Nambu—Jona-Lasinio—type effective Lagrangian: Anomalies and nonlinear Lagrangian of low-energy, large-N QCD

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We present a qualitative derivation of the chiral model from QCD. This is based on using a Nambu—Jona-Lasinio—type effective Lagrangian as an intermediate step. A detailed derivation of the anomalous low-energy Wess-Zumino term is presented. This includes vector, axial-vector, and pseudoscalar particles. The low-energy scale is set by $\overline{H} \sim \langle \overline{\Psi}\Psi \rangle$. We also present the low-energy nonlinear chiral model which generalizes the Skyrme model. The possibility of soliton solutions is indicated. There is a possible application of these ideas to electroweak theory.

I. INTRODUCTION

A well-known fact about strong interactions is that pions are approximately Nambu-Goldstone bosons associated with the spontaneous breakdown of $SU(3)\times SU(3)$ chiral symmetry. This is amply substantiated by the phenomenological success of current-algebra soft-pion theorems and chiral models.[1] In QCD, the microscopic theory of the strong interactions, numerical calculations indicate the spontaneous breakdown of chiral symmetry.[2] There are also qualitative arguments that in the long-wavelength and large-N limit QCD can be approximated by a weakly coupled local field theory of mesons.[3] The strength of the coupling in this effective field theory is proportional to $1/N$, N being the number of colors. Baryons appear as solitons in this effective field theory, their mass being proportional to N.[4]

More recently there has been a revival of interest in current-algebra anomalies, the Wess-Zumino[5] term, and its topological properties.[6] These developments are especially interesting because the anomaly coefficients contained in the Wess-Zumino term are universal: they are identical for the low-energy theory (written in terms of composite smooth fields) and the microscopic theory. These coefficients are solely determined by the fermion representation of the microscopic theory. The Wess-Zumino term also fixes the quantum numbers of the topological solitons.[7]

At present there is no calculation that derives this field theory of mesons, the chiral model plus Wess-Zumino term, from first principles. The issue is similar to a derivation of hydrodynamics from the principles of atomic physics. The theoretical framework is that of the renormalization group,[8] and there is a possibility that Monte Carlo renormalization-group calculations can include the combined gauge-field light-quark systems and attempt a derivation of the chiral model. In particular, the numerical coefficients occurring in the chiral model will have to be predicted from the microscopic theory. Of these universal anomaly coefficient is perhaps the least difficult to calculate.

Some time ago we had outlined the proposal that the Nambu—Jona-Lasinio— (NJL) type Lagrangian[9] considered as a cutoff field theory may adequately reproduce the main features of the low-energy chiral model including its topological anomalies and higher-order nonlinear terms which can support soliton solutions.[10–12] In this paper we present a detailed exposition of this proposal. We have also included vector and axial-vector couplings in the NJL Lagrangian. Tensor and higher couplings are omitted since they correspond to higher mass excitations. The plan of the paper is as follows: in Sec. II we present the arguments for the NJL Lagrangian in the context of QCD. Section III reviews the standard discussion of spontaneous breakdown of chiral symmetry. Section IV presents a discussion of current-algebra anomalies including scalar, pseudoscalar, vector, and axial-vector mesons. The Wess-Zumino term of current algebra is the phase of the determinant of the Dirac operator in the presence of these fields in the long-wavelength limit. The anomaly is reflected in the noninvariance of this phase under local chiral (gauge) transformations. The anomaly equation expresses this fact as a linear equation in the configuration space of the local chiral group. This equation is integrated along a path in this configuration space to obtain the Wess-Zumino term. In Sec. V we present the long-wavelength expansion of the modulus of the determinant of the Dirac operator. By long wavelength we mean that the typical space-time variation of the scalar, pseudoscalar, vector, and axial-vector fields entering the Dirac operator is slow compared to the inverse mass of the fermion in the broken-symmetry phase of the NJL theory. Section VI is devoted to the large-N perturbative spectrum of the theory and in Sec. VII we discuss the possibility of soliton solutions to the nonlinear σ model which contains all nonlinear four-dimensional operators involving the current $l_\mu(x)=i\partial_\mu U(x)U^\dagger(x)$. In Sec. VIII we present arguments to identify these solitons with baryons. We conclude in Sec. IX with remarks on the possibility of a topological soliton in the Glashow-Salam-Weinberg model as a result of the existence of heavy quarks. We reserve the phenomenological implications of this work for another publication.

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II. NAMBU—JONA-LASINIO LAGRANGIAN AND QCD

We have mentioned that the theoretical framework to discuss phenomenological Lagrangians is provided by the Kadanoff-Wilson renormalization group. In the absence of a calculation we present qualitative reasoning to motivate the Nambu—Jona-Lasinio—type effective Lagrangian. Let us begin with four-dimensional QCD on a hypercubical lattice. When the lattice spacing is small (large momentum cutoff) we assume that the theory is described by Wilson's lattice action for gauge fields and fermions. The color group is SU(N) and the flavor group is $U(n)\times U(n)$. To obtain an effective Lagrangian at a larger lattice spacing (smaller momentum cutoff) we have to integrate out high momentum fluctuations of the gauge and fermion fields. Now suppose that after a few iterations we reach a lattice spacing of the size of the correlation length of the gauge fields, i.e., the momentum cutoff is approximately equal to the glueball mass. The lattice action at this length scale will be more complicated than the original Wilson action. Besides minimal terms of the Wilson action, depending on the blocking procedure, higher-order nonminimal terms will be included in the pure gauge and fermion parts of the action.[13,27] A possible parametrization of the action is

$$S_\Lambda=\frac{1}{g^2}\sum \mathrm{tr}U_{4l}(P)+\frac{1}{g_l^{\,2}}\sum \mathrm{tr}U_{6l}+\frac{1}{g_p^{\,2}}\sum \mathrm{tr}U_{6p}$$
$$+\frac{1}{g_t^{\,2}}\sum \mathrm{tr}U_{6t}+K_1\sum\bar\psi(1+\gamma_\mu)U_{x,x+\hat\mu}\psi$$
$$+K_2\sum_{\langle x,y\rangle}(\bar\psi\Gamma_a U^a_{x,y}\psi)(\bar\psi\Gamma_b U^b_{x,y}\psi)\ ,$$

$\Gamma_a=\{1,\gamma_5,\gamma_\mu,\gamma_5\gamma_\mu,\ldots\}$, $\langle x,y\rangle$ extend over a few lattice spacings.

