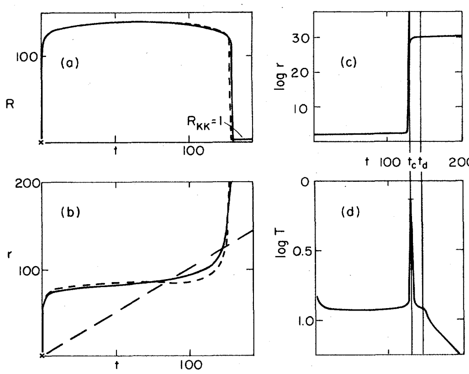

FIG. 2. The behavior of the scale factors and the temperatures as functions of time corresponding to the parameter choice $A=117.0$, $a=69.95$, and $R_{\rm KK}=1.0$, all measured in Planck units. The collapse time t_c and decoupling time t_d are indicated. The solid curves correspond to the results of the numerical integration, while the short-dashed curves represent the analytic "matching" results. (a) The scale factor R of the extra D dimensions; (b) the scale factor r of the ordinary d dimensions where the long-dashed curve corresponds to the horizon "size" ct; (c) the logarithm of the scale factor r; (d) the logarithm of the temperature T.

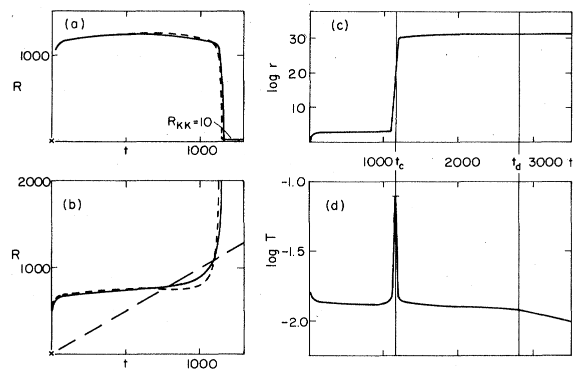

FIG. 3. The behavior of the scale factors and the temperature as functions of time corresponding to the parameter choice $A=949.0$, $a=567.5$, and $R_{\rm KK}=10.0$, all measured in Planck units. The collapse time t_c and decoupling time t_d are indicated. The solid curves correspond to the results of the numerical integration, while the short-dashed curves represent the analytic "matching" results. (a) The scale factor R of the extra D dimensions; (b) the scale factor r of the ordinary d dimensions where the long-dashed curve corresponds to the horizon "size" ct; (c) the logarithm of the scale factor r; (d) the logarithm of the temperature T.

specific value of r at the end of the Kaluza-Klein epoch and therefore on our detailed assumptions about classical behavior during the collapse and decoupling regimes. Thus its specific value in the present calculations is highly suspect. However, it is clearly of order the time of the collapse and thus *much* smaller than the present time (of order 10^{61} in Planck units). Hence the subsequent history of the universe is insensitive to our inability to evaluate this number accurately. Similar statements apply to the integration constant $\hat{t}$.

The behavior of the temperature is much less dramatic. Initially it falls, is then constant with everything else and then there is some reheating at the time of collapse. However, as noted above, this reheating does not take the temperature above the Planck scale. For the specific parameter values illustrated in this figure the peak reheating temperature, which occurs just as R reaches $R_{\rm KK}$, still is bounded by 0.7 $M_{\rm Pl}$ (as we shall discuss more fully below). Of considerably more immediate interest is the fact that, for the choice of initial conditions chosen, equal values of a and A, the scale r, which represents the early evolution of our presently observable universe, does not fall inside the causal horizon [indicated by the short-dashed curve in Fig. 1(b)] during the Kaluza-Klein epoch. However, while one intuitively expects R and r to behave similarly during the very early times, i.e., the universe is in some sense symmetrical in all n dimensions, any detailed equality of R and r is not appropriate. Note, in particular, that, while the logarithmic derivatives of R and r are appropriately equal [recall Eq. (12)], the actual *magnitudes* of the two scales are defined in very different fashions. The scale R is, within a factor $\sqrt{D-1}$, the radius of a compact space while r describes the "size" of that portion of a flat space which is presently observable. The actual magnitudes, as noted above, are controlled by the magnitude of the input entropy, the size of $R_{\rm KK}$, and our choice of $D=40$ to guarantee approximately equal, but relatively small excitation of the various degrees of freedom at early times. In any case we find that, for initial conditions $A \gtrsim 116R_{\rm KK}$ and [correspondingly from Eq. (13)] $a \lesssim 72R_{\rm KK}$, the scale r is indeed smaller than a causal horizon at some time during this early epoch. This is illustrated in Fig. 2 which corresponds to $A=117.0$ and $a=69.95$ with $R_{\rm KK}=1$. The results for $R_{\rm KK}=10$ and $A=949.0(=117.0\times 10^{10/11})$ are indicated by the solid curves in Fig. 3. This figure exhibits the result noted earlier that the "size" of the Kaluza-Klein epoch, i.e., the maximum magnitude of R and the time until collapse, scales essentially linearly with $R_{\rm KK}$. Thus cosmological considerations place no constraints on its value as long as it is not too much larger than the Planck scale. The long-term evolution of the universe is insensitive.

