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Axial-gauge formulation of a three-dimensional field theory

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Since the non-Abelian version of a recently formulated gauge theory in two spatial dimensions gives rise to a nonlinear constraint upon the fields in the radiation-gauge approach, one is motivated to attempt a description in terms of the axial gauge. This is accomplished in the Abelian version of the model, with results similar to those encountered in the radiation gauge. The non-Abelian case is then formally solved in the same gauge, it being subsequently shown, however, that the theory is not covariant. It is argued on the basis of perturbation theory that such noncovariance is a real effect which is not readily circumvented by modification of the field transformation properties.

I. INTRODUCTION

The impact of gauge theories upon the development of particle physics in recent years would be difficult to overstate. Yet progress in fully exploiting the gauge concept has been limited except at the more phenomenological levels by the mathematical complexities of such theories. This situation makes the development of either soluble or simplified models particularly welcome despite the fact that these are frequently constructed in other than four-dimensional space-times. One such effort in a world of two spatial dimensions that has recently been advanced[1] is based upon the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\phi^\mu\epsilon_{\mu\nu\alpha}\partial^\alpha\phi^\nu+g\phi^\mu j_\mu+\mathscr{L}' , \tag{1}$$

where j_μ is a conserved current constructed out of fields contained in $\mathscr{L}'$. In Ref. 1 it was shown that (1) describes a theory which is unusual in many respects, including, in particular, the following.

(a) Although (1) is invariant (up to divergence) under gauge transformations, it does not imply a gauge particle (i.e., a photon). This follows from the fact that in the radiation gauge

$$\partial_i\phi_i=0 , \quad i=1,2$$

the fields ϕ^μ can be explicitly evaluated in terms of the current j^μ as

$$\phi_i(x)=-g\epsilon_{ij}\partial_j\int d^2x'\mathscr{D}(x-x')j^0(x') \tag{2}$$

and

$$\phi^0(x)=g\int d^2x'\,\vec{j}(x')\times\vec{\nabla}'\mathscr{D}(x-x') , \tag{3}$$

where

$$-\nabla^2\mathscr{D}(x)=\delta(x) .$$

Thus the system described by (1) may be said to be one in which a spatially nonlocal current-current interaction has disguised itself as a gauge-field coupling. The absence of a photon may be particularly useful in applications where it is helpful to be able to neglect radiation losses in a rigorous way.

(b) Although the theory described by (1) has been shown to be covariant, it was found that the charge-bearing field has an anomaly in its spatial rotation properties. Stated slightly differently the commutator of that field with the (single) angular momentum operator includes a term which depends upon the coupling constant g.

(c) A bound state was found to develop in the propagator of the ϕ^μ field for small g^2 in the case of a coupling to a spinor field for the appropriate sign of the spinor mass. This result depends crucially on space being two-dimensional, but is a rigorous result for arbitrarily small g^2.

It is clear from these results that there is much content in this theory. One issue, in particular, which immediately presents itself is the matter of the non-Abelian version of (1). This is formally accomplished by allowing ϕ_a^μ to transform as the adjoint representation of some unspecified non-Abelian group and replacing (1) by

$$\mathscr{L}=\tfrac{1}{2}\phi^\mu\epsilon_{\mu\nu\alpha}\partial^\alpha\phi^\nu-\frac{i}{6}g\phi^\mu\epsilon_{\mu\nu\alpha}t^a\phi_a^\alpha\phi^\nu+g\phi_a^\mu j_\mu^a ,$$

where t^a are a set of imaginary matrices which comprise the structure constants of the group and j_a^μ is a current operator which need not be specified for now. The equation for ϕ^μ thus becomes

$$\epsilon_{\mu\nu\alpha}\partial^\alpha\phi_a^\nu+i\tfrac{1}{2}g\epsilon_{\mu\nu\alpha}\phi^\alpha t_a\phi^\nu+gj_\mu^a=0 ,$$

which for the $\mu=0$ component reads

$$-\vec{\nabla}\times\vec{\phi}_a-i\tfrac{1}{2}g\vec{\phi}\times t_a\vec{\phi}=gj_a^0 . \tag{4}$$

