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Gauge invariance and the finite-element solution of the Schwinger model

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We apply the method of finite elements to two-dimensional quantum electrodynamics. We construct gauge-invariant operator difference equations and compute the chiral anomaly in the Schwinger model. The relative error between the exact answer and the finite-element prediction vanishes like M^{-2}, where M is the number of finite elements.

I. INTRODUCTION

In a 1970 Physics Today article, P. A. M. Dirac wrote:[1]

> The phenomena of high-energy physics have stimulated the development of several new mathematical approaches to calculate and explain the experimental results. Many of these approaches bear little relation to methods used in other areas of physics and many have incomplete or unsatisfactory aspects to them. They have been used with varying success. Methods based on the equations of motion, so necessary for low-energy physics, have been largely abandoned as being intractable to this latest branch of physics. Yet if we believe in the unity of physics, we should believe that the same basic ideas universally apply to all fields of physics. Should we not then use the equations of motion in high-energy as well as low-energy physics? I say we should. A theory with mathematical beauty is more likely to be correct than an ugly one that fits some experimental data.

Our intention here is to respond to Dirac's challenge.

In recent papers it was proposed that the method of finite elements could be used to solve operator quantum field equations on a Minkowski lattice. A first paper[2] showed that this method reduces the Heisenberg operator equations of ordinary quantum mechanics and of two-dimensional self-interacting boson theories to operator difference equations which can be solved explicitly. The solution exactly preserves the equal-time commutation relations of the field operators. A second paper[3] applied the same ideas to free fermion field theories. The operator Dirac equation was solved using the method of finite elements and the solution was shown to be consistent with the canonical equal-time anticommutation relations. Moreover, this Minkowski lattice solution preserves unitarity (and chiral invariance in the massless theory), there is no fermion doubling, and the lattice Lagrangian (but *not* the Hamiltonian[4]) is local.

by using the method of finite elements. In this paper we implement Abelian gauge invariance. Specifically, we formulate electrodynamics on a two-dimensional Minkowski lattice and construct operator difference equations which are manifestly gauge invariant and possess the correct continuum limit. In the massless lattice theory (the Schwinger model) chiral symmetry is broken and a mass is generated. We compute this symmetry-breaking mass and obtain answers in excellent agreement with the known continuum results; the relative error between the exact answer and the finite-element prediction vanishes like M^{-2}, where M is the number of finite elements.

II. GAUGE INVARIANCE ON A FINITE-ELEMENT LATTICE

The field equations in two-dimensional continuum electrodynamics are

$$\partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} , \tag{1}$$

$$\partial_\nu F^{\mu\nu} = J^\mu , \tag{2}$$

$$(i\partial\!\!\!/ + eA\!\!\!/ + \mu)\psi = 0 . \tag{3}$$

Classically, the electric current is $J^\mu = e\bar{\psi}\gamma^\mu\psi$. However, in quantum field theory it is necessary to define J^μ as a limit of a point-separated current.

An infinitesimal local gauge transformation is

$$\psi \to \psi + \delta\psi, \quad \delta\psi = ie\delta\Lambda\psi , \tag{4}$$

$$A_\mu \to A_\mu + \delta A_\mu, \quad \delta A_\mu = \partial_\mu \delta\Lambda . \tag{5}$$

Under this gauge transformation the left side of (1) does not change; this implies that the field strength $F_{\mu\nu}$ is gauge invariant. The current J^μ is formally gauge invariant [under (4), $J^\mu \to J^\mu + O(\delta\Lambda^2)$]. Thus, both sides of (2) are gauge invariant. Individual terms on the left side of (3) are not gauge invariant, but their sum is if ψ satisfies (3).

