

Critical behavior of doubling approximations of the $:\phi^4:_2$ lattice field theories

D. Isaacson

Rensselaer Polytechnic Institute, Troy, New York 12181

E. L. Isaacson

University of Wyoming, Laramie, Wyoming 82071

D. Marchesin and P. J. Paes-Leme

*Pontificia Universidade Católica do Rio de Janeiro,
Rio de Janeiro, Brasil 22453*

(Received 1 January 1984)

We present graphs of the lower part of the spectrum of the K -state doubling approximation to a $g:\phi^4:_2$ lattice field theory. The graphs exhibit critical behavior in the infinite-volume limit.

INTRODUCTION

We are interested in studying the lower part of the spectrum of the N -coupled anharmonic-oscillator operators

$$H^{(N)} = \frac{1}{2} \left[\sum_{j=1}^N \left(-\frac{\partial^2}{\partial q_j^2} + gq_j^4 + (3 - \omega g)q_j^2 \right) \right] - \sum_{j=1}^{N-1} q_j q_{j+1}$$

in the infinite-volume limit (i.e., the limit as N tends to ∞). In this paper, we present the results for the lower part of the spectrum given by the K -state doubling approximation for K equal to 4, 8, and 12. This approximation, which is described in the next section, exhibits critical behavior in the infinite-volume limit. Here, g is a positive parameter, and as explained in Ref. 1, we take the Wick constant ω to be 1.927913. The operators $H^{(N)}$ result from the quantization of finite-difference approximations to the classical Hamiltonian

$$\frac{1}{2} \int_{-l/2}^{l/2} (\phi_t^2 + \phi_x^2 + \phi^2 + g\phi^4) dx$$

We take the lattice spacing to be 1 and use Dirichlet boundary conditions.

The spectra of the operators $H^{(N)}$ for $N=2, 4, 8, 16$ were studied in Refs. 1-3, where additional references can be found. Variational methods were used to study these operators in Refs. 4-6. A generalization of the moment recursion method was used in Ref. 7 to study a two-oscillator operator. A theoretical analysis of the two-state doubling approximation in the infinite-volume limit was made in Ref. 8.

THE K -STATE DOUBLING APPROXIMATION

We will briefly describe the K -state approximation of $H^{(N)}$ for $N=2^n$, $n=2, 3, \dots$. This approximation is natural when the doubling algorithm³ is used to compute the energy levels of $H^{(N)}$. Details of the doubling algorithm are given in Ref. 3.

Let $E_j^{(N)}$ ($j=1, 2, \dots$) denote the eigenvalues of $H^{(N)}$ in increasing order and $\Psi_j^{(N)}(q_1, \dots, q_N)$ denote the corresponding eigenfunctions. Assume that the eigenvalues $E_j^{(N)}$

and the matrix elements

$$X_{lm}^{(N,\alpha)} = \int \Psi_l^{(N)}(q_1, \dots, q_N) \times q_\alpha \Psi_m^{(N)}(q_1, \dots, q_N) dq_1 \cdots dq_N$$

for $\alpha=1$ and N are known. We compute $E_j^{(2N)}$, $X_{lm}^{(2N,1)}$, and $X_{lm}^{(2N,2N)}$ as follows. Expand $\Psi_j^{(2N)}(q_1, \dots, q_{2N})$ in the basis

$$\{\Psi_i^{(N)}(q_1, \dots, q_N) \Psi_l^{(N)}(q_{N+1}, \dots, q_{2N})\}$$

so that

$$\Psi_j^{(2N)} = \sum_{i,l=1}^{\infty} C_{il}^j \Psi_i^{(N)}(q_1, \dots, q_N) \Psi_l^{(N)}(q_{N+1}, \dots, q_{2N})$$

Then the equation $H^{(2N)}\Psi_j^{(2N)} = E_j^{(2N)}\Psi_j^{(2N)}$ becomes

$$\sum_{i,l=1}^{\infty} \left[(E_i^{(N)} + E_l^{(N)})\delta_{i,l} \delta_{i,l} - X_{i'l}^{(N,N)} X_{i'l}^{(N,1)} \right] C_{il}^j = E_j^{(2N)} C_{i'l}^j$$

for $i', l'=1, 2, \dots$. The K -state approximation is obtained by replacing the infinite sum above by a finite sum, that is, $i, l=1, \dots, K$ and numerically computing the lowest eigenvalues and corresponding eigenvectors of the resulting $K^2 \times K^2$ matrix eigenvalue problem. Having computed approximations to $E_j^{(2N)}$ and to $\Psi_j^{(2N)}$, we compute approximations to $X_{lm}^{(2N,1)}$, $X_{lm}^{(2N,2N)}$ by using the truncated formulas

$$X_{lm}^{(2N,1)} = \sum_{i,n=1}^K \sum_{i',n'=1}^K C_{in}^l C_{i'n'}^m X_{ii'}^{(N,1)} \delta_{nn'}$$

$$X_{lm}^{(2N,2N)} = \sum_{i,n=1}^K \sum_{i',n'=1}^K C_{in}^l C_{i'n'}^m \delta_{ii'} X_{nn'}^{(N,N)}$$

