Majorana spinors in higher-dimensional theories

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The formulation of Majorana spinors in higher-dimensional spaces is discussed.

The Kaluza-Klein formulations of supergravity require the corresponding extension of Majorana spinors to higher dimensions. According to Scherk,[1] Majorana spinors may be defined for, and only for, $D=2,3,4$ mod 8 dimensions, while van Nieuwenhuizen[2] has found that $D=8,9$ are also possible. We present here another simple consideration of this question based on a slightly different extension of the definition of charge conjugation.

The condition for the existence of Majorana spinors in any number of dimensions is

$$\psi^{\dagger}\gamma_4=\psi^T C \tag{1a}$$

or

$$(\gamma_4 C^{-1})^*(\gamma_4 C^{-1})=1 \ , \tag{1b}$$

where

$$\gamma_4 S^{\dagger}\gamma_4=S^{-1} \ , \tag{2}$$

$$CS^T C^{-1}=S^{-1} \ , \tag{3}$$

and S is the spin representation of the Lorentz transformation generalized to the multidimensional space.

Equation (1) is preserved if γ_μ and C are transformed as follows:

$$\gamma'_\mu=U^{-1}\gamma_\mu U \ , \tag{4a}$$

$$C'=U^T C U \ . \tag{4b}$$

Consider a particular representation in which all matrices are Hermitian and either real or imaginary. Define C to be the product of the (p) imaginary matrices only.[3] Then

$$C=\prod_1^p \gamma_k \ , \tag{5}$$

$$C^{\dagger}=C^{-1}=\epsilon(p)C \ , \tag{6}$$

where

$$\epsilon(p)=(-)^{p(p-1)/2} \ . \tag{7}$$

The transpose of C is

$$C^T=\epsilon(p)(-)^p C \tag{8}$$

and

$$C\gamma_k C^{-1}=(-)^p\gamma_k^T \ . \tag{9}$$

Then (1b) may be rewritten

$$\gamma_4^T=(-)^p\epsilon(p)C\gamma_4 C^{-1} \ . \tag{10}$$

By (9) and (10)

$$\epsilon(p)=1 \ . \tag{11}$$

This is our condition for the existence of a Majorana spinor in any number of dimensions.

Charge conjugation will be defined here by (5) in any representation composed of matrices that are either real or imaginary. It will then be defined in any other representation by (4). Then (3) will be correct by the usual argument.

The defining representation for the γ_k will be constructed according to the following rule. If D is even, then to go from $D+1$ to $D+2$ dimensions, choose

$${}^{D+2}\gamma_k=\begin{pmatrix} {}^{D+1}\gamma_k & 0 \\ 0 & -{}^{D+1}\gamma_k \end{pmatrix}, \quad k=1,\ldots,D+1 \ . \tag{12}$$

The last matrix may be taken to be either

$${}^{D+2}\gamma_{D+2}(a)=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{13a}$$

or

$${}^{D+2}\gamma_{D+2}(b)=\begin{pmatrix} 0 & -i1 \\ i1 & 0 \end{pmatrix} \ . \tag{13b}$$

To continue from $D+2$ to $D+3$ one then includes both $\gamma(a)$ and $\gamma(b)$. If one starts this sequence with either (σ_z,σ_x) or (σ_z,σ_y) for $D=2$, one generates Table I for $D=2n$.

TABLE I. Majorana spinors in even-dimensional spaces.

D	n	p_a	p_b	$\epsilon(p_a)$	$\epsilon(p_b)$	$(-)^{p_a}\epsilon(p_a)$	$(-)^{p_b}\epsilon(p_b)$
2	1	0	1	1	1	1	−1
4	2	1	2	1	−1	−1	−1
6	3	2	3	−1	−1	−1	1
8	4	3	4	−1	1	1	1
10	5	4	5	1	1	1	−1
12	6	5	6	1	−1	−1	−1

For $D = 2n$, the numbers of imaginary matrices according to the two choices are then

$$p_a = n - 1 \ , \tag{14a}$$

$$p_b = n \ . \tag{14b}$$

According to the condition (11) and the table, Majorana spinors may exist in 2, 4, 8 mod 8 dimensions. In 2 mod 8 dimensions this condition is satisfied in two ways. The symmetry or antisymmetry of C is indicated in column $(-)^p\epsilon(p)$. For $D = 4$, Eqs. (8) and (9) take their usual form.

For $D =$ odd both $\gamma(a)$ and $\gamma(b)$ must be included. Then we have

$$p = n \tag{15}$$

and Table II. According to this table Majorana spinors are possible in $D = 3$, 9 mod 8 dimensions.

In contrast to Ref. 2 all results are here based on the single generic representation (12) and (13). Our treatment also differs from Ref. 1 in that it is not necessary to compute the number of symmetric matrices in the generic representation, since the existence of C is always assured by (5). However, we shall compute one example of such a sum.

