QCD chiral-symmetry breaking in a Rayleigh-Ritz variational calculation

P. Castorina*
Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

So-Young Pi
Department of Physics, Boston University, Boston, Massachusetts 02215
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Within a Rayleigh-Ritz variational calculation for QCD, we establish the occurrence of chiral-symmetry breaking if the force is sufficiently strong. The logarithmic behavior of gauge coupling is essential for the result.

Quantum chromodynamics (QCD) is widely believed to be the theory which describes strong interactions. One of the important questions about QCD is whether it leads to dynamical chiral-symmetry breaking for massless fermions. A common approach to this problem is through a gap equation for the generated mass, which is then discussed analytically in a linearized approximation.[1]

In this paper we study the energetics of chiral-symmetry breaking, using an effective potential for composite operators, developed by Cornwall, Jackiw, and Tomboulis (CJT).[2] This method provides a variational approximation scheme that preserves some of the nonlinear features of field theory, which are presumably crucial for dynamical symmetry breaking: by performing an arbitrary variation of quark propagators in the effective potential, a nonlinear gap equation for the generated mass emerges. We improve this gap equation by allowing the coupling constant to run according to the renormalization group. Then the equation in linearized approximation possesses the chiral-symmetry-breaking solution obtained by Politzer.[3] We evaluate our effective potential, using the running coupling constant, with a specific parameter-dependent ansatz for the generated mass, consistent with the solution of the improved linearized gap equation. By varying the parameters involved, we show that for QCD with massless fermions a stable minimum exists in which chiral symmetry is spontaneously broken. Furthermore, we have found that the logarithmic behavior of the running coupling constant and the generated mass are crucial for the stability of the spontaneously broken chiral-symmetry phase.

I. EFFECTIVE POTENTIAL

We consider an $SU(N)$ gauge theory with massless fermions in the fundamental representation. The Lagrangian is

$$L=\frac{1}{2g^2}\mathrm{tr}F^2+i\overline{\Psi}\,\not{D}\Psi \,, \tag{1}$$

where the potentials A_μ and field strengths $F_{\mu\nu}$ are anti-Hermitian, Lie-algebra-valued matrices in the fundamental representation. Following CJT we construct the Hartree-Fock approximation to the generalized effective potential $V(G,\Delta)$, which depends on the complete fermion propagator G and gluon propagator Δ for the above Lagrangian:

$$V(G,\Delta)=-i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}[\ln S^{-1}(p)G(p)-S^{-1}(p)G(p)+1]+\tfrac{1}{2}i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}[\ln D^{-1}(p)\Delta(p)-D^{-1}(p)\Delta(p)+1]+V_2(G,\Delta)\,. \tag{2}$$

[Space-time and group indices are suppressed in Eq. (2).] We use the Landau gauge; the free propagators $D^{\mu\nu}$ and $S(p)$ are conventional, unit matrices in group space,

$$D^{\mu\nu}(p)=-\frac{i}{p^2}(g^{\mu\nu}-p^\mu p^\nu/p^2)I\,,$$
$$S(p)=\frac{i}{\not{p}}I\,. \tag{3}$$

The first two terms in Eq. (2) are one-loop contributions. $V_2(G,\Delta)$ is the two-loop contribution given by the graphs of Fig. 1 in the Hartree-Fock approximation. [The solid

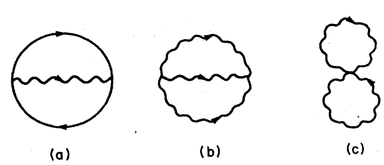

FIG. 1. Hartree-Fock approximation to two-loop effective potential. The solid line is the fermion propagator G; the wavy line is the gluon propagator Δ.

lines represent $G(p)$; the wavy lines represent $\Delta(p)$. In addition to these diagrams there are ghost contributions.] The analytic expression for Fig. 1(a), the only relevant graph for our problem, as we shall see later, is

$$V_2^{(a)}(G,\Delta)=\tfrac{1}{2}i\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\Gamma_\mu{}^a G(p)\Gamma_\nu{}^b G(k)\times\Delta_{ab}^{\mu\nu}(p-k)\ , \tag{4}$$

where $\Gamma_\mu{}^a$ is a Lie-algebra-valued vertex function.

