Exact gravitational field of a string

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The exact spacetime metric representing the exterior of a static cylindrically symmetric string is found. The geometry is conical, with a deficit angle of $8\pi G\mu$, where μ is the linear energy density of the string. The results of Vilenkin, obtained using linearized gravity, are thus shown to be correct to all orders in $G\mu$. Strings with $G\mu \geq \frac{1}{4}$ are found to collapse the exterior spacetime, resulting in dimensional reduction.

If gauge theories with spontaneous symmetry breaking correctly describe elementary particle physics, then the Universe may have undergone a number of phase transitions since the big bang.[1] Topological structures such as vacuum domain walls, strings, and monopoles produced in these phase transitions may possibly have survived to the present day.[2,3] Cosmic strings seem to be of particular interest, both as a possible "seed" for galaxy formation[4,5] and as a possible gravitational lens.[6]

A string can be either infinite in length or a closed loop. In either case, the string tension will generally cause it to oscillate at velocities close to the speed of light, yielding an asymmetric, highly dynamic structure whose gravitational field cannot be easily calculated. Vilenkin[6] has taken an important first step in studying the gravitational effects of strings by calculating the gravitational field of a static, cylindrically symmetric string in the linear approximation to general relativity. He found that the spacetime exterior to a string is conical in nature, with the deficit angle of the cone equal to $8\pi\mu$, to first order in μ, where μ is the linear energy density (mass per unit length) of the string.[7]

The purpose of this paper is to extend the results of Vilenkin's analysis to the full, exact theory of general relativity, i.e., to find the exact exterior spacetime metric of a static, cylindrically symmetric string. There are two motivations for this work. The first is to reproduce Vilenkin's first-order (in μ) results with more rigor. Conical singularities in a spacetime are termed "removable" singularities, and generally are removed so as to maximally extend the spacetime.[8] In the case at hand, with a string present on the z axis, we clearly do not want to remove the singularity entirely, for to do so would remove the string as well. In order to separate the true, physical string-induced conical deficit angle from any spurious conical singularity which should be removed by extending the spacetime, it is necessary to match the exterior metric to an interior metric representing the string. This cannot be done in the linear approximation, as the approximation fails near the string. It is thus necessary to study an exact solution to determine the exterior geometry rigorously even at first order. The second motivation for this work is to determine the exterior geometry to all orders in μ. While a standard grand unified theory has $\mu \sim 10^{-6}$, it is quite possible that values of μ much closer to one will be of interest in future theories, possibly incorporating gravity in their unification scheme.

The string spacetime is assumed to be static and cylindrically symmetric, with the string lying along the axis of symmetry. The most general static, cylindrically symmetric metric has the form[9]

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}d\phi^2 + e^{2\lambda}(d\rho^2 + dz^2) \ , \tag{1}$$

where ν, ψ, and λ are functions of ρ, and $\phi = 0$ and $\phi = 2\pi$ are identified.

The transverse dimensions of a string are of the order of the Compton wavelength of the Higgs fields. Within the classical theory of general relativity, it is then appropriate to treat the string as a δ-functional source in its transverse dimensions. In order to avoid any possible ambiguity arising from the use of a singular source, the string will instead be chosen to have a uniform density, $\epsilon > 0$, out to some cylindrical radius ρ_0. The end results will prove to be independent of ρ_0, so that a "classical limit" may be taken by reducing the string's transverse dimensions to zero, yielding an unambiguous exact exterior metric for the string.

Following the arguments of Vilenkin,[6] the stress-energy tensor of the string is given by

$$T^t{}_t = T^z{}_z = -\epsilon \quad (\rho < \rho_0) \ , \tag{2}$$

with all other components equal to zero.

The components of the Einstein tensor for the metric given in Eq. (1) are easily found;[9] the resulting Einstein field equations are

$$G^t{}_t = e^{-2\lambda}(\psi_{,\rho\rho} + \psi_{,\rho}{}^2 + \lambda_{,\rho\rho}) = -8\pi\epsilon \ , \tag{3}$$

$$G^\phi{}_\phi = e^{-2\lambda}(\nu_{,\rho\rho} + \nu_{,\rho}{}^2 + \lambda_{,\rho\rho}) = 0 \ , \tag{4}$$

$$G^\rho{}_\rho = e^{-2\lambda}(\nu_{,\rho}\psi_{,\rho} + \nu_{,\rho}\lambda_{,\rho} + \psi_{,\rho}\lambda_{,\rho}) = 0 \ , \tag{5}$$

$$G^z{}_z = e^{-2\lambda}(\nu_{,\rho\rho} + \nu_{,\rho}{}^2 - \nu_{,\rho}\lambda_{,\rho} + \psi_{,\rho\rho} + \psi_{,\rho}{}^2 + \psi_{,\rho}\nu_{,\rho} - \psi_{,\rho}\lambda_{,\rho}) = -8\pi\epsilon \ . \tag{6}$$