In the Abelian case (4) could readily be solved in the radiation gauge to yield (2), but in the present application one is confronted with the necessity of solving a nonlinear constraint for the fields $\phi_i^a(x)$. There exists, however, a straightforward approach which can avoid this difficulty, namely, the use of the axial gauge as defined by the condition

$$\vec{n}\cdot\vec{\phi}_a=0 , \tag{5}$$

where $\vec{n}$ is a unit two-vector, i.e.,

$$n^2=1 .$$

It is evident that (5) effectively reduces ϕ_i^a to a single-component vector in the two-dimensional space, thereby trivializing the cross product of ϕ_i^a with itself and making (4) a linear equation for the nonvanishing component $\vec{n}\times\vec{\phi}_a$. On the other hand, since the earlier treatment of the Abelian model was based on the radiation gauge, it is not immediately clear that there exists a consistent operator formulation in the axial gauge. Because of this the task of extending the work of Ref. 1 to the axial gauge will first be considered. Having once derived and verified the consistency of the results it is relatively straightforward to extend it to the non-Abelian case. The resulting operator formalism is found, however, to fail the test of covariance, a result which is reinforced by perturbative calculations.

II. THE ABELIAN AXIAL GAUGE

The axial-gauge condition evidently allows the replacement of ϕ_i by the combination

$$\phi_i - n_i\vec{n}\cdot\vec{\phi}$$

which, because of the two-dimensional identity

$$\delta_{ij}=n_i n_j+\epsilon_{ik}\epsilon_{jl}n_k n_l ,$$

is equivalent to the combination

$$-\epsilon_{ij}n_j\vec{n}\times\vec{\phi} .$$

Thus the equation

$$-\vec{\nabla}\times\vec{\phi}=gj^0$$

for the Abelian case easily reduces to

$$-\vec{n}\cdot\vec{\nabla}(\vec{n}\times\vec{\phi})=gj^0$$

with the solution

$$\vec{n}\times\vec{\phi}(x)=g\int d^2x'd_n(x-x')j^0(x') , \tag{6}$$

where d_n satisfies the equation

$$-\vec{n}\cdot\vec{\nabla}d_n(x)=\delta(\vec{x}) . \tag{7}$$

A solution of (7) is given by

$$d_n(x)=-\tfrac{1}{2}\epsilon(\vec{x}\cdot\vec{n})\delta(\vec{n}\times\vec{x}) , \tag{8}$$

where

$$\epsilon(x)=\frac{x}{|x|} .$$

Although one could in principle modify (8) by adding a constant to the alternating function, there are at least two good reasons for not doing so. One is the fact that such a term would violate the symmetry of (7) and (8) under the reflection $\vec{n}\to-\vec{n}$. Second, and more to the point, is the observation that in the Abelian case it is not needed, while in the non-Abelian case it fails to solve the covariance problem.

Additional arbitrariness in the expression for $\phi_i(x)$ arises from the fact that one can add to solution (6) a function of $\vec{n}\times\vec{x}$ and x^0. However, the axial-gauge condition $\vec{n}\cdot\vec{\phi}=0$ has not exhausted all gauge-fixing possibilities as there yet remains the class of transformations

$$\phi^\mu(x)\to\phi^\mu(x)+\partial^\mu\Lambda(\vec{n}\times\vec{x},x^0) .$$

This freedom is consequently used to fix (6) as the unique gauge choice.

The spatial components of the field equations can be resolved into two scalar equations by taking projections parallel and perpendicular to the vector $\vec{n}$. The latter yields

$$-\vec{n}\cdot\vec{\nabla}\phi^0=g\vec{n}\times\vec{j}$$

or

$$\phi^0(x)=g\int d^2x'd_n(x-x')\vec{n}\times\vec{j}(x')+\xi(\vec{n}\times\vec{x},x^0) . \tag{9}$$