II. THE GAP EQUATION AND LARGE-p^2 ASYMPTOTE TO $M(p)$

Demanding that $V(G,\Delta)$ be stationary against variation of G gives from Eqs. (2) and (4)

$$G^{-1}(p)=S^{-1}(p)-\Sigma(p)\ , \tag{5a}$$

$$\Sigma(p)=A(p)\not p+\Sigma_V(p)=\int\frac{d^4k}{(2\pi)^4}C_f(N)\Gamma_\mu G(k)\Gamma_\nu\Delta^{\mu\nu}(p-k)\ , \tag{5b}$$

where $C_f(N)=(N^2-1)/2N$. $A(p)$ is that part of the self-energy which exists in the normal solution, while $\Sigma_V=iM(p)$ is a possible chiral-symmetry-breaking contribution. By setting in Eq. (5) $\Delta^{\mu\nu}=D^{\mu\nu}$, $A(p)=0$, and $\Gamma_\mu=g\gamma_\mu$, i.e., their lowest values, we obtain a nonlinear equation for the generated mass:

$$iM(p)=\int\frac{d^4k}{(2\pi)^4}3g^2C_f(N)\frac{M(k)}{k^2-M^2(k)}\frac{1}{(p-k)^2}\ . \tag{6}$$

Next we improve Eq. (6) by the renormalization group to account for asymptotic freedom. It has been shown[4] from the behavior of the vertex function and the fermion and gluon propagators that for large $P^2=-p^2$, the gap equation (5b) when linearized becomes, after the angular integration,

$$M(P)=\int_0^{P^2}dK^2\frac{1}{P^2}g^2(P)M(K)\left[1+O\left[\frac{\ln K}{\ln P}\right]\right]+\int_{P^2}^{\infty}dK^2\frac{1}{K^2}g^2(K)M(K)\left[1+O\left[\frac{\ln P}{\ln K}\right]\right]\ , \tag{7}$$

where $K^2=-k^2$. The running coupling constant $g^2(P)$ is given by

$$g^2(P)=\frac{24\pi^2}{11N-2n_f}(\ln P/\Lambda)^{-1}\quad\text{for large } P\ . \tag{8}$$

Here $\Lambda\approx$ few hundred MeV is the QCD scale and n_f is the number of flavors. Therefore, we shall improve our lowest-order gap equation by introducing an effective coupling constant $\bar g^2(P,K)$ into Eq. (6):

$$\bar g^2(P,K)=g^2(P)\theta(P-K)+g^2(K)\theta(K-P)\ . \tag{9}$$

With this effective coupling, Eq. (6) when linearized is the same as Eq. (7), without the logarithmic corrections. Our improved gap equation possesses, in the large-P^2 limit, a solution[5] which is consistent with the large-P asymptote to $M(P)$ obtained by Politzer[3] using an operator-product expansion:

$$M(P)=4\langle\bar\Psi\Psi(\mu)\rangle\frac{g^2(P)}{P^2}(\ln P/\mu)^{A/2}\quad\text{for large } P\ , \tag{10}$$

where μ is a renormalization point and $A=18C_f(N)/(11N-2n_f)$. Equation (10) corresponds to the Goldstone realization of chiral-symmetry breaking.

III. RAYLEIGH-RITZ APPROXIMATION

In order to study the nonlinear aspects of the problem we evaluate

$$\Omega\equiv V(G,\Delta)-V(G_N,\Delta_N)\ , \tag{11}$$

which is the difference of the effective potential between the normal solution and symmetry-breaking solution. The equation for the normal, symmetry-preserving solution has the form

$$G_N^{-1}=S^{-1}-\Sigma_N\ ,\qquad \Sigma_N(p)=\int\frac{d^4k}{(2\pi)^4}\bar g^2C_f(N)\gamma_\mu G_N(k)\gamma_\nu D^{\mu\nu}(p-k)\ . \tag{12}$$

In the lowest approximation we shall set $\Sigma_N=0$ and $G_N{}^{-1}=S^{-1}$. The gluon propagator plays no role in chiral-symmetry breaking, hence we ignore its difference between the symmetric and asymmetric phases:

$$\Delta^{-1}(p)=\Delta_N{}^{-1}(p)=D^{-1}(p)-\Pi_N(p)\ . \tag{13a}$$

We make a further approximation of ignoring the radiative corrections to the gluon propagator (although we do use a momentum-dependent coupling constant):

$$\Pi_N(p)=0,\quad \Delta^{\mu\nu}(p)=\Delta_N^{\mu\nu}(p)=D^{\mu\nu}(p)\ . \tag{13b}$$