IV. ANALYTIC RESULTS

To achieve a more complete appreciation of the numerical results let us compare them to the approximate analytic expressions introduced in Ref. 1. The two-term approximations corresponding to Eq. (12) and useful for small times, just after the big bang, can be written as

$$R(t)\simeq At^{\alpha}(1+A't^{\delta}) \tag{19a}$$

and

$$r(t)\simeq at^{\beta}(1+a't^{\delta}) \;, \tag{19b}$$

where A and a are related by Eq. (13). Note that the two terms in each expression correspond to approximately solving Einstein's equations including the effects of *both* the energy density and the curvature. In the discussion that follows we shall present the various coefficients and exponents in terms of simple expressions when such expressions exist and, since they generally do not exist, in terms of explicit numerical constants. Since these constants are often raised to large powers or appear as exponents of large numbers in subsequent calculations, we shall carry along the requisite high level of numerical precision. This will guarantee that the reader can easily reproduce our subsequent results. For $D=3$ and $d=40$ the various exponents and coefficients in Eq. (19) are found[1] to be

$$\alpha=\beta=\frac{1}{n+1}=0.045\,46\,, \tag{20a}$$

$$\delta=2\left[\frac{n-1}{n+1}\right]=1.9091\,, \tag{20b}$$

$$A'=-0.096\,51\left[\frac{1}{A^2}\right]\,, \tag{20c}$$

and

$$a'=0.086\,41\left[\frac{1}{A^2}\right]\,. \tag{20d}$$

Note that A' is negative so that the feature that $R(t)$ has a maximum is already present in this simple approximation. Thus the time when the maximum is reached is easily approximated by setting $\dot{R}=0$ in Eq. (19a) to find

$$t_m=\left[-\frac{1}{nA'}\right]^{(n+1)/2(n-1)}=0.4745A^{1.0476} \tag{21a}$$

for $d=3$ and $D=40$. The corresponding value of R is

$$R_m=At_m{}^{2/(n+1)}\left[1-\frac{1}{n}\right]=0.9442A^{1.0476}\,. \tag{21b}$$

The corresponding expressions valid for times near the singular collapse time t_0 are written in terms of the variable $\tau=(t_0-t)$. The explicit expressions are

$$R(t)\simeq B\tau^{\gamma}(1+B'\tau^{\epsilon}) \tag{22a}$$

and

$$r(t)\simeq b\tau^{\eta}(1+b'\tau^{\epsilon})\,. \tag{22b}$$

For $d=3$ and $D=40$ some arithmetic[1] yields

$$\gamma=\frac{1+\left[\frac{d}{D}(n-1)\right]^{1/2}}{n}=0.064\,53\,, \tag{23a}$$

$$\eta=\frac{1-\left[\frac{D}{d}(n-1)\right]^{1/2}}{n}=-0.527\,08\,, \tag{23b}$$

and

$$\epsilon=\frac{n-1}{n}=0.9767\,. \tag{23c}$$

Further algebra[1] yields the first subleading terms B' and b' in terms of the ratios (A/B) and (a/b). For $d=3$ and $D=40$ these relationships are

$$B'=-0.018\,36\left[\left[\frac{A}{B}\right]^D\left[\frac{a}{b}\right]^d\right]^{(n+1)/n} \tag{24a}$$

and

$$b'=0.5669\left[\left[\frac{A}{B}\right]^D\left[\frac{a}{b}\right]^d\right]^{(n+1)/n}\,. \tag{24b}$$