On a finite-element Minkowski lattice, derivatives are averaged forward differences.[2,3] Thus, the infinitesimal gauge transformation (5) becomes

$$(\delta A_0)_{m,n}=\frac{1}{2h}(\delta\Lambda_{m+1,n+1}+\delta\Lambda_{m,n+1}-\delta\Lambda_{m+1,n}-\delta\Lambda_{m,n}),$$
$$(\delta A_1)_{m,n}=\frac{1}{2h}(\delta\Lambda_{m+1,n+1}+\delta\Lambda_{m+1,n}-\delta\Lambda_{m,n+1}-\delta\Lambda_{m,n}), \quad (6)$$

where h is the spacing on a square Minkowski lattice, m is the space variable, $m=0,1,2,\ldots,M$, and n is the time variable, $n=0,1,2,\ldots,N$. We take the boson fields and their variations to be periodic:

$$(A_\mu)_{0,n}=(A_\mu)_{M,n},\quad (\delta A_\mu)_{0,n}=(\delta A_\mu)_{M,n},$$
$$\delta\Lambda_{0,n}=\delta\Lambda_{M,n}. \quad (7)$$

The left side of the finite-element transcription of (1) is invariant under the transformation (6). Thus $(F_{\mu\nu})_{m,n}$ is gauge invariant.

Constructing a gauge-invariant form of the Dirac equation on the lattice is the nontrivial part of this discussion. The free Dirac equation $(i\partial\!\!\!/+\mu)\psi=0$, by the finite-element prescription, becomes[3]

$$0=i\gamma^0(\psi_{m+1,n+1}+\psi_{m,n+1}-\psi_{m+1,n}-\psi_{m,n})/(2h)$$
$$+i\gamma^1(\psi_{m+1,n+1}+\psi_{m+1,n}-\psi_{m,n+1}-\psi_{m,n})/(2h)$$
$$+\mu(\psi_{m+1,n+1}+\psi_{m+1,n}+\psi_{m,n+1}+\psi_{m,n})/4. \quad (8)$$

Equation (8) is, of course, invariant under the global phase transformation

$$\delta\psi_{m,n}=ie\delta\Omega\psi_{m,n}, \quad (9)$$

where $\delta\Omega$ is independent of the lattice site. The appropriate local generalization of (9) is

$$\delta\Psi_{m,n}=ie\delta\Omega_{m,n}\Psi_{m,n}, \quad (10)$$

where

$$\Psi_{m,n}=\tfrac{1}{4}(\psi_{m,n}+\psi_{m,n+1}+\psi_{m+1,n}+\psi_{m+1,n+1}),$$

for then the mass term in the Dirac equation transforms covariantly under an infinitesimal local gauge transformation.

A strict interpretation of the finite-element prescription in Refs. 2 and 3 suggests the connection between the two infinitesimal quantities $\delta\Omega_{m,n}$ and $\delta\Lambda_{m,n}$:

$$\delta\Omega_{m,n}=(\delta\Lambda_{m+1,n+1}+\delta\Lambda_{m+1,n}+\delta\Lambda_{m,n+1}+\delta\Lambda_{m,n})/4. \quad (11)$$

We solve (10) for $\delta\psi_{m,n}$:

$$\delta\psi_{m,n}=(-1)^n\delta\psi_{m,0}+2ie\sum_{n'=0}^{n-1}\left[\sum_{m'=1}^{m-1}-\sum_{m'=m}^{M}\right](-1)^{m+m'+n+n'}\delta\Omega_{m',n'}\Psi_{m',n'}. \quad (12)$$

Note that (12) is not the most general solution to (10); it has been simplified by imposing the boundary conditions[5]

$$\psi_{M,n}=(-1)^{M+1}\psi_{0,n},\quad \delta\psi_{M,n}=(-1)^{M+1}\delta\psi_{0,n}. \quad (13)$$

In addition to these spatial boundary conditions, the variations $\delta\psi_{m,n}$ must also satisfy boundary conditions at $t=0$ ($n=0$). We discuss these after (15).