We take advantage of the fact that $H^{(N)}$ commutes with the unitary operators R and S , where

$$R\Psi(q_1, \dots, q_N) = \Psi(q_N, \dots, q_1)$$

and

$$S\Psi(q_1, \dots, q_N) = \Psi(-q_1, \dots, -q_N)$$

Thus, in each of the four invariant subspaces

$$\mathcal{H}_{\pm\pm} = \{\Psi: R\Psi = \pm\Psi \text{ and } S\Psi = \pm\Psi\}$$

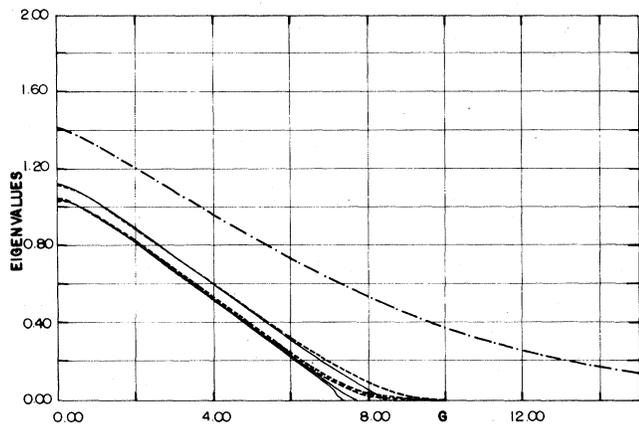


FIG. 1. From the top down: masses for the 4-, 8-, and 12-state approximations as a function of g . The dot-dashed line is for 2 coupled oscillators (1 doubling); they coincide within graphical accuracy. The dashed line is for 16 oscillators (4 doublings), and the solid line is for 128 oscillators (7 doublings).

we compute the $K/4$ lowest eigenvalues of the corresponding restriction of $H^{(N)}$. Therefore, we take K to be a multiple of 4.

We view the algorithm as follows. The operators $H^{(N)}$ approximate $g:\phi^4_2$ lattice field theories for $g \geq 0$. The K -state approximation produces operators $H_K^{(N)}$ which tend to $H^{(N)}$ in the limit as K tends to infinity. Thus, the $H_K^{(N)}$ also approximate the field theories. For example, the mass $m^{(N)}(g)$ of $H^{(N)}$ is the difference between its two lowest eigenvalues: $m^{(N)}(g) = E_2^{(N)}(g) - E_1^{(N)}(g)$. In the infinite-volume limit ($N \rightarrow \infty$), $m^{(N)}(g)$ tends to $m(g)$, the mass of the field theory. Similarly, the K -state masses $m_K^{(N)}$ tend to $m^{(N)}$ as $K \rightarrow \infty$, so that

$$m(g) = \lim_{N \rightarrow \infty} \lim_{K \rightarrow \infty} m_K^{(N)}(g)$$

In the K -state algorithm, we interchange the limit processes: first we calculate $m_K(g) = \lim_{N \rightarrow \infty} m_K^{(N)}(g)$, and then we increase K .

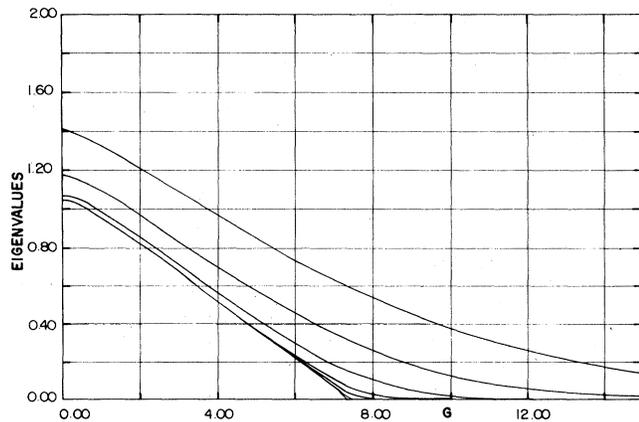


FIG. 2. From top down: masses for the 12-state approximation for 1, 2, . . . , 7 doublings.