Note the interesting (if misleading) feature that B' is negative (similarly to A') even though this two-term approximation corresponds to a leading term which includes *neither* the energy-density or curvature contributions and a second term which includes the influence of *only* the energy-density contribution. Thus the approximate solution again exhibits a maximum for R even though an *exact* solution of the equations without the curvature contribution would *not* have such a maximum. Ignoring this detail we can obtain an approximate analytic solution for all t in this earlier epoch by simply "matching" the two approximate expressions above at their respective maxima for R as suggested in Ref. 1. After some algebra, again for $d=3$ and $D=40$, we find

$$B=1.0681A^{0.9800}\,, \tag{25a}$$

$$\begin{aligned}b&=0.2087aA^{0.5998}\\&=5.495\times10^{28}A^{-12.734}R_{\mathrm{KK}}{}^{13.03}\end{aligned} \tag{25b}$$

and for the square brackets in Eq. (24),

$$\left[\left[\frac{A}{B}\right]^D\left[\frac{a}{b}\right]^d\right]=\frac{7.8913}{A}\,. \tag{25c}$$

Note, for example, that in the symmetric limit $A=a$ of Eq. (14) the corresponding values are

$$\widetilde{B}=109.66R_{\mathrm{KK}}{}^{0.891} \tag{26a}$$

and

$$\widetilde{b}=399.42R_{\mathrm{KK}}{}^{1.45}\,, \tag{26b}$$

again indicating that for large D all quantities essentially scale with R_{KK}. This allows us to calculate τ_m (when $\dot{R}=0$ as a function of τ) in a similar way to t_m. An approximation to the collapse time t_0 can now be obtained by simply adding

$$t_0\simeq t_m+\tau_m=0.8736A^{1.0476}\,. \tag{27}$$

Using these expressions with the specific values of A and R_{KK} used to create the solid curves in Figs. 1—3, as discussed above, yields the short-dashed curves for R and r exhibited in Figs. 1—3. A careful comparison of the solid and dashed curves indicates that the simple analytic "matching" approximation to R is remarkably good. For the scale r there is a small deviation in the region just after t_m which is not surprising given the above proviso concerning the $\tau\sim0$ solution. Overall these approximations are quite reliable and offer a helpful tool for understanding the numerical results above. This is especially true near the singular points.

As an interesting example let us consider the relationship between the collapse and decoupling times discussed above. In this approximation the former is given by

$$\tau_c = \left[\frac{R_{KK}}{B}\right]^{1/\gamma} = 0.3603 R_{KK}{}^{15.497} A^{-15.187} , \quad (28a)$$

while from Eq. (17b) the latter is given by

$$\tau_d = 0.002\,31 R_{KK}{}^{22.025} A^{-23.180} . \quad (28b)$$

Thus the relevant ratio is

$$\frac{\tau_d}{\tau_c} = 0.006\,41 R_{KK}{}^{6.53} A^{-7.99} . \quad (29)$$

Thus, as noted earlier, R reaches R_{KK} before the extra dimensions decouple [as defined by Eq. (17)], $\tau_d/\tau_c < 1$, for $A > 0.532 R_{KK}{}^{0.82}$. But, also as discussed earlier, the interesting regime of parameters is $a \sim A \sim 10^2 R_{KK}{}^{0.909} (S/10^{88})^{0.023}$. Thus, if we require[6] that $\tau_d \equiv \tau_c$ in order to avoid any problems with thermal excitation preventing the stabilization at R_{KK}, we find that we must require $S/10^{88} \sim 10^{-98} R_{KK}{}^{-3.95}$. Hence, with this constraint, either S or R_{KK} (or *both*) must be *very* small. For example, the requirement that both $S = 10^{88}$ and $\tau_c = \tau_d$ yields the disastrous result that $R_{KK} \sim 10^{-25} R_{Pl}$, in agreement with the conclusion of Ref. 6. However, the attitude of the present analysis is that this is too restrictive. If instead we relax the constraint on τ_c and τ_d, we can proceed as above and simply evaluate the "amount of thermal excitation" indicated by the quantity RT at the stabilization time τ_c, we find

$$RT\big|_{\tau_c} = 0.149 R_{KK}{}^{-0.269} A^{0.330} . \quad (30)$$

So for $A \sim 10^2 R_{KK}{}^{0.909}$ we find $RT(\tau_c) \sim 0.7$ which is clearly larger than the corresponding decoupling value in Eq. (17a), $RT(\tau_d) \sim 0.12$, but certainly does not correspond to a large excitation. Recall that for the specified entropy we have at the time of the maximum R value that $RT(t_m) \sim 10^2$. Perhaps the most important consideration is that R and r are varying so rapidly in this time interval (that is why the above constraint is so restrictive) that classical considerations cannot be valid in detail. However, as discussed below we do not expect quantum effects to change the general classical structure presented above.