Transforming the space and time derivatives in (8) using (12) we find that

$$\delta(\psi_{m+1,n+1}+\psi_{m+1,n}-\psi_{m,n+1}-\psi_{m,n})=ie\delta\Omega_{m,n}(\psi_{m+1,n+1}+\psi_{m+1,n}-\psi_{m,n+1}-\psi_{m,n})$$
$$+ie\left[\sum_{m'=1}^{m}-\sum_{m'=m+1}^{M}\right](-1)^{m+m'}(\delta\Omega_{m',n}-\delta\Omega_{m'-1,n})(\psi_{m',n+1}+\psi_{m',n}), \quad (14)$$

$$\delta(\psi_{m+1,n+1}+\psi_{m,n+1}-\psi_{m+1,n}-\psi_{m,n})=ie\delta\Omega_{m,n}(\psi_{m+1,n+1}+\psi_{m,n+1}-\psi_{m+1,n}-\psi_{m,n})$$
$$+2ie\sum_{n'=1}^{n}(-1)^{n+n'}(\delta\Omega_{m,n'}-\delta\Omega_{m,n'-1})(\psi_{m,n'}+\psi_{m+1,n'})$$
$$+2ie(-1)^n\delta\Omega_{m,0}(\psi_{m,0}+\psi_{m+1,0})-2(-1)^n(\delta\psi_{m,0}+\delta\psi_{m+1,0}). \quad (15)$$

The continuum limit[6] of (15) must agree with the time derivative of (4):

$$\partial_0(\delta\psi)=ie\partial_0(\delta\Lambda)\psi+ie\delta\Lambda(\partial_0\psi).$$

This requirement determines the boundary conditions on the variation of the lattice Dirac field at $n=0$:

$$\delta(\psi_{m,0}+\psi_{m+1,0})=ie[\delta\Omega_{m,0}-(\delta\Omega_{m,1}-\delta\Omega_{m,0})/2](\psi_{m,0}+\psi_{m+1,0}).$$

A first approximation to the interaction term on the lattice is

$$(I_1)_{m,n}=\frac{e\gamma^0}{2}\sum_{n'=1}^{n}(-1)^{n+n'}[(A_0)_{m,n'}+(A_0)_{m,n'-1}](\psi_{m+1,n'}+\psi_{m,n'})$$
$$+\frac{e\gamma^0}{4}(-1)^n[(A_0)_{m,1}+(A_0)_{m,0}](\psi_{m,0}+\psi_{m+1,0})$$
$$+\frac{e\gamma^1}{4}\left[\sum_{m'=1}^{m}-\sum_{m'=m+1}^{M}\right](-1)^{m+m'}[(A_1)_{m',n}+(A_1)_{m'-1,n}](\psi_{m',n+1}+\psi_{m',n})\;, \tag{16}$$

because if we perform an infinitesimal gauge transformation by varying (16) with respect to A, according to (6), and combine the result with the terms obtained by varying the free Dirac equation (8), then all terms of order $\delta\Lambda_{m,n}$ cancel. This cancellation depends on ψ satisfying the free Dirac equation. However, there are two problems: first, we neglected to vary (16) with respect to ψ; second, when the interaction term (16) is included, the free Dirac equation is no longer satisfied. We therefore vary I_1 with respect to ψ and obtain new first-order terms. Then we invent a second interaction term I_2 so chosen that, if it is varied with respect to A and combined with the terms obtained by varying I_1 with respect to ψ, all first-order terms once again vanish, provided that ψ satisfies the Dirac equation with interaction term I_1. Again we have neglected to vary I_2 with respect to ψ. Also, with the interaction term I_2 included, ψ no longer satisfies the Dirac equation with I_1 only. Thus, we invent a third interaction term I_3, where the variation of I_3 with respect to A cancels the variation of I_2 with respect to ψ, given that ψ satisfies the Dirac equation including both I_1 and I_2. This process is continued *ad infinitum* with the nth interaction term I_n proportional to n powers of $e, n-1$ powers of h, and n powers of A. Fortunately, the infinite series of interaction terms $I=I_1+I_2+I_3+\cdots$ exponentiates and can be summed in closed form:[7]