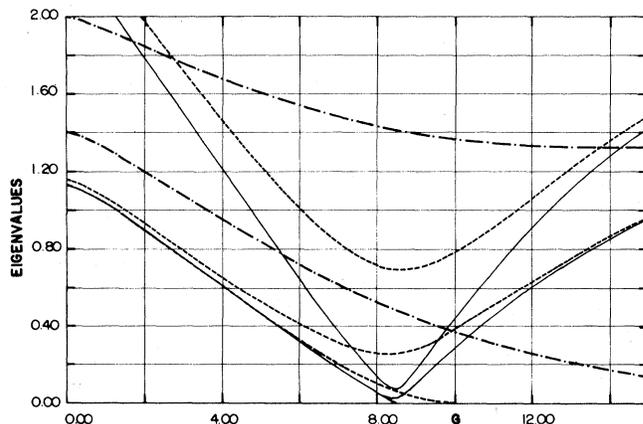


FIG. 3. Spectrum of the 4-state approximation in the range 0.0–2.0. The lowest eigenvalue is subtracted from all eigenvalues (vacuum renormalization). The dot-dashed line is for 1 doubling, the dashed line is for 4 doublings, and the solid line for 7 doublings.

RESULTS

We now exhibit the behavior of the lower part of the spectrum of the infinite-volume limit of the K -state approximation.

The dot-dashed line in Fig. 1 shows the masses for 2 coupled oscillators (1 doubling) in the 4-, 8-, and 12-state approximations. They are equal within graphical accuracy. A rigorous proof of the rapid convergence of this method and rigorous error bounds are given in Ref. 1. In fact, the error analysis in Ref. 1 indicates that the rate of convergence is exponentially fast with respect to the number of states. The dashed and solid lines show the same quantities for 16 and 128 oscillators (4 and 7 doublings). Note that in the critical region (where the mass becomes zero) the rate of convergence of the mass as K increases seems slower for larger numbers of coupled oscillators.

We show in Fig. 2 the convergence to the infinite-volume behavior of the K -state approximation by displaying the masses for 1 through 7 doublings with $K = 12$. Again, the convergence is slower in the critical region. For K equal to

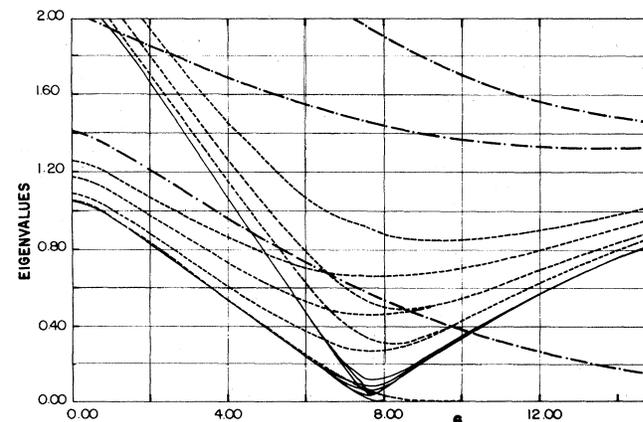


FIG. 4. Same as Fig. 3, but for the 8-state approximation.

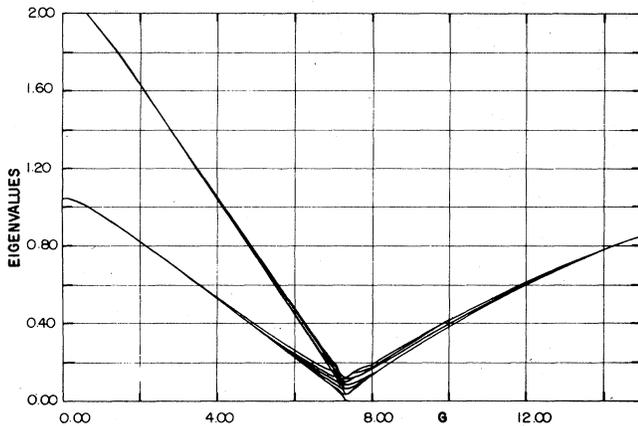


FIG. 5. Same as the solid line in Fig. 3, but for the 12-state approximation.

4 and 8, curves with similar behavior are obtained. This clearly exhibits the critical behavior of the models: the infinite doubling mass becomes zero for a finite critical value g_c of the parameter g . We see from the solid lines in Fig. 1 that the value g_c decreases with K .

The behavior of the eigenvalues above the mass as a function of the number of doublings and the number of states is similar to the behavior of the mass. This can be seen in Figs. 3, 4, and 5. We also see from the solid lines in these figures that for $g < g_c$ (away from the critical region) the spectra converge to two excited states with energies m and $2m$.

In the infinite-volume limit, the spectrum becomes continuous above the one-particle threshold if we let K tend to infinity before N . This behavior is clearly seen in the detailed graphs of Ref. 2. Since here we use a small number of trial functions, the spectra cluster about the one- and two-particle thresholds in the infinite-volume limit.