These nonlinear equations for the metric functions are easily solved in the case of the uniform-density string. Conservation of stress-energy ($T_\alpha{}^\beta{}_{;\beta} = 0$) yields

$$(\nu_{,\rho} + \lambda_{,\rho})\epsilon = 0 \ . \tag{7}$$

This implies, through Eq. (4), that ν and λ are constant, and may be set equal to zero by an appropriate rescaling of the coordinates t, ρ, z. Equation (5) is then satisfied automatically and Eqs. (3) and (6) become identical:

$$\psi_{,\rho\rho} + \psi_{,\rho}{}^2 = -8\pi\epsilon \ . \tag{8}$$

Equation (8) is easily solved by the substitution $R = e^\psi$

($g_{\phi\phi}=R^2$) to yield

$$R=A\cos(\rho/\rho_*)+B\sin(\rho/\rho_*) \quad , \tag{9}$$

where

$$\rho_*=(8\pi\epsilon)^{-1/2} \quad . \tag{10}$$

The metric on the axis will be flat (no cone singularity) if $A=0$, $B=\rho_*$. The interior metric of a uniform-density string is then

$$ds^2=-dt^2+d\rho^2+dz^2+\rho_*{}^2\sin^2(\rho/\rho_*)d\phi^2 \quad . \tag{11}$$

The exterior metric for the string spacetime must be a static, cylindrically symmetric, vacuum solution of the Einstein equations. The most general such solution was found in 1917 by Levi-Civita:[10,11]

$$ds^2=-r^{2m}dT^2+r^{-2m}[r^{2m^2}(dr^2+dZ^2)+a^2r^2d\phi^2] \quad , \tag{12}$$

where m and a are freely chosen constants. As pointed out by Vilenkin,[6] the string is Lorentz invariant in the z direction. Requiring the metric of Eq. (12) to be Lorentz invariant in the z direction restricts the values of m in Eq. (12) to just two: $m=0$ and $m=2$. The former value is a flat, possibly conical (depending on the value of a) space; the latter is more unusual. As r decreases, the circumference of a circle ($r=$constant) increases, diverging as $r\to 0$. As r becomes large, the circumference asymptotically approaches zero and the spacetime becomes effectively three dimensional.

Now that the interior and exterior metrics have been found, they must be joined together along the surface of the string at $\rho=\rho_0$, $r=r_0$. Einstein's equations, reduced to a set of junction conditions by Israel,[12] require that the intrinsic metrics induced on the junction surface by the interior and exterior metrics be identical (up to coordinate transformations), and that the discontinuity in the extrinsic curvature of the surface be related to the stress-energy of the surface (if any). Consider first the $m=0$ flat exterior case. The intrinsic metrics can then be matched by requiring $t=T$, $z=Z$, and (setting $g^+_{\phi\phi}=g^-_{\phi\phi}$)

$$ar_0=\rho_*\sin\left(\frac{\rho_0}{\rho_*}\right) \quad . \tag{13}$$

The extrinsic curvature tensor is defined by

$$K_{ij}{}^{\pm}=-e_i^\alpha e_j^\beta n^{\pm}_{\alpha;\beta} \quad , \tag{14}$$

where e_i^α is an orthonormal triad lying in the junction surface, and $n_\alpha^{\pm}$ is the unit outward-normal vector in the interior ($-$) or exterior ($+$) metric. Calculating the extrinsic curvature tensors and equating them to each other (so as to have no surface stress-energy present), one obtains the relation

$$a^2=\rho_*{}^2/(\rho_*{}^2+r_0{}^2) \quad . \tag{15}$$

Combining this with the intrinsic metric constraint, Eq. (13), to eliminate r_0, yields

$$a=\cos(\rho_0/\rho_*) \quad . \tag{16}$$

The exterior metric of the string is then given by Eq. (12) with $m=0$, and a given by Eq. (16). The geometry is conical with a deficit angle of $\delta\phi=2\pi[1-\cos(\rho_0/\rho_*)]$. The circumference of a circle with proper radius $r=r_1=$constant is $2\pi r_1\cos(\rho_0/\rho_*)$.