$$I_{m,n}=\frac{-2i\gamma^0}{h}\sum_{n'=1}^{n}(-1)^{n+n'}\exp\left[ieh\sum_{n''=n'+1}^{n}B_{m,n''}\right](e^{iehB_{m,n'}}-1)\phi_{m,n'}$$
$$-\frac{2i\gamma^0}{h}(-1)^n\exp\left[ieh\sum_{n''=1}^{n}B_{m,n''}\right](e^{iehB_{m,1}/2}-1)\phi_{m,0}+\frac{i\gamma^1}{h}\sec\left[\frac{eh}{2}\sum_{m'=1}^{M}C_{m',n}\right]\sum_{m''=1}^{M}\operatorname{sgn}(m''-m)(-1)^{m+m''}$$
$$\times\exp\left[\frac{ieh}{2}\sum_{m'''=1}^{M}\operatorname{sgn}(m'''-m)\operatorname{sgn}(m'''-m'')\operatorname{sgn}(m'''-m)C_{m''',n}\right](e^{iehC_{m'',n}}-1)\theta_{m'',n}\;, \tag{17}$$

where

$$B_{m,n}=[(A_0)_{m,n}+(A_0)_{m,n-1}]/2\;, \tag{18}$$

$$C_{m,n}=[(A_1)_{m,n}+(A_1)_{m-1,n}]/2\;, \tag{19}$$

$$\phi_{m,n}=(\psi_{m,n}+\psi_{m+1,n})/2\;, \tag{20}$$

$$\theta_{m,n}=(\psi_{m,n}+\psi_{m,n+1})/2\;, \tag{21}$$

and $\operatorname{sgn}(x)$ is 1 if $x>0$ and -1 if $x\leq 0$. This interaction term completely solves the gauge-invariance problem. The full lattice Dirac equation is (8) with $I_{m,n}$ included on the right side. This equation is invariant under the gauge transformation (6) and (12). It is also invariant under finite gauge transformations. In the limit $h\to 0$ we recover the continuum Dirac equation (3).[6] Moreover, the point separation mentioned after (3) (point separation is required to give rise to chiral-symmetry breaking and mass generation in the massless theory) has arisen in a natural way in the interaction term I. It is crucial that (17) involves only fields at time $n+1$ and earlier, so that we can solve the Dirac and Maxwell equations interactively in terms of the initial fields by time-stepping through the lattice.

III. THE SCHWINGER MODEL FOR ONE FINITE ELEMENT

The Schwinger model[8] is *ab initio* massless $(1+1)$-dimensional quantum electrodynamics. Although essentially trivial, it points the way to dynamical symmetry breaking and mass generation because manifest gauge invariance is seen to be compatible with a "photon" mass. This feature, together with the solvability of the model, has made it a popular laboratory for studying the chiral anomaly in field theory. The finite-element procedure leads to an interaction which is exactly gauge invariant, but which exhibits the nonlocal, point-separation structure necessary to break chiral symmetry. It is precisely the breaking of chiral symmetry which in the continuum gives rise to the chiral anomaly

$$\partial_\mu J_5^\mu=-\frac{e^2}{\pi}E\;, \tag{22}$$

where $E=F_{01}$ is the single nonvanishing component of the field-strength tensor. One then infers the dynamically generated boson mass

$$\omega=e\pi^{-1/2}\;. \tag{23}$$

In this section we present a simple example of a fieldtheoretic calculation using the method of finite elements. We consider the massless Schwinger model for the case of one finite element $M=1, N=1$. We choose a gauge in which $A_0=0$ and $A_1=\mathscr{A}$. The periodicity condition (7) implies that there is no space variation on one finite element, and that only the time index $n=0,1$, need be displayed. The difference equation corresponding to (1) is

$$(E_1+E_0)/2=(\mathscr{A}_1-\mathscr{A}_0)/h \ . \tag{24a}$$

The simplest choice for a gauge-invariant current is

$$J^{\mu}=e(\bar{\psi}_0+\bar{\psi}_1)\gamma^{\mu}(\psi_0+\psi_1)/4 \ .$$

From this definition the two components $\mu=0$ and $\mu=1$ of (2) become

$$(\psi_0^{\dagger}+\psi_1^{\dagger})(\psi_0+\psi_1)=0 \ , \tag{24b}$$

$$(E_1-E_0)/h=e(\psi_0^{\dagger}+\psi_1^{\dagger})\gamma^0\gamma^1(\psi_0+\psi_1)/4 \ . \tag{24c}$$

We will return to the question of how to construct a conserved gauge-invariant current for a general lattice in Sec. IV.