The mass for the field theories near the critical point g_c is expected to behave like

$$m(g) \approx c|g - g_c|^\nu \quad (\text{for } g < g_c),$$

where c is a constant and $\nu = 1$.⁹ The 4- and 8-state models yield $\nu = 0.9691 \pm 0.0001$ and $\nu = 1.00 \pm 0.01$, respectively. The corresponding values of g_c are $8.489861941 \pm 0.000000001$ and $7.624437432 \pm 0.000000001$.

These values were obtained from a table of infinite-volume masses constructed for values of g slightly smaller than the critical value. These masses were computed by doubling sufficiently many times to ensure convergence (typically 20 to 40 times to obtain an accuracy of 7 decimal places in the masses).

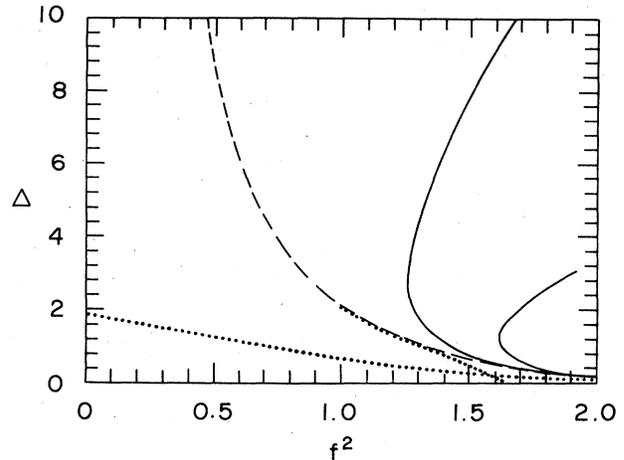


FIG. 6. Phase diagrams for $H(f^2, \Delta)$ (see Ref. 4). The two solid lines are our 4- and 8-state approximations (respectively, right and left). The three remaining curves are those from Richardson and Blankenbecler (Ref. 4).

Finally, in Fig. 6 we compare our computations of the critical point using the 4- and 8-state approximations with the results obtained by Richardson and Blankenbecler:⁴

$$H(f^2, \Delta) = \sum_{j=1}^{\infty} \left[-\frac{\partial^2}{\partial q_j^2} + (q_j^2 - f^2)^2 + \Delta(q_{j+1} - q_j)^2 \right].$$

To make the comparison, we appropriately modified the coefficients in our definition of $H^{(N)}$. Our phase diagrams appear to approach the one in Ref. 4 in the small- f^2 (large- Δ) region as we increase the number of states. Only a few states are needed to obtain an accurate phase diagram when f^2 is large (Δ small).

ACKNOWLEDGMENTS

D.I. was supported in part by NSF Grants No. MCS-80-02938 and No. INT-79.207728. E.L.I. was supported in part by NSF Grant No. PHY-80-09179 and NSF Postdoctoral Fellowship SPI-80-09169 while at Rockefeller University, New York, New York. D.M. was supported in part by NSF Grant No. PHY-80-09179 while at Courant Institute of Mathematical Sciences, New York, New York and Conselho Nacional de Desenvolvimento Científico e Tecnológico Grant No. CC-I.01891/81. P.J.P.-L. was supported in part by Conselho Nacional de Desenvolvimento Científico e Tecnológico Grants No. CC-I.01891/81 and No. 302482-82MA.

¹D. Isaacson, E. L. Isaacson, D. Marchesin, and P. J. Paes-Leme, *SIAM J. Numer. Anal.* **19**, 126 (1982).

²D. Isaacson, E. L. Isaacson, D. Marchesin, and P. J. Paes-Leme, *Phys. Rev. D* **27**, 3036 (1983).

³D. Isaacson, E. L. Isaacson, D. Marchesin, and P. J. Paes-Leme, *SIAM J. Sci. Stat. Comput.* (to be published).

⁴J. L. Richardson and R. Blankenbecler, *Phys. Rev. D* **20**, 1351 (1979).

⁵J. B. Bronzan and R. L. Sugar, *Phys. Rev. D* **21**, 1564 (1980).

⁶A. F. Pacheco, *Phys. Rev. D* **23**, 1845 (1981).

⁷R. Blankenbecler, T. DeGrand, and R. L. Sugar, *Phys. Rev. D* **21**, 1055 (1980).

⁸D. Isaacson, E. L. Isaacson, D. Marchesin, and P. J. Paes-Leme, *J. Math. Phys.* **24**, 41 (1983).

⁹J. Glimm and A. Jaffe, *Quantum Physics—A Functional Integral Point of View* (Springer, New York, 1981).