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Interaction of electric and magnetic charges: Addendum

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The problem of a head-on collision between a spherically symmetric electric and magnetic charge is reconsidered. The assumption of the equality of the electric and magnetic forces made earlier is abandoned as it does not follow from the model. It is shown that the rotational angular momentum a charge acquires is determined by the amount of the angular momentum which the charge removes from the field by virtue of its extension. The interaction energy of each charge with the dipole it induces in the other charge is equal to the rotational energy of the charge itself. This results in simple expressions for the effective potential, and the distance of closest approach as given by classical electrodynamics and nonrelativistic mechanics. A mechanism is suggested for the transfer of the angular momentum from the field to the monopole.

I. INTRODUCTION

In an earlier paper,[1] referred to here as I, we investigated classically the problem of a head-on collision between an electric and a magnetic charge, each of a finite radius and a spherical charge distribution, to remove a long-standing difficulty with the conservation of the angular momentum. Recent interest in magnetic monopoles, and the possible far-reaching implications of their existence,[2,3] prompted us to reconsider this problem and report our new results which correct, simplify, and generalize the results of I, and illuminate the exchange of angular momentum between particles and fields.

We consider the charges to move on the x axis from infinite separation with an initial velocity of approach v_0, and with the electric charge e on the left and the magnetic charge g on the right. Initially, all the angular momentum of the system is in the electromagnetic field and equals $(eg/c)\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is a unit vector pointing from e to g. As the finite charges approach each other, the field angular momentum $\mathbf{L}$ is decreased by an amount which is converted into a rotational angular momentum $\mathbf{l}$ for the charges.

The electric charge induces in the magnetic charge an electric dipole which repels the electric charge by a force which is inversely proportional to the fifth power of their separation. A similar repulsive force is obtained between the magnetic charge and the magnetic dipole it induces in the electric charge. *It was assumed in I that these two repulsive magnetic and electric forces were equal, or inequivalently the electric and magnetic interaction energies were equal.*[4] This is an *ad hoc* assumption which cannot hold in general, and can be looked upon, at best, as an external constraint on the system. We shall here abandon this assumption, and consider the torque one charge exerts on the other, and equate the torque to the rate of change of the angular momentum of the charge. In Sec. II the torque equation is derived. In Sec. III it is shown that the rotational angular momentum of the charge is determined by the amount of the field angular momentum which the charge removes by virtue of its extension. This will render the angular momentum conservation of the system of field and charges self-evident. In Sec. IV a simple but general expression is derived for the minimum distance of closest approach, and a mechanism is suggested for the transfer of the angular momentum from the field to the monopole.

II. TORQUE ON A SPHERICAL CHARGE

Figure 1 shows the geometry for the calculation of the torque the magnetic charge exerts on the electric charge. The center of the electric charge 0 is chosen as origin for a spherical polar coordinate system. A is the center of the magnetic charge at a distance x from 0 (and $0A$ is taken as the polar axis). An element of electric charge $e\rho(r)\,d^3r$ centered at $P(r,\theta,\phi)$ and moving with velocity $v\hat{\mathbf{x}}$ relative to A experiences the Lorentz force $e\rho\,d^3r\,(v/c)(g/s^2)\sin\alpha$ perpendicular to the plane of the paper. Here $s=AP$ and ρ is a spherical charge distribution normalized to unity and $\rho(r)=0$ for $r>R$ the radius of the sphere. By the sine law $\sin\alpha=r\sin\theta/s$. The only component of the torque $\mathbf{\Gamma}$ which survives is in the x direction and $\mathbf{\Gamma}=\Gamma\hat{\mathbf{x}}$, where

$$\Gamma=\int_0^R \frac{eg\,v}{c}2\pi r^4\rho(r)\int_0^\pi \frac{\sin^3\theta}{s^3}\,d\theta\,dr \quad . \tag{2.1}$$

The $(\sin\theta/s)^3$ integration is characteristic of this probem.[5] It is performed by a partial integration with respect to s, since $s\,ds=xr\sin\theta\,d\theta$ followed by an expansion of $1/s$ in spherical harmonics which selects only the $\cos\theta$ term. The result is

$$\begin{aligned}\int_0^\pi \frac{\sin^3\theta}{s^3}\,d\theta &= \tfrac{4}{3}x^{-3}, \quad x>r\\ &= \tfrac{4}{3}r^{-3}, \quad x<r \ ,\end{aligned} \tag{2.2}$$

which when used in (2.1) gives

$$\Gamma=\frac{eg}{c}\lambda R^2\frac{2v}{x^3} \quad , \tag{2.3}$$

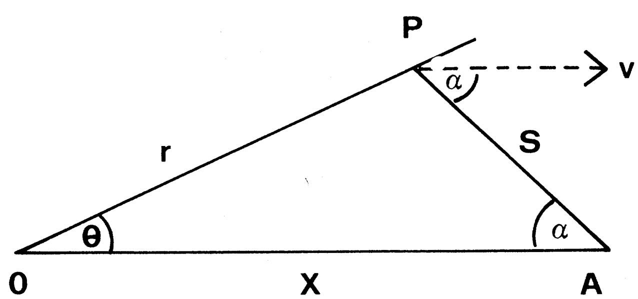

FIG. 1. Geometry of the model.