With these approximations diagrams (b) and (c) in Fig. 1 do not contribute to Ω as mentioned earlier and we have

$$\Omega=-i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}[\ln S^{-1}(p)G(p)-S^{-1}(p)G(p)+1]+\tfrac{1}{2}i\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\bar g^2C_f(N)\mathrm{tr}[\gamma_\mu(G(p)-S(p))\gamma_\nu(G(k)-S(k))D^{\mu\nu}(p-k)]\ , \tag{14}$$

where $\bar g^2$ is the effective coupling given in Eq. (9). Using explicit forms for the propagators G, S, and D, we obtain, after some algebra,

$$\Omega=2in_f\int\frac{d^4p}{(2\pi)^4}\ln\left[1-\frac{M^2(p)}{p^2}\right]+4in_f\int\frac{d^4p}{(2\pi)^4}\frac{M^2(p)}{p^2-M^2(p)}$$
$$-2n_f\int\frac{d^4p}{(2\pi)^4}\frac{M(p)}{p^2-M^2(p)}\int\frac{d^4k}{(2\pi)^4}3\bar{g}^2C_f(N)\frac{M(k)}{[k^2-M^2(k)](p-k)^2}\ . \tag{15}$$

Now we shall evaluate Ω with a specific, parameter-dependent ansatz for $M(p)$ and vary these parameters. First we assume that the large-distance behavior leading to confinement in QCD is irrelevant to chiral-symmetry breaking. Thus we take the following form for the running coupling constant at small P:

$$g^2(P)=\frac{24\pi^2}{11N-2n_f}(\ln P_c/\Lambda)^{-1}\equiv g_c^{\,2},\quad P\le P_c\ , \tag{16a}$$

while for large P we remain with the expression Eq. (8),

$$g^2(P)=\frac{24\pi^2}{11N-2n_f}(\ln P/\Lambda)^{-1},\quad P\ge P_c\ . \tag{16b}$$

P_c is a momentum defining the infrared region. Then it is reasonable to use the following form for the generated mass, as a variational ansatz:

$$M(P)=m\quad\text{for } P\le P_c$$
$$=m\frac{P_c^{\,2}}{P^2}\ln\left[\frac{P_c}{\Lambda}\right]^{1-A/2}\left[\ln\frac{P}{\Lambda}\right]^{A/2-1}\quad\text{for } P\ge P_c\ , \tag{17}$$

which is consistent with the asymptotic solution given in Eq. (10).

With the above expression for $g^2(P)$ and $M(P)$, we find that our effective potential Ω/Λ^4 is a function of m/Λ and P_c/Λ whose minimum determines the solution m/Λ for various values of P_c/Λ. Our order parameter m plays the role of some suitably regularized expectation value $\langle\bar{\Psi}\Psi\rangle$ and the value of P_c is related to the maximum strength of the coupling constant involved in the calculation. We take Λ as a fixed parameter in this problem.

IV. RESULTS

We have analyzed our effective potential numerically and obtained the following results: For SU(3) with $n_f=3$, dynamical symmetry breaking occurs when $P_c/\Lambda\lesssim 1.5$ and a stable minimum exists at $m/\Lambda\lesssim 1$. For example, when $P_c/\Lambda=1.3$, the minimum occurs at $m=0.5\Lambda$. The effective potential Ω for $P_c/\Lambda=1.3$ is shown in Fig. 2(a). As P_c/Λ decreases, the value of m at the minimum increases. For $P_c/\Lambda>1.5$, $m=0$ is always the global minimum [see Fig. 2(b)]. This tells us that a large coupling constant,

$$\frac{3g_c^{\,2}C_f(N=3)}{8\pi^2}\gtrsim 1 \tag{18}$$

is necessary for dynamical chiral-symmetry breaking. When we change the number of flavors, for example $n_f=9$, dynamical symmetry breaking occurs when $P_c/\Lambda\lesssim 2.25$, which again implies the requirement in Eq. (18). Our results agree with the numerical study of the direct solution ot the renormalization-group-improved nonlinear integral equation carried out by Higashijima,[6] and confirm the validity of the variational Rayleigh-Ritz method for dynamical symmetry breaking developed by CJT.

As a check that dynamical symmetry breaking occurs for $P_c/\Lambda\lesssim 1.5$ we have made the following observation. Instead of evaluating the effective potential first and then varying m to look for the solution, we simplify the effective potential by using the minimization condition, Eq.