The lattice Dirac equation on one finite element is

$$i\gamma^0(\psi_1-\psi_0)=\gamma^1(\psi_0+\psi_1)\tan(eh\mathscr{A}_0/2) \ . \tag{24d}$$

The equal-time anticommutation relations are $[\psi_n^{a\dagger},\psi_n^b]_+ =\delta^{ab}/h$, where the superscripts $a,b=1,2$ are Dirac indices.

To calculate the chiral-symmetry-breaking mass ω, we make the approximation that there exist six states in this field theory: the vacuum state $|0\rangle$, a boson state $|B\rangle$ of mass ω, and four fermion states. Following the procedure used in Ref. 2 to compute masses in quantum oscillator problems, we use the Heisenberg equation for operators, which in the continuum reads

$$i\frac{d}{dt}\mathscr{A}=[\mathscr{A},H] \ ,$$

to relate the time dependence of an operator to its mass spectrum. Taking the expectation value of the above commutator between $|0\rangle$ and $|B\rangle$ states gives, on the lattice,

$$(\langle B|\mathscr{A}_1|0\rangle-\langle B|\mathscr{A}_0|0\rangle)/h \sim i\omega\langle B|\mathscr{A}_0|0\rangle \ . \tag{24e}$$

This asymptotic relation becomes exact in the limit $h\to 0$.

We now solve the five equations (24) in the asymptotic limit $h\to 0$. In this limit the difference between $\mathscr{A}_0$ and $\mathscr{A}_1$, E_0 and E_1, and ψ_0 and ψ_1 is $O(h)$, and can be approximated by using equations like (24e). We also drop the time index and replace $\psi_0+\psi_1$ by 2ψ. The anticommutation relation sets the scale for the Fermi field: $\psi=O(h^{-1/2})$. Finally, we use (24b) to eliminate the lower component of the Fermi field β, in favor of the upper component α, where $\psi=h^{-1/2}(\alpha,\beta)$. Equation (24b) is particularly interesting because it implies that the metric is not positive definite. If the metric were positive definite, the theory would be trivial. We perform the indicated algebra and reduce (24) to

$$0=\tan(e^2\langle B|\alpha^{\dagger}\alpha|0\rangle/\omega^2) \ . \tag{25}$$

The anticommutation relation $[\alpha^{\dagger},\alpha]_+=1$ implies that the matrix element in (25) equals c, a number of order one.[9] There is some ambiguity here but choosing the lowest nontrivial zero of the tangent function, we have $\omega^2=ce^2/\pi$. This result agrees well with the known result e^2/π for the continuum Schwinger model.[8] It is not clear why such good agreement is achieved for only one finite element. One possible reason is that the model is free in the continuum. Another is that while commutation relations only have infinite-dimensional representations, anticommutation relations do have finite-dimensional representations, in this case six-dimensional (six states).

IV. THE CHIRAL ANOMALY IN THE SCHWINGER MODEL

In this section we perform a direct calculation of the chiral anomaly by computing the divergence of the axial-vector current. We begin by noting that in the gauge $A_0=0$ the interaction term (17) satisfies the recursion relation

$$I_{m,n}=-e^{iehC_{m,n}}I_{m-1,n}-\frac{2i\gamma^1}{h}(e^{iehC_{m,n}}-1)\theta_{m,n} \ . \tag{26}$$