If Γ is a product of g matrices, $\gamma_{i_1} \cdots \gamma_{i_g}$, then

$$C\Gamma = \lambda(p,g)(C\Gamma)^T \ , \tag{16}$$

where

$$\lambda(p,g) = (-)^{p(g+1)}\epsilon(g)\epsilon(p) \ . \tag{17}$$

Then the total number of symmetric matrices in the algebra is

$$N_s(D) = \sum_0^D \tfrac{1}{2}[1 + \lambda(p,g)]C_g^D \ . \tag{18}$$

Also

$$N_s(D) = \tfrac{1}{2}2^n(2^n + 1) \ . \tag{19}$$

These expressions for $N_s(D)$ must agree and must be independent of p. Here $D = 2n$. For our example choose $p_a = n - 1$. Then

$$\lambda(p,g) = (-)^{(n-1)g}\epsilon(n)\epsilon(g) \tag{20}$$

and

$$N_s(D) = \tfrac{1}{2}2^{2n} + \tfrac{1}{2}\epsilon(n)\sum_0^{2n}(-)^{(n-1)g}\epsilon(g)C_g^{2n} \tag{21}$$

TABLE II. Majorana spinors in odd-dimensional spaces.

D	p	$\epsilon(p)$	$(-)^p\epsilon(p)$
3	1	1	-1
5	2	-1	-1
7	3	-1	1
9	4	1	1
11	5	1	-1

by (18). We find

$$\sum_0^{2n}(-)^{(n-1)g}\epsilon(g)C_g^{2n} = 2^n\epsilon(n) \ \text{ for } p \text{ even } (n \text{ odd}) \ , \tag{22}$$

$$\sum_0^{2n}(-)^{(n-1)g}\epsilon(g)C_g^{2n} = (-)^n 2^n\epsilon(n) \ \text{ for } p \text{ odd } (n \text{ even}) \ . \tag{23}$$

Then

$$N_s(D) = \tfrac{1}{2}2^{2n} + \tfrac{1}{2}2^n$$

by either (22) or (23), in agreement with (19).

C is a product of p imaginary matrices only in the special defining representation. In any other representation, C is determined by (4). Suppose that C in the new representation is still of the form (5), then

$$\gamma_1 \cdots \gamma_{p'} = (U)^T(\gamma_1 \cdots \gamma_p)U \ , \tag{24}$$

where p and p' may be different. But (24) implies by transposition

$$(-)^{p'}\epsilon(p') = (-)^p\epsilon(p) \ , \tag{25}$$

and by (11), the condition for the existence of a Majorana spinor implies

$$p' - p = \text{even} \ . \tag{26}$$

To satisfy both (11) and (26) one requires

$$p' = p \ \text{mod} \ 4 \ . \tag{27}$$

However, we should find all the allowed values of p without making the assumption (24) in order to determine when Majorana spinors exist according to (11). In order to do this let us designate as standard any representation in which the matrices are Hermitian and either real or imaginary and in which C is defined by (5). We shall now show that a Majorana spinor is allowed in any standard representation according to (11) if it is allowed in our reference representations (Tables I and II).

Let U be a unitary matrix that connects two standard representations and may change p. Then

$$\gamma_k' = U^{-1}\gamma_k U \tag{28}$$

and the complex conjugate equation is

$$\epsilon_k'\gamma_k' = (U^{-1})^*\epsilon_k\gamma_k U^* \ , \tag{29a}$$

where

$$\gamma_k^* = \epsilon_k\gamma_k \ \text{and} \ \epsilon_k = \pm 1 \ . \tag{29b}$$

Then

$$\eta_k\gamma_k = V\gamma_k V^{-1} \ , \tag{30a}$$

where

$$\eta_k = \epsilon_k\epsilon_k' \tag{30b}$$

and

$$V = UU^T \ . \tag{31}$$

The reflection matrix V is symmetric and unitary.

If $\eta_k = 1$ for all k, then $UU^T = I$ and U is a rotation matrix. Let $\eta_k = -1$ for $k = 1, \ldots, K$. Then

$$-\gamma_k = V\gamma_k V^{-1}, \ \ k = 1, \ldots, K \ , \tag{32a}$$

$$\gamma_k = V\gamma_k V^{-1}, \ \ k = k+1, \ldots, D \ , \tag{32b}$$

and

$$p'=p-I+R=p+K-2I\ , \tag{33}$$

where there are I (R) imaginary (real) matrices in the original representation that change to R (I) in the new representation.