where the parameter λ is defined in terms of the second moment of the charge distribution by

$$\lambda R^2 = \frac{1}{3} \int_0^R r^2 \rho(r)\, d^3r \quad . \tag{2.4}$$

Since $\mathbf{\Gamma} = d\mathbf{l}/dt$, where $\mathbf{l}$ is the rotational angular momentum of the sphere, and since $v = -dx/dt$, we see that $\mathbf{l} = l\hat{\mathbf{x}}$ and

$$\frac{dl}{dt} = \frac{eg}{c} \lambda R^2 \frac{d}{dt} \frac{1}{x^2} \quad . \tag{2.5}$$

With the initial condition $l = 0$, for $x = \infty$, we have

$$\mathbf{l} = \frac{eg}{c} \frac{\lambda R^2}{x^2} \hat{\mathbf{x}} \quad . \tag{2.6}$$

It should be emphasized that the torque $\mathbf{\Gamma}$ is the same whether the magnetic charge is a point charge or spherical charge, as long as we assume that we deal with nonoverlapping spheres, since $\mathbf{B}$ is the same at the location of the electric charge elements.

III. FIELD ANGULAR MOMENTUM IN THE SPHERE

By spreading the electric charge over a sphere we wish to show that the field angular momentum is decreased from its value for a point charge by precisely the amount given on the right-hand side of Eq. (2.6). The calculation is a generalization of the results of Sec. V of Ref. 5. In Fig. 1, the electric field $\mathbf{E}$ is radial and has the magnitude $(e/r^2)\int_0^r \rho(r')\,d^3r'$, $\mathbf{B}$ equals g/s^2 in the AP direction, and $\mathbf{E}\times\mathbf{B}$ brings in a factor $\sin P$, which equals $(x/s)\sin\theta$ by the sine law. This together with the definition of the field angular momentum element $d\mathbf{L} = (1/4\pi c)\mathbf{r}\times(\mathbf{E}\times\mathbf{B})\,d^3r$, and the observation that $\mathbf{L}$ is in the x direction gives

$$L(R) = \frac{egx}{2c} \int_0^R r \int_0^r \rho(r')\, d^3r' \int_0^\pi \frac{\sin^3\theta}{s^3}\, d\theta\, dr \quad , \tag{3.1}$$

for the magnitude of the field angular momentum inside the sphere. The angular integration is performed as in Sec. II and the integral is reduced to

$$L(R) = \frac{eg}{3cx^2} \int_0^R \left[\int_0^r \rho(r')\, d^3r' \right] d(r^2) \quad , \tag{3.2}$$

which by a partial integration gives

$$\mathbf{L}(R) = \frac{eg}{cx^2} \left(\tfrac{1}{3} - \lambda\right) R^2 \hat{\mathbf{x}} \quad . \tag{3.3}$$

Outside the sphere, $\mathbf{E}$ is that of a point charge, and the only reduction to L by the electric charge takes place inside the space occupied by the charge and equals $(eg/c)\lambda R^2/x^2$, which according to Eq. (2.6) is acquired by the charge as a rotational angular momentum. Thus, within the sphere $r < R$, the total angular momentum is the same as that of a point charge, with the fraction $\frac{1}{3} - \lambda$ in the field, and the fraction λ in the charge. For $\lambda = \frac{1}{3}$, which corresponds to a uniform surface charge distribution, all the angular momentum is in the charge. It should be evident also that $\mathbf{L}$ for all space plus $\mathbf{l}$ for both charges is $(eg/c)\hat{\mathbf{x}}$.

IV. RESULTS

We assume that the electric spherical charge is rigid and rotates with an angular frequency Ω. By the Biot-Savart law or the vector-potential method, the magnetic field outside the sphere is that of a dipole $\mu\hat{\mathbf{x}}$, where

$$\mu = \frac{e\Omega}{c} \lambda R^2 \quad , \tag{4.1}$$

and $\Omega = l/I$, where I is the moment of inertia of the charge about the x axis. If the mass density is $mf(r)$, with $f(r)$ normalized to unity, we have

$$I = \frac{2m}{3} \int_0^R r^2 f(r)\, d^3r \equiv 2m\lambda' R^2 \quad . \tag{4.2}$$

By the above and Eq. (2.6),

$$\mu = \frac{e^2 g}{2mc^2} \frac{\lambda^2 R^2}{\lambda' x^2} \quad . \tag{4.3}$$

Thus *the magnetic moment* $\boldsymbol{\mu}$ *does not equal* $(e/2mc)\mathbf{l}$ *unless the second moment of* r *with respect to the charge distribution is the same as it is for the mass distribution.*[6]