Using this gauge we can solve the massless ($\mu=0$) Dirac equation very easily because the chiral components decouple. Denoting the ± 1 eigenvalues of $\gamma_5\equiv\gamma^0\gamma^1$ by superscripts, we find that

$$\phi_{m,n+1}^{(+)}=e^{iehC_{m,n}}\phi_{m-1,n}^{(+)} \ ,$$
$$\phi_{m-1,n+1}^{(-)}=e^{-iehC_{m,n}}\phi_{m,n}^{(-)} \ , \tag{27}$$

which differ from solutions of the free massless Dirac equation by simple phase factors depending on the vector potential A contained in C. Evidently, it is appropriate to regard the field ϕ as canonical, for the solution (27) preserves the canonical equal-time anticommutation relations,

$$[\phi_{m,n}^{(\pm)\dagger},\phi_{m',n}^{(\pm)}]_+=\frac{1}{h}\delta_{m,m'} \ . \tag{28}$$

To proceed we need an expression for the current J^{μ}, the source of the field E. On the lattice the Maxwell equation (2) becomes

$$-\frac{1}{2h}(E_{m+1,n+1}+E_{m+1,n}-E_{m,n+1}-E_{m,n})=J_{m,n}^0 \ ,$$
$$\frac{1}{2h}(E_{m+1,n+1}+E_{m,n+1}-E_{m+1,n}-E_{m,n})=J_{m,n}^1 \ . \tag{29}$$

Thus, the analog of $\partial^{\mu}J_{\mu}=0$ is

$$\frac{1}{2h}(J_{m+1,n+1}^0+J_{m,n+1}^0-J_{m+1,n}^0-J_{m,n}^0+J_{m+1,n+1}^1+J_{m+1,n}^1-J_{m,n+1}^1-J_{m,n}^1)=0 \ . \tag{30}$$

It is extremely fortunate that the finite-element transcription of the current

$$J_{m,n}^{\mu}=e\Psi_{m,n}^{\dagger}\gamma^0\gamma^{\mu}\Psi_{m,n} \tag{31}$$

satisfies the discrete conservation law in (30). Equation (31) is also gauge invariant, according to (10). Using (27) we easily find the lattice divergence, Eq. (30), of (31) to be

$$(\text{“}\partial_\mu J^\mu\text{”})_{m,n}=\frac{ie}{2h}\sin\left[\frac{eh^2}{2}\widetilde{E}_{m,n}\right]e^{ieh(C_{m+1,n+1}+C_{m+1,n})/2}\times\phi^\dagger_{m+1,n+1}\phi_{m,n+1}+\text{H.c.}\ , \tag{32}$$

where

$$\widetilde{E}_{m,n}=\frac{1}{2h}[(A_1)_{m+1,n+1}+(A_1)_{m,n+1}-(A_1)_{m+1,n}-(A_1)_{m,n}]\ .$$

Equation (32) has vanishing vacuum expectation value because the Dirac equation implies that

$$\langle\phi_{m,n}\phi^\dagger_{m+1,n}\rangle\propto\gamma_5\ . \tag{33}$$

Hence the trace on Dirac indices in (32) vanishes and (31) is an acceptable conserved current.[10]

The axial-vector current is defined by

$$(J_5^\mu)_{m,n}\equiv e\Psi^\dagger_{m,n}\gamma^0\gamma^\mu\gamma_5\Psi_{m,n}=-\epsilon^{\mu\nu}(J_\nu)_{m,n}\ , \tag{34}$$

where $\epsilon^{\mu\nu}=-\epsilon^{\nu\mu}$, $\epsilon^{01}=1$. The lattice divergence of (34) differs from (32) by the replacement

$$\phi^\dagger_{m+1,n+1}\phi_{m,n+1}\rightarrow\phi^\dagger_{m+1,n+1}\gamma_5\phi_{m,n+1}\ . \tag{35}$$

Thus, Eq. (33) implies a nonzero value for the vacuum expectation value of the lattice divergence of (34). The canonical anticommutation relations imply that[11,12]

$$\langle\phi_{m,n}\phi^\dagger_{m+1,n}\rangle=-\frac{i\gamma_5}{Mn\sin(\pi/M)} \tag{36}$$

[see Eq. (49)]. Thus (32) and (35) yield the axial anomaly

$$(\text{“}\partial_\mu J_5^\mu\text{”})_{m,n}=-e^2\left[M\sin\frac{\pi}{M}\right]^{-1}\widetilde{E}_{m,n}\ . \tag{37}$$