In order to find the general solution V of (32) consider the most general U. It may be built up as product of unitary transformations of the type

$$U=e^{i\Gamma\phi}\ , \tag{34}$$

where ϕ is real and Γ is a Hermitian product of s different γ_i:

$$\Gamma=i^{[1-\epsilon(s)]/2}\gamma_{i_1}\cdots\gamma_{i_s}\ . \tag{35}$$

Here $\epsilon(s)$ is the function defined in Eq. (7). Then the transpose of Γ is

$$\Gamma^T=\epsilon(s)(-)^q\Gamma\ , \tag{36}$$

where q is the number of antisymmetric or imaginary matrices in the set $(i_1,\ldots,i_s)$. Then

$$V=UU^T=\exp\{i[1+\epsilon(s)(-)^q]\Gamma\phi\}\ . \tag{37}$$

In the nontrivial case

$$\epsilon(s)(-)^q=1\ . \tag{38}$$

Then

$$\epsilon(s)=1 \text{ and } q \text{ is even} \tag{39a}$$

or

$$\epsilon(s)=-1 \text{ and } q \text{ is odd}\ . \tag{39b}$$

Then

$$V=e^{2i\Gamma\phi} \tag{40}$$

and

$$V\gamma_kV^{-1}=\cos^2 2\phi\gamma_k+\sin^2 2\phi\Gamma\gamma_k\Gamma+\frac{i}{2}\sin 4\phi(\Gamma,\gamma_k)\ . \tag{41}$$

Equation (32) now requires

$$4\phi=m\pi\ . \tag{42}$$

Since $\eta_k=1$ for all k if m is even, Eq. (32) also requires that m be odd. Then (41) becomes

$$V\gamma_kV^{-1}=\Gamma\gamma_k\Gamma=\beta_k(s)\gamma_k\ , \tag{43}$$

where

$$\beta_k(s)=(-)^{s-1} \tag{44a}$$

if γ_k is included in Γ, and

$$\beta_k(s)=(-)^s \tag{44b}$$

if γ_k is not included in Γ.

By (32) and (44)

$$\begin{aligned} &s=\text{even},\quad k=1,\ldots,K\ ,\\ &s=\text{odd},\quad k=K+1,\ldots,D\ . \end{aligned} \tag{45}$$

Also

$$V=e^{im\pi\Gamma/2}\ , \tag{46}$$

$$U=\cos\frac{m\pi}{4}+i\sin\frac{m\pi}{4}\Gamma\ . \tag{47}$$

By (45) and (46), V is a product of the set of matrices $(i_1,\ldots,i_s)$ that is either the same as the set $(1,\ldots,K)$ if s is even, or is complementary to the set $(1,\ldots,K)$ if s is odd.

The number of matrices that satisfy (32a) is s if s is even and $D-s$ if s is odd. In either case this number is

$$\mathscr{S}=\tfrac{1}{2}[1-(-)^s]D+(-)^s s\ . \tag{48}$$

The number of imaginary matrices that satisfy (32a) is q if s is even and $p-q$ if s is odd. In either case

$$I=\tfrac{1}{2}[1-(-)^s]p+(-)^s q\ . \tag{49}$$

By (33), the change in the number of imaginary matrices is

$$\begin{aligned}\Delta p&=\mathscr{S}-2I\\ &=\tfrac{1}{2}[1-(-)^s](D-2p)+(-)^s(s-2q)\ . \end{aligned} \tag{50}$$

If D is even, one may take $D=2p$ as in Table I. Then

$$\Delta p=(-)^s(s-2q)\ . \tag{51}$$

There is an increasing (decreasing) sequence corresponding to even (odd) s. By (39) this series progresses in steps of 4. If D is odd one may take $D-1=2p$ as in Table II. Then

$$\Delta p=\tfrac{1}{2}[1-(-)^s]+(-)^s(s-2q)\ . \tag{52}$$

This equation for odd D is different from (51) only in the case that s is odd. Then

$$\Delta p=1-(s-2q)\ . \tag{53}$$

This series begins at $s=1$ with $\Delta p=0$.

It follows from (50) and (52) that the possible values of p are

$$\begin{aligned} &p=n,n-1 \bmod 4\ \text{ if } D=2n\ ,\\ &p=n \bmod 4\ \text{ if } D=2n+1\ . \end{aligned} \tag{54}$$

It follows from (54) and (11) that if a Majorana spinor is allowed in our reference representations, then it is also allowed in any other standard representation. We conclude that Majorana spinors (as defined here) exist only in 2, 3, 4, 8, 9 mod 8 dimensions.

[1]J. Scherk, in *Recent Developments in Gravitation,* proceedings of a Summer Institute held in Cargèse, France, 1978, edited by M. Lévy and S. Deser, NATO Advanced Study Institute, Series B, Vol. 44 (Plenum, New York, 1979), p. 479.

[2]P. van Nieuwenhuizen, in *Supergravity 1981,* proceedings of conference held in Trieste, Italy, 1981, edited by S. Ferrara and J. G. Taylor (Cambridge Univ. Press, Cambridge, England, 1982), p. 151.

[3]R. Finkelstein, Nuovo Cimento 1, 1104 (1955).