The former implies after using (9) that

$$\vec{n}\times\vec{\nabla}\xi(\vec{n}\times\vec{x},x^0)=\frac{g}{2}[\vec{n}\cdot\vec{j}(\vec{x}\cdot\vec{n}=\infty)+\vec{n}\cdot\vec{j}(\vec{x}\cdot\vec{n}=-\infty)]$$

or

$$\xi(\vec{n}\times\vec{x},x^0)=\frac{g}{4}\int d(\vec{n}\times\vec{x}\,')\epsilon(\vec{n}\times(\vec{x}-\vec{x}\,'))\times[\vec{n}\cdot\vec{j}(\vec{x}\cdot\vec{n}=\infty)+\vec{n}\cdot\vec{j}(\vec{x}\cdot\vec{n}=-\infty)] .$$

Despite the apparent ease with which the operator ξ has entered the operator formalism, one finds that it is necessary for consistency (i.e., in order that the Hamiltonian generate time displacements) that $\xi=0$ and thus that

$$\vec{n}\cdot\vec{j}(\vec{x}\cdot\vec{n}=\infty)+\vec{n}\cdot\vec{j}(\vec{x}\cdot\vec{n}=-\infty)=0 . \tag{10}$$

The validity of (10) must then be ascertained by reference to the solution of the model as holding for all matrix elements.

One now proceeds to the verification of the consistency of the theory with regard to the existence of a suitable set of Poincaré-group generators. For the case of a coupling to a Hermitian spinor field as implied, for example, by the Lagrangian

$$\mathscr{L}=\tfrac{1}{2}\phi^\mu\epsilon_{\mu\nu\alpha}\partial^\alpha\phi^\nu+\frac{i}{2}\psi\alpha^\mu\partial_\mu\psi-\frac{m_0}{2}\psi\beta\psi+g\phi^\mu j_\mu ,$$

where

$$j^\mu=\tfrac{1}{2}\psi\beta\gamma^\mu q\psi$$

and q is a charge matrix, the relevant components of the conserved symmetric energy-momentum tensor are

$$T^{00}=-\frac{i}{2}\psi\alpha^k(\partial_k-igq\phi_k)\psi+\frac{m_0}{2}\psi\beta\psi \tag{11}$$

and

$$T^{0k}=-\frac{i}{2}\psi(\partial^k-igq\phi^k)\psi-\frac{i}{8}\psi[\alpha^k,\alpha^l]\partial_l\psi . \tag{12}$$

All definitions and conventions concerning the Dirac matrices are identical to those of Ref. 1.

With the Poincaré generators defined by

$$P^\mu = \int d^2x\, T^{0\mu}\ ,$$
$$J^{\mu\nu} = \int d^2x\,(x^\mu T^{0\nu} - x^\nu T^{0\mu})\ ,$$

the transformation properties of the fields can be determined. One readily establishes that the Hamiltonian P^0 satisfies

$$[P^0,\psi] = -i\,\partial_0\psi$$

and consequently that P^0 generates time displacements for all operators of the theory. Equally straightforward is the result

$$[P^k,\psi] = i\,\partial_k\psi\ .$$

The commutator of P^k with ϕ^μ is of the same form provided that (10) is generalized to

$$j^\mu(\vec{x}\cdot\vec{n}=\infty) + j^\mu(\vec{x}\cdot\vec{n}=-\infty) = 0\ . \qquad (10')$$

The boost operators $K_i \equiv J^{0i}$ have the property of inducing operator gauge transformations upon the gauge fields as displayed by the results

$$[K_i,\psi(x)] = i(x^0\partial^i - x^i\partial^0)\psi(x) + \frac{i}{2}\alpha_i\psi(x) + gqn_i\Lambda(x)\psi(x)$$

and

$$[K_i,\phi^\mu(x)] = i(x^0\partial^i - x^i\partial^0)\phi^\mu(x) - i[g^{\mu i}\phi^0(x) - g^{\mu 0}\phi^i(x)] - i\partial^\mu n_i\Lambda(x)\ ,$$

where

$$\Lambda(x) = -g\int d^2x'\,d_n(x-x')\vec{n}\cdot(\vec{x}-\vec{x}\,')\vec{n}\times\vec{j}(x')$$

and the condition

$$\vec{j}\cdot\vec{n}(\vec{x}\cdot\vec{n}=\infty) - \vec{j}\cdot\vec{n}(\vec{x}\cdot\vec{n}=-\infty) = 0$$