This formula, which is the main result of this paper, rapidly approaches the continuum result (22) as M, the number of lattice sites in the space direction, approaches ∞. The $M=2$ result, $\omega=2^{-1/2}e$, is in error by only 25%.[13]

To prove (36) we work in momentum space. At the nth time step let

$$\psi_{m,n}=u_n e^{ip_k m}\ , \tag{38}$$

where $1\le k\le M$ and $p_k=2\pi k/M$ if ψ satisfies periodic spatial boundary conditions and $p_k=(2k+1)\pi/M$ if ψ satisfies antiperiodic spatial boundary conditions. Then

$$\psi_{m,n+1}=u_{n+1}e^{ip_k m}\ , \tag{39}$$

where the free Dirac equation implies that[3]

$$u_{n+1}=Tu_n$$

with

$$T=[\epsilon^2(1+e^{ip_k})^2+e^{ip_k}]^{-1}\times\begin{bmatrix}-\epsilon^2(1+e^{ip_k})^2+1 & \epsilon(1+e^{ip_k})^2\\ -\epsilon(1+e^{ip_k})^2 & -\epsilon^2(1+e^{ip_k})^2+e^{2ip_k}\end{bmatrix} \tag{40}$$

in a representation where γ_5 is diagonal, and $\epsilon=\mu h/4$. The energies of the modes are given by the eigenvalues t of T,

$$t=e^{-i\omega h}\ , \tag{41}$$

where

$$\tan^2(\omega h/2)=4\epsilon^2+\tan^2(p_k/2)\ . \tag{42}$$

Note that (42) is just the dispersion relation found in Ref. 3.

Here we are interested in the simple case $\epsilon=\mu=0$; then

$$T=\begin{bmatrix}e^{-ip_k} & 0\\ 0 & e^{ip_k}\end{bmatrix},\quad \omega=\pm p_k/h\ . \tag{43}$$

In terms of the eigenvectors of T,

$$v^{(+)}=\begin{bmatrix}1\\0\end{bmatrix},\quad v^{(-)}=\begin{bmatrix}0\\1\end{bmatrix}\ , \tag{44}$$

the free Dirac field has the expansion

$$\psi_{m,n}=\sum_{k=1}^{M}e^{ip_k m}(e^{-ip_k n}v^{(+)}a_k^{(+)}+e^{ip_k n}v^{(-)}a_k^{(-)})\ . \tag{45}$$

Since

$$\sum_{k=1}^{M}e^{ip_k(m-m')}=M\delta_{mm'}\ , \tag{46}$$

we infer from the canonical anticommutation relations of $\psi_{m,n}$ that

$$[a_k^{(\pm)},a_{k'}^{(\pm)\dagger}]_+=\frac{1}{Mh}\delta_{kk'}\ . \tag{47}$$

For definiteness let us suppose that M is even; from (13) we see that $\psi_{m,n}$ satisfies antiperiodic boundary conditions. Then the physical interpretation of $a_k^{(\pm)},a_k^{(\pm)\dagger}$ as creation and annihilation operators is as follows:

$$\begin{aligned}&\text{for } 0\le k\le\frac{M}{2}-1\left[\frac{\pi}{M}\le p_k\le\frac{\pi(M-1)}{M}\right],\\ &\quad a_k^{(+)}|0\rangle=a_k^{(-)\dagger}|0\rangle=0\ ;\\ &\text{for } -\frac{M}{2}\le k<0\left[-\frac{\pi(M-1)}{M}\le p_k\le-\frac{\pi}{M}\right],\\ &\quad a_k^{(+)\dagger}|0\rangle=a_k^{(-)}|0\rangle=0\ .\end{aligned} \tag{48}$$

We easily conclude that, for $m\neq m'$,

$$\langle \psi_{m,n}\psi^\dagger_{m',n}\rangle = \frac{\gamma_5}{2Mh}\left[\sum_{k=0}^{M/2-1} - \sum_{k=-M/2}^{-1}\right] e^{i\pi(2k+1)(m-m')/M} = \frac{i\gamma_5}{2Mh}(1-e^{i\pi(m-m')})\frac{1}{\sin[\pi(m-m')/M]} . \tag{49}$$

Equation (36) and the axial anomaly (37) follow immediately.