We can also consider the interval between t_c and t_d using the expressions of this section and of Sec. III. From Eq. (18) we know the behavior of r in this interval and may use this to deduce that

$$T = \frac{h_n}{(R_{KK}{}^D r^d)^{1/n}} = \left[\frac{2(d-1)n^2}{d(n+1)^2 f_n}\right]^{1/(n+1)} (t-\hat{t})^{-2/(n+1)} . \quad (31)$$

Now at t_d, $T = 0.121/R_{KK}$ and so we find that

$$t_d - \hat{t} = 16.85 R_{KK}{}^{(d+1)/2} . \quad (32)$$

In a similar fashion we may find

$$t_c - \hat{t} = 0.172 \left[\frac{R_{KK}{}^{1.09}}{A}\right]^{(n+1)/2} , \quad (33)$$

which is extremely small for appropriate values of A. Thus we recover the result given in Sec. II, that the interval $[t_c, t_d]$ scales essentially as $R_{KK}{}^2$.

V. CONCLUSIONS

Let us conclude by considering in more detail the issue of the reliability of our results. Clearly the *very* rapid inflation of the ordinary universe near the collapse time suggests[1] that equilibrium and purely classical behavior are unlikely. What will change? If nonequilibrium conditions lead to the production of entropy during this period, this can be accounted for by simply reducing the excitation (for example, the temperature) during the early epoch and/or the number of extra dimensions (i.e., $D < 40$). Perhaps more interesting, and certainly less well understood, is the potential role of quantum gravity effects. One reasonable[7] effect is the slowing down of the rapid variation in r via an effective "viscosity" term in the equations coupling R and r. The essential point is that, for the present purposes, we need not understand any such effect in detail. Unless it participates in such a dramatic way as to exclude the basic Kaluza-Klein scenario, whereby some dimensions compactify while others do not (a disaster for all cosmological Kaluza-Klein ideas except for a static $R = R_{KK}$ scenario), such an effect can only delay somewhat the onset of the inevitable Robertson-Walker epoch. This general attitude is illustrated pictorially in Fig. 4. At early times, when the classical analysis is presumed to be appropriate, the figure is identical to the earlier figures. Note, in particular, that the issue of whether the scale factor r is within a causal horizon at early times is answered (in the affirmative) during this presumably classical regime. However, during the *short*

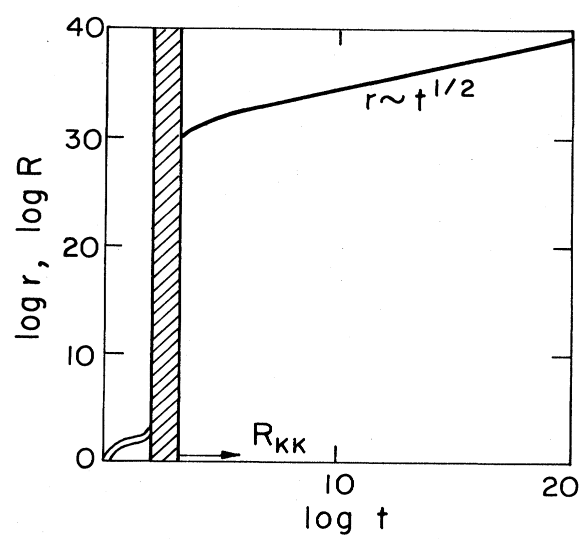

FIG. 4. Idealized view of the evolution of the scale factors as in the previous figures, but with the "black box" of ignorance clearly indicated. Note the *short* relative duration of this regime.

time interval surrounding the collapse quantum effects are essential, but poorly understood. Thus this region is indicated by a "black box." The important point is that, unless the collapse is completely stopped by the quantum levers and gears in the "black box," the universe as it emerges from the box must evolve as indicated. In this case the *only* issue is the value of $\bar{t}$. While the specific value of $\bar{t}$ may be in considerable doubt due to our ignorance of the interior of the "box," the overall evolution of the universe will be unchanged. Such details are simply lost on the logarithmic time scale on which the subsequent cosmological events occur. Thus we conclude, as in Ref. 1, that Kaluza-Klein scenarios with ~ 40 extra dimensions offer an explanation of the usual entropy, inflation, and fine-tuning problems. The more detailed issues of the matter spectrum, the return to equilibrium after inflation, and the role of fluctuations require considerable further study.

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