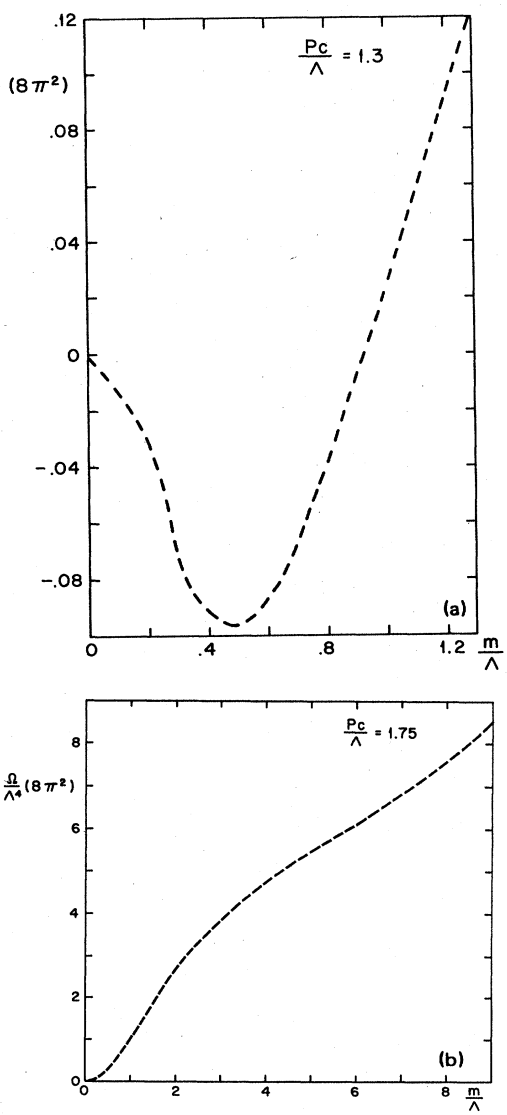

FIG. 2. (a) Effective potential Ω for P_c/Λ = 1.3, $N=3$, and $n_f=3$. (b) Effective potential Ω for P_c/Λ = 1.75, $N=3$, and $n_f=3$.

(6), which results from an arbitrary variation of G: the two-loop contribution in Eq. (15) can be written in a simple form if we use Eq. (6):

$$-2\int \frac{d^4p}{(2\pi)^4}\frac{M(p)}{p^2-M^2(p)}\int \frac{d^4k}{(2\pi)^4}3\bar{g}^2C_f(N)\times\frac{M(k)}{k^2-M^2(k)}\frac{1}{(p-k)^2}=-2i\int \frac{d^4p}{(2\pi)^4}\frac{M^2(p)}{p^2-M^2(p)}\ . \quad (19a)$$

Then, together with the one-loop contribution, we obtain

$$\bar{\Omega}=2in_f\int \frac{d^4p}{(2\pi)^4}\ln\left[1-\frac{M^2(p)}{p^2}\right]+2in_f\int \frac{d^4p}{(2\pi)^4}\frac{M^2(p)}{p^2-M^2(p)}\ . \quad (19b)$$

The physical meaning of $\bar{\Omega}$, which is different from Ω, can be given as follows: $\bar{\Omega}$ is the effective potential for every value of m which is a solution of the gap equation, Eq. (6). Therefore, for a symmetry-breaking solution, $m\neq 0$, $\bar{\Omega}$ must be negative in order that the solution have lower energy than the symmetric phase $m=0$. Indeed, we find that for $P_c/\Lambda \lesssim 1.5$, $\bar{\Omega}<0$ for all m (see Fig. 3). Ω becomes more negative for larger m, implying that for larger values of m the corresponding minimum potential energy is lower. Notice that $\bar{\Omega}$, in Fig. 3, is unbounded from below. However, this does not mean that there is an instability, since $\bar{\Omega}$ does not determine the minimum. Minimization is carried out on Ω, which is bounded below, and $\bar{\Omega}$ is the value of Ω at the stationary point. That $\bar{\Omega}$ is negative is a clear signal for dynamical symmetry breaking. The value of Ω and $\bar{\Omega}$ at the stationary point, for example, $m=0.5\Lambda$ for $P_c/\Lambda=1.3$ should be in principle the same. However, in Figs. 2(a) and 3, they are somewhat different. This discrepancy comes from the fact that for Ω an ansatz for $M(p)$ has been used, while for $\bar{\Omega}$, the exact equation for $M(P)$ in the approximation which we use has been employed to reexpress the two-loop term in the same form as the one-loop term. The ansatz has been used only at the later stage to evaluate the single loop. Thus, the value of $\bar{\Omega}$ at $m=0.5\Lambda$ is a better approximation to the true vacuum energy than Ω.