must be imposed. Finally, turning to $J \equiv \frac{1}{2}\epsilon_{kl}J^{kl}$ one obtains

$$[J,\psi(x)] = i(\vec{x}\times\vec{\nabla})\psi(x) + \frac{i}{8}\epsilon_{kl}[\alpha^k,\alpha^l]\psi(x) + g^2\int d^2x'\,d_n(x-x')\vec{n}\cdot(\vec{x}-\vec{x}\,')j^0(x') \qquad (13)$$

which displays the anomaly shown in Ref. 1 to be characteristic of the spatial rotations of the charge-bearing field. Although the radiation-gauge result displayed there is considerably simpler than that of Eq. (13), the presence of a rotational anomaly is seen to be quite generally characteristic of the model.

The specific conclusion which one achieves as a result of these considerations is that the axial-gauge formulation is consistent. This is made particularly clear by the demonstration that the operators (11) and (12) satisfy the Dirac-Schwinger covariance condition

$$-i[T^{00}(x),T^{00}(x')] = -[T^{0k}(x)+T^{0k}(x')]\partial_k\delta(\vec{x}-\vec{x}\,')\ . \qquad (14)$$

Since the proof of (14) is tedious but not particularly illuminating, the calculation will not be presented here. Instead, attention will now be directed to the principal motivation of this paper, namely, the non-Abelian version of the model.

III. EXTENSION TO THE NON-ABELIAN CASE

The discussion subsequent to Eq. (4) has already indicated that the axial-gauge choice has the property of linearizing (4). Thus one needs only to solve the equation

$$-\vec{\nabla}\times\vec{\phi}_a = gj_a^0\ ,$$

where j_a^μ is formally defined by

$$j_a^\mu = \tfrac{1}{2}\psi\beta\gamma^\mu T_a\psi$$

and the matrices T_a are required to provide a representation of the relevant symmetry group. Following essentially the same sequence of steps as in the Abelian case one arrives at the result

$$\vec{n}\times\vec{\phi}_a = g\int d^2x'\,d_n(x-x')j_a^0(x')\ .$$

Similarly, upon decomposing the equation

$$\epsilon_{i\nu\alpha}\partial^\alpha\phi_a^\nu + ig\epsilon_{i\nu\alpha}\phi^\alpha t_a\phi^\nu + gj_a^i = 0$$

into components parallel to and perpendicular to $\vec{n}$ there follows

$$\phi_a^0 = g\int d^2x'\,d_n(x-x')\vec{n}\times\vec{J}_a\ ,$$

provided that

$$\vec{n}\cdot\vec{J}_a(\vec{n}\cdot\vec{x}=\infty) + \vec{n}\cdot\vec{J}_a(\vec{n}\cdot\vec{x}=-\infty) = 0\ ,$$

where

$$J_a^i \equiv j_a^i + i\epsilon_{ik}\phi^0 t_a\phi_k$$

and the internal-symmetry index of ϕ_a^μ has been suppressed whenever it is summed.

One is now confronted with the task of demonstrating the covariance of the non-Abelian formulation. Although the use of (11) and (12) with appropriate matrix replacements leads to an admissible set of operators P^μ, it is not possible to establish Poincaré invariance. The crucial point here is the fact that in examining the commutator of $T^{00}(x)$ with $T^{00}(x')$, there occur terms from

$$[gj_a^k(x)\phi_k^a(x), gj_b^l(x')\phi_l^b(x')] \qquad (15)$$

which do not conform to the Dirac-Schwinger condition (14). Since the commutator (15) is proportional to g^4 it is apparent that it cannot be canceled by any of the other terms in the energy-density commutator.