For nonrelativistic velocities the magnetic field of μ at the center of the magnetic charge is $(2\mu/x^3)\hat{\mathbf{x}}$, which leads to a repulsive interaction potential energy given by

$$V_{\mu_e,g} = \frac{e^2 g^2}{m_e c^2} \frac{\lambda_e^2}{\lambda_e'} \frac{R_e^2}{4x^4} \quad . \tag{4.4}$$

The introduction of the subscripts e and g for the electric and magnetic charges in the quantities $\mathbf{l}$, λ, λ', m, ρ, f, R, and I is essential to avoid any confusion. The rotational energy of the electric charge is $E_e^r = l_e^2/2I_e$, which equals the potential energy $V_{\mu_e,g}$. Similarly the electric dipole p_g induced in g by e interacts with e by the repulsive potential

$$V_{p_g,e} = \frac{e^2 g^2}{m_g c^2} \frac{\lambda_g^2}{\lambda_g'} \frac{R_g^2}{4x^4} \quad , \tag{4.5}$$

which also equals the rotational energy of the magnetic charge. *Thus the rotational energy of the system is equal to its potential energy.*

The energy conservation equation now simplifies to

$$\tfrac{1}{2} M v_0^2 = 2(V_{\mu_e,g} + V_{p_g,e}) + \tfrac{1}{2} M v^2 \quad , \tag{4.6}$$

where M is the reduced mass. The distance of closest approach x_0 is given by

$$x_0^2 = \frac{eg}{cv_0} \left[\frac{1}{M} \left(\frac{\lambda_e^2}{m_e \lambda_e'} R_e^2 + \frac{\lambda_g^2}{m_g \lambda_g'} R_g^2 \right) \right]^{1/2} \quad . \tag{4.7}$$

For $R_g \to 0$ or $m_g \to \infty$, x_0 is the same as is given by Eq. (9) of *I* with $\alpha = \alpha' = \lambda = \lambda'$ and $eg/c = \hbar/2$.

As an application, let us consider the head-on collision of a point charge particle ($R_e = 0$) with a magnetic monopole of the type currently discussed in the grand unification theories.[7] If in Eq. (4.7) we set $\lambda_g' \sim \lambda_g$, and $eg/c = n\hbar/2$, where n is an integer, we obtain

$$x_0^2 = \frac{n}{2(v_0/c)} \left(\frac{\hbar}{Mc} \frac{\hbar}{m_g c} \lambda_g \right)^{1/2} R_g \quad . \tag{4.8}$$

If R_g is assumed to be of order $g^2/(m_g c^2)$ and $m_g c^2 \sim 10^{16}$ GeV, then $R_g \sim 10^{-26}$ cm for $n \sim 20$ and $\hbar/m_g c \sim 10^{-30}$

cm, and if $\hbar/Mc \sim 10^{-12}$ cm, we obtain $x_0 \sim 10^{-23}$ cm $\sim 10^3 R_g$. The point is that the repulsive x^{-5} interaction does not preclude the projectile from penetrating the outer layers of the monopole. If the projectile should recede from the monopole after reaching the distance of closest approach, then the angular momentum of Eq. (2.6) which was acquired from the field by the monopole will be returned to the field and nothing interesting happens except for the prevention of the violation of the conservation of the angular momentum. If on the other hand the projectile is captured by one of the fields of the three unified interactions (electroweak and strong), then the angular momentum acquired by the monopole will remain in the monopole system, and thus this model suggests a mechanism for exciting the rotational states of the monopole.

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[1] I. Adawi, Phys. Rev. D **16**, 1232 (1977); see also Ref. 5 for introduction.

[2] B. Cabrera, Phys. Rev. Lett. **48**, 1378 (1982).

[3] V. A. Rubakov, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 658 (1981) [JETP Lett. **33**, 644 (1981)]; C. G. Callan, Jr., Phys. Rev. D **25**, 2141 (1982).

[4] Even if this assumption is accepted, the potential energy in Eq. (16) of I should be doubled, and the factor $\frac{1}{2}$ at the beginning of the square brackets in Eqs. (18) and (19) should be replaced by 1.

[5] See I. Adawi, Am. J. Phys. **44**, 762 (1976).

[6] The diamagnetic and diaelectric polarizabilities of the charges as can be read from Eqs. (4.3) and the corresponding equation for the electric dipole induced in g are $(e^2/2m_e c^2)(\lambda_e{}^2/\lambda_e')R_e{}^2$ and $(g^2/2m_g c^2)(\lambda_g{}^2/\lambda_g')R_g{}^2$. It is interesting to observe that this classical model contains in a natural way induced electric and magnetic moments. This is to be contrasted with the infinitesimal magnetic moment which was added to the relativistic Hamiltonian of a charged particle in the field of a magnetic monopole by Y. Kazama, C. N. Yang, and A. S. Goldhaber, in Phys. Rev. D **15**, 2287 (1977), and by Y. Kazama and C. N. Yang, in Phys. Rev. D **15**, 2300 (1977).

[7] See, for example, *Magnetic Monopoles,* edited by R. A. Carrigan, Jr. and W. P. Trower (Plenum, New York, 1983); P. Goddard and D. I. Olive, Rep. Prog. Phys. **41**, 1357 (1978).