The concept of a mass per unit length for a cylindrically symmetric source in general relativity is not unambiguously defined (unlike the case of spherical symmetry). For a static, cylindrically symmetric spacetime, a simple definition which will be useful here is to simply integrate the energy density ϵ over the proper volume of the source (the string). The mass per unit length (or linear energy density) is then

$$\mu=\int_0^{\rho_0}\int_0^{2\pi}\epsilon\rho_*\sin(\rho/\rho_*)\,d\phi\,d\rho=2\pi\epsilon\rho_*{}^2\left[1-\cos\left(\frac{\rho_0}{\rho_*}\right)\right] \quad , \tag{17}$$

or, finally, using Eq. (10),

$$4\mu=1-\cos(\rho_0/\rho_*) \quad . \tag{18}$$

Combining Eqs. (16) and (18), we find

$$a=1-4\mu \quad , \tag{19}$$

so that the conical deficit angle is

$$\delta\phi=8\pi\mu \quad , \tag{20}$$

exactly as in Vilenkin's linearized analysis,[6] but now exact to all orders in μ.

The exact exterior metric is

$$ds^2=-dt^2+dz^2+dr^2+(1-4\mu)^2r^2\,d\phi^2 \quad . \tag{21}$$

While Vilenkin's expression for the deficit angle is correct to all orders, his value for $g_{\phi\phi}$ is correct only to first order, as would be expected in the linear approximation. Since the exterior metric now depends only on μ, not ρ_0 or ρ_*, the string source may now be shrunk to a δ function, letting $\rho_0\to 0$ while holding ρ_0/ρ_*, and hence μ, constant. In this way we avoid any semiclassical complications in dealing with the classical gravitational field of the quantum Higg's fields over Compton-wavelength dimensions.

Since the exterior metric given by Eq. (12) with $m=0$ and a given by Eq. (19) is exact to all orders in μ, we can ask what happens for large values of μ (of order 1). As $\mu\to\frac{1}{4}$, $a\to 0$, and the exterior becomes a cylinder of radius ρ_*:

$$ds^2=-dt^2+d\rho^2+dz^2+\rho_*{}^2\,d\phi^2 \quad . \tag{22}$$

For $\frac{1}{2}>\mu>\frac{1}{4}$, a match to a conical exterior is again possible, but now the cone closes as one moves away from the string; the exterior coordinate r decreases away from the string. There is now an *exterior* conical singularity a distance

$$R=\rho_*\tan(-\rho_0/\rho_*) \tag{23}$$

from the surface of the string. As $\rho_0/\rho_*\to\pi/2$, $\mu\to\frac{1}{4}$, and $R\to\infty$, and the spacetime becomes cylindrical [Eq. (22)]. As $\rho_0/\rho_*\to\pi$, $\mu\to\frac{1}{2}$, and the string closes upon itself; no exterior is needed or possible. These matches with $\mu\geqslant\frac{1}{4}$ are of little interest, since as the $\rho_0\to 0$ limit is taken, the ρ (or r), ϕ two-space collapses, reducing to a point (if $\mu>\frac{1}{4}$) or a line (if $\mu=\frac{1}{4}$).

It is also possible to join the uniform-density string interior [Eq. (11)] with the $m=2$ vacuum metric. In this case there is always a nonzero surface stress-energy tensor at the junction. It is possible to join the metrics so that r increases or decreases as one moves away from the string. The surface stress-energy diverges as the $\rho_0 \rightarrow 0$ classical limit is taken, except in one special case where r is chosen to decrease away from the string and $\mu = \frac{1}{4}$. In this case the metric parameter a is not determined by the junction conditions. The exterior metric is given by Eq. (12) with $m=2$ and $0 < r < r_0$. The surface energy density in this case is always negative, violating the weak energy condition. As the classical limit $\rho_0 \rightarrow 0$ is taken, the surface stress-energy vanishes and $r_0 \rightarrow \infty$. This case is unphysical, however, because as $r_0 \rightarrow \infty$ the string is removed to physical infinity.

In summary, the results of Vilenkin, obtained within the linear approximation to general relativity, have been shown to be correct to all orders in μ. The exact exterior geometry of a static, cylindrically symmetric string with $\mu < \frac{1}{4}$ is that of a cone with deficit angle $\delta\phi = 8\pi\mu$, and a metric given by Eq. (21). In terms of light bending, a photon traveling past the string will be deflected through an angle $\delta\phi = 4\pi\mu$. If $\mu \geqslant \frac{1}{4}$, the exterior has collapsed onto the string; this is probably related to the fact that $r \leqslant 2M$ for any radius sphere centered on a string with $\mu \geqslant \frac{1}{4}$.

Note added in proof. Since this paper was written, I have become aware of a preprint by J. R. Gott III, which has since appeared in print [Astrophys. J. **288**, 422 (1985)], which independently derives the exact static, cylindrically symmetric string solution for the $m=0$ exterior, with particular application to the gravitational lens problem.

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