Direct calculation leads to the result that (15) has the form

$$g^4 \int d^2x'' t_{acb} j_c^0(x'')\{\vec{n}\times \vec{j}_a(x)\vec{n}\times \vec{j}_b(x')[\tfrac{1}{2}d_n(x-x'')d_n(x'-x'')+d_n(x-x'')d_n(x-x')]$$
$$+\vec{n}\times \vec{j}_b(x')\vec{n}\times \vec{j}_a(x)[\tfrac{1}{2}d_n(x-x'')d_n(x'-x'')+d_n(x'-x'')d_n(x'-x)]\} .$$

By means of the identity

$$[\tfrac{1}{2}d_n(x-x'')d_n(x'-x'')+d_n(x-x'')d_n(x-x')]=\tfrac{1}{2}d_n(x-x')[d_n(x-x'')+d_n(x'-x'')]$$
$$+\tfrac{1}{8}\delta(\vec{n}\times\vec{x}-\vec{x}\,'))\delta(\vec{n}\times(\vec{x}-\vec{x}\,''))$$

it is readily shown that the commutator (15) can be rewritten as

$$\tfrac{1}{8}g^4 \int d^2x'' j_c^0(x'') t_{acb}\{\vec{n}\times \vec{j}_a(x), \vec{n}\times \vec{j}_b(x')\}\delta(\vec{n}\times(\vec{x}-\vec{x}\,'))\delta(\vec{n}\times(\vec{x}-\vec{x}\,'')) .$$

Since this term gives a nonvanishing contribution to the commutator of the Lorentz boost with the Hamiltonian, one is led to the conclusion that the axial gauge is not a truly covariant approach to the non-Abelian version of the model.

There does exist an escape from this conclusion, however, if an allowance is made for the possibility of modifying the energy density by the inclusion of one or more additional terms. Such terms must, of course, consist of total divergences if the Hamiltonian and equations of motion are to be preserved. (A modification of this type was used by Schwinger[2] to demonstrate covariance of the standard non-Abelian gauge theory in the radiation gauge.) In the following section, however, it is argued on the basis of perturbation theory that the noncovariance of the operator formalism is a real effect which appears in fourth order in the coupling constant g.

IV. PERTURBATIVE RESULTS

As has been seen, the formal operator structure of the non-Abelian model in the radiation gauge has not been successfully reconciled with the requirements of Lorentz invariance. One could in principle search for a modification of the energy-density operator (as in Ref. 2) which would have the structure of a two-dimensional divergence and thus alter the Lorentz transformation properties while leaving intact the Hamiltonian equations of motion. In this section the unlikelihood of such a discovery is argued on the basis of a demonstration that such formal tinkering cannot be a solution of the covariance problem unless there also occurs a modification of the Feynman rules.

The approach to be used here is based on a specialization of the internal invariance group to the case of U(N) in the large-N limit. This technique has been recognized for some time as being useful in suppressing many diagrams in the large-N limit provided that the combination g^2N remains finite. Furthermore, since the axial gauge has the property that there are no three-boson vertices, one has the result that through order g^4N^2 one has only to consider the same diagrams which characterize the conventional two-dimensional non-Abelian model.[3] The g^2N and g^4N^2 contributions to the fermion propagator are shown in Fig. 1.

Since the covariance-breaking terms are of order g^4, it is anticipated that a calculation of the diagram in Fig. 1(a) will be compatible with Poincaré invariance while Fig. 1(b) will not. To this end one calculates the mass operator to second order using

$$M^{(2)}=m_0-ig^2N\int\frac{dk}{(2\pi)^3}\gamma_\mu\frac{1}{m+\gamma(p-k)}\gamma_\nu\mathcal{G}^{\mu\nu}(k) , \tag{16}$$

where

$$\mathcal{G}^{\mu\nu}(k)=-\epsilon^{\mu\nu\alpha}n_\alpha(nk)^{-1} .$$

This leads by direct calculation to the result

$$M^{(2)}=m_0+\frac{g^2N}{2\pi}(\Lambda-m) -\frac{g^2N}{2\pi}[(\vec{p}\cdot\vec{n})-im\beta\vec{\gamma}\times\vec{n}]\tan^{-1}\frac{\vec{p}\cdot\vec{n}}{m} , \tag{17}$$

where the linear divergence of (16) has led to the appearance of the Pauli-Villars cutoff Λ. Because of the appearance of the vector n, it is not clear by inspection of (17) whether it is compatible with relativity. As is pointed out in Ref. 3 and in more detail elsewhere,[4] however, one expects a modification of the fermion propagator in the form of terms which are not *manifestly* covariant. What this means is that in the present case Lorentz invariance is established if (17) can be brought to the form