Finally, we explain how our method breaks chiral symmetry. Certainly, the free massless Dirac equation (8) is invariant not only under the global transformation (9), but under the global chiral transformation

$$\delta\psi_{m,n} = ie\gamma_5\delta\Omega\psi_{m,n'} . \tag{50}$$

Just as in the continuum this implies that the lattice chiral current is conserved, but *only to zeroth order in powers of* A_μ. Since the interaction is not simply linear in A_μ, the nongauged chiral current is not conserved in first order. This is, of course, the same reason that point separation breaks chiral symmetry in the continuum.

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[4]It is the nonlocality of the Minkowski lattice Hamiltonian which allows this finite-element formulation to evade the no-go theorems [L. H. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981); H. B. Nielsen and M. Ninomiya, *ibid.* **B185**, 20 (1981); J. M. Rabin, Phys. Rev. D **24**, 3218 (1981)], which require fermion doubling in a unitary, chiral-invariant, local,

[5]We know from the free Dirac equation studied in Ref. 2 that the canonical anticommutation relations require that $\psi_{m,n}$ be either periodic or antiperiodic in space: $\psi_{M,n}=\pm\psi_{0,n}$. We have not pursued the consequences of the alternative boundary condition to (13), namely, $\psi_{M,n}=(-1)^M\psi_{0,n}$.

[6]The generic problem of taking the continuum limit on a finite-element lattice concerns expressions of the type $2\sum_{n=1}^{N}(-1)^{n+N}f_n+(-1)^N f_0$, where $f_n=f(nh)$. When N is even, the above expression has the form

$$2[(f_N-f_{N-1})+(f_{N-2}-f_{N-3})+\cdots+(f_2-f_1)]+f_0 ,$$

which as $h\to 0$ is asymptotic to $2h[f'_N+f'_{N-2}+\cdots+f'_2]+f(0)$. This can be represented as a Riemann integral $\int_0^x f'(t)dt+f(0)=f(x)$, where $x=Nh$. When N is odd we obtain the same result.

[7]We ignore in this paper any of the subtle questions involving operator ordering in the interaction term I.

[8]J. Schwinger, Phys. Rev. **128**, 2425 (1962); J. H. Lowenstein and J. A. Swieca, Ann. Phys. (N.Y.) **68**, 172 (1971), and references therein.

[9]It is not obvious how to fix c in this asymptotic limit $h\to 0$. If we determine c by a free-field representation of the anticommutation relations we obtain $c=\frac{1}{2}$.

[10]We certainly do not allege that our choice of lattice current in (31) is unique. There may be many currents which are gauge invariant, are conserved on the lattice in the sense of (30), and have the correct continuum limit. We choose to use the current in (31) because it is especially simple.

[11]The calculation leading to (36) uses the free-field vacuum state corresponding to the canonical fields $\phi_{m,n}$. Replacing the physical vacuum by this Fock vacuum is valid here because the underlying continuum theory is free. This kind of ansatz is conventionally used in the continuum Schwinger model [see, for example, L. S. Brown, Nuovo Cimento **29**, 617 (1963)].

[12]When M is odd, we obtain a slightly more complicated result: the right side of (36) is multiplied by the extra factor $\frac{1}{2}[1+\cos(\pi/M)]$. This does not change the large-M limit.

[13]Observed that for large M the relative error is $\pi^2/(6M^2)$. This M^{-2} behavior of the relative error is characteristic of finite-element approximations to solutions of classical differential equations. For example, the solution to $y'=y$ has a relative error of $1/(12M^2)$. See C. M. Bender, in *Workshop on Non-Perturbative Quantum Chromodynamics,* edited by K. A. Milton and M. A. Samuel (Birkhäuser, Boston, 1983), p. 157.