In fact, when the integrals in (19b) are continued in Euclidean space and M^2 is assumed to be positive there, one sees that $\bar{\Omega}$ is an integral of a negative integrand, hence always negative, on a solution, even outside our ansatz.[7]

V. DISCUSSIONS

We emphasize that the logarithmic behavior of g^2 and $M(P)$ in Eqs. (16) and (17) is crucial for the stability of dynamical chiral-symmetry breaking. When we evaluate Ω ignoring the logarithmic behavior *both* in the coupling constant and in $M(P)$, i.e., using the ansatz

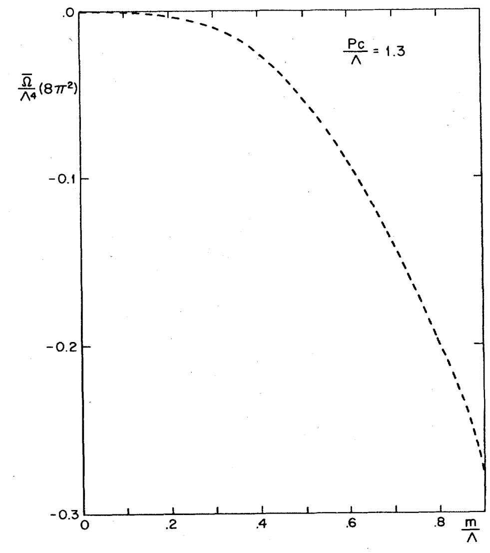

FIG. 3. Effective potential $\bar{\Omega}$ for $P_c/\Lambda = 1.3$, $N=3$, and $n_f=3$.

$$M(P)=m,\quad P\le P_c$$
$$=mP_c{}^2/P^2,\quad P\ge P_c \quad (20)$$

and a *constant* coupling constant, we find that dynamical symmetry breaking occurs for $3g^2C_f(N)/8\pi^2>\frac{2}{3}$, but a stable minimum does not exist; the effective potential decreases monotonically as m increases from zero and is not bounded from below. For $3g^2C_f(N)/8\pi^2<\frac{2}{3}$, $m^2=0$ is a stable minimum, indicating no symmetry breaking. Our interpretation of this is that by taking a *constant* coupling constant [and Eq. (20)] in the lowest-order calculation of Ω to which only Fig. 1(a) contributes, one is considering a theory which is not asymptotically free and for that theory no stable minimum exists with chiral-symmetry breaking. We stress that there is no connection between the two Ω's: one cannot obtain the Ω with constant g^2 and $M(P)$ in Eq. (20) from our Ω for QCD, by varying in $M(P)$ the power in the logarithm away from $A/2-1$, because there is further logarithmic dependence in g^2 for the QCD Ω. Our study of Ω using constant g^2 and Eq. (20) shows the importance of the logarithmic behavior, which is a hallmark of an asymptotically free theory.

In the course of our study we received a paper by Casalbuoni *et al.*,[8] in which chiral-symmetry breaking in QCD is examined by a method similar to ours, omitting the logarithmic behavior.

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*Permanent address: Instituto Dipartmentale di Fisica dell' Universita, 57 Corso Italia, I-95129 Catania, Italy.

[1]For example, K. Lane, Phys. Rev. D **10**, 2605 (1974); S. H. Tye, E. Tomboulis, and E. C. Poggio, *ibid.* **11**, 2839 (1975); H. Pagels, *ibid.* **19**, 3080 (1979).

[2]J. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D **10**, 2428 (1974).

[3]H. D. Politzer, Nucl. Phys. **B117**, 397 (1976).

[4]Tye *et al.*, in Ref. 1.

[5]When we convert Eq. (7) to a differential equation we obtain

$$8M'(P)+4P^2M''(P)=[-\lambda(P)+\lambda'(P)P^2]\frac{M(P)}{P^2}+\frac{4\lambda''(P)M'(P)P^4}{-\lambda(P)+\lambda'(P)P^2},$$

where

$$\lambda(P)\equiv\frac{1}{4\pi^2}3g^2(P)C_f(N)=\frac{A}{\ln P/\Lambda}$$

and the prime denotes differentiation with respect to P^2.

[6]K. Higashijima, Phys. Rev. D **29**, 1228 (1984). We emphasize that our calculation concerns the energetics of the problem, which are not addressed in this reference; i.e., we show that the solution of the gap equation has lower energy than the symmetric solution.

[7]We see no evidence of the saddle-point behavior claimed by J. Perez-Mercader and R. W. Haymaker, Phys. Rev. D **27**, 1353 (1983).

[8]R. Casalbuoni, S. De Curtis, D. Dominici, and R. Gatto, Phys. Lett. **140B**, 357 (1984).