$$\gamma p+M=e^{a\beta\vec{\gamma}\times\vec{n}}(m+\gamma p)e^{a\beta\vec{\gamma}\times\vec{n}} \tag{18}$$

with m being a covariant function of p^2 and a being arbitrary. It is readily established that

$$m=m_0+\frac{g^2N}{4\pi}(\Lambda-m)$$

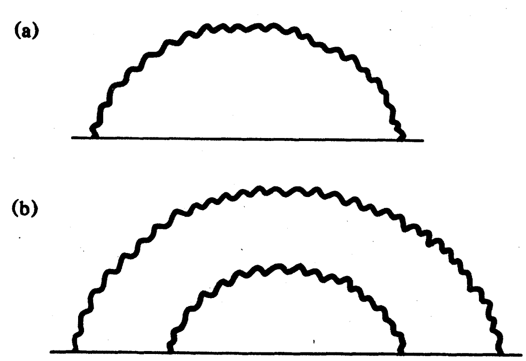

FIG. 1. (a) Second-order contribution to M. (b) Fourth-order contribution to M.

and

$$a = i\frac{g^2 N}{4\pi}\tan^{-1}\frac{\vec{p}\cdot\vec{n}}{m}$$

in complete conformity with the stated requirements on these functions.

The corresponding calculation of Fig. 1(b) is made more manageable by the recognition that since the fermion bare mass plays no meaningful role, it is convenient to set it to zero. This leads to the significantly simplified result for the g^4N^2 term:

$$\Lambda\frac{g^2N}{2\pi}\frac{\partial}{\partial m}M^{(2)}\bigg|_{m=0} + \frac{ig^4N^2}{4}\int\frac{dk}{(2\pi)^3}\gamma_\mu\frac{(\vec{p}-\vec{k})\cdot\vec{n}\epsilon(\vec{p}\cdot\vec{n}-\vec{k}\cdot\vec{n})}{(p-k)^2}\gamma_\nu\mathcal{G}^{\mu\nu}(k)$$

which can be reduced to

$$i\frac{g^4N^2}{8\pi}\Lambda\beta\vec{\gamma}\times\vec{n}\epsilon(\vec{p}\cdot\vec{n}) - \frac{g^4N^2}{2}\beta\vec{\gamma}\times\vec{n}\int\frac{dk}{(2\pi)^3}\frac{|\vec{p}\cdot\vec{n}-\vec{k}\cdot\vec{n}|}{(p-k)^2(\vec{k}\cdot\vec{n})} .$$

The remaining integral is logarithmically ultraviolet divergent and cannot be accommodated by the structure (18) or the most general possible form:

$$\begin{aligned}\gamma p + M = A(p)\exp[a(p)\beta\vec{\gamma}\times\vec{n} + c(p)\beta \\ + \beta\vec{\gamma}\cdot\vec{b}(p)](\gamma p + m) \\ \times\exp[-a(-p)\beta\vec{\gamma}\times\vec{n} - c(-p)\beta \\ -\beta\vec{\gamma}\cdot\vec{b}(-p)] .\end{aligned}$$

One is thus forced to conclude that the standard Feynman rules lead to a noncovariant structure for the two-point function which is totally consistent with the results of the preceding section.

V. CONCLUDING REMARKS

It has been seen that the quantization of the model of Ref. 1 can be carried out in the axial gauge. However, the axial gauge (which was motivated primarily to enable one to handle the complications of the non-Abelian case) has not passed the crucial test of covariance in this latter case. To set this remark in its proper perspective it should be noted that similar difficulties have been found to characterize the axial-gauge formulation of the usual Yang-Mills theory.[5]

On the other hand, the possibility of a radiation-gauge formulation is not ruled out. Although the nonlinear constraints for the fields ϕ_μ may not allow an explicit construction of those fields, it could be possible to establish covariance despite this notable shortcoming. In view of the obvious desirability of having a non-Abelian model which is presumably simpler than the standard one in four dimensions, this question merits further investigation.

ACKNOWLEDGMENT

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