Now let us focus on Green's functions of gauge-invariant local chiral operators at this scale. An example is

$$\mathcal{O}_a^{ij}(x)=\sum_\alpha \bar\psi^{\,\alpha}_{xi}(x)\Gamma_a\psi^\alpha_{xj}(x)\ .$$

If we further restrict our attention to properties of these Green's functions for distances much larger than the correlation length of the gauge fields (which in this case is one lattice spacing) we are effectively dealing with a theory of fermions with contact interactions. *The main point in the above reasoning is that the non-Abelian gauge field develops a finite correlation length and the nonlinear fermion theory only evaluates correlations of gauge-invariant fermionic operators.* With this understanding of the model there is no conflict with local gauge invariance of the underlying theory and also the question of quark confinement. A similar though far simpler situation, where a massive gauge field leads to a Fermi theory with contact interactions, is the familiar Glashow-Salam-Weinberg model. Here the gauge symmetry is realized in the Higgs phase which is manifest in the unitary gauge. For wavelengths much larger than the Compton wavelength of the W boson the gauge theory is well approximated by the Fermi theory without violation of gauge invariance.

With these remarks we write down the effective NJL Lagrangian which evaluates gauge-invariant correlations of purely fermionic operators for distances larger than the correlation length of the gauge fields,

$$\begin{aligned}-i\mathcal{L}=&\bar\psi_{L\alpha j}i\partial\!\!\!/\psi_{L\alpha j}+\bar\psi_{R\alpha j}i\partial\!\!\!/\psi_{R\alpha j}\\&+g_1^{\,2}(\bar\psi_{L\alpha j}\psi_{R\alpha k})(\bar\psi_{R\beta k}\psi_{L\beta j})\\&-i\frac{g_2^{\,2}}{4}[(\bar\psi_{L\alpha j}\gamma_\mu\psi_{L\alpha k})(\bar\psi_{L\beta k}\gamma_\mu\psi_{L\beta j})\\&+L\leftrightarrow R]+\cdots\ .\end{aligned}\tag{1}$$

$\alpha=1,\ldots,N$ are color indices; $j,k=1,\ldots,n$ are flavor indices and the subscripts L and R refer to the left and right chiral projections of the quarks. g_1 and g_2 are constants of mass dimension -1. We treat them as phenomenological parameters. We denote the cutoff implicit in (1) by Λ, which is much less than the glueball mass.

The effective Lagrangian retains the global $U(n)\times U(n)$ symmetry of the QCD Lagrangian corresponding to independent $U(n)$ rotations of left and right fermions. However in QCD the axial U(1) symmetry is broken by an anomaly.[14] To incorporate this, in a manner consistent with the large-N expansion, we add to (1) the term

$$|\ln\det(\bar\psi_{R\alpha j}\psi_{L\alpha k})|^2\tag{2}$$

which manifestly breaks the axial U(1) symmetry. It can be shown that it leads to a mass for the η meson proportional to $1/\sqrt{N}$. In the limit of large N, which is of present interest to us, (2) is unimportant compared to (1) and hence we shall ignore it in the following discussion.

The NJL Lagrangian (1) can be recast in terms of color gauge-invariant collective variables, M, $M^\dagger$, L_μ, R_μ. M is a complex scalar and L_μ and R_μ are vector fields that couple left and right fermions, respectively,

$$\begin{aligned}-i\mathcal{L}=&\bar\psi_L i\partial\!\!\!/\psi_L+\bar\psi_R i\partial\!\!\!/\psi_R+\bar\psi_L M\psi_R+\bar\psi_R M^\dagger\psi_L\\&+i(\bar\psi_L L\!\!\!/\psi_L+\bar\psi_R R\!\!\!/\psi_R)+\frac{i}{g_1^{\,2}}\mathrm{tr}M^\dagger M\\&+\frac{i}{g_2^{\,2}}\mathrm{tr}(L_\mu^{\,2}+R_\mu^{\,2})\ .\end{aligned}\tag{3}$$

Evaluating the Gaussian integral over the collective variables we can regain (1).

Under the chiral group $U(n)\times U(n)$, the quarks and the collective variables transform as

$$\begin{aligned}&\psi_L\to V_1\psi_L,\quad \psi_R\to V_2\psi_R\ ,\\&M\to V_1MV_2^\dagger\ ,\\&L_\mu\to V_1L_\mu V_1^\dagger,\quad R_\mu\to V_2R_\mu V_2^\dagger\ ,\end{aligned}\tag{4}$$

where $(V_1,V_2)\in U(n)\times U(n)$. Here (V_1,V_2) is x independent.

III. CHIRAL-SYMMETRY BREAKING

In this section we review the single most important fact about the NJL Lagrangian, that chiral $U(n)\times U(n)$ symmetry is spontaneously broken down to diagonal U(n). The method uses the large-N limit.[15]

Integrating over the fermions in the path integral corresponding to (3), we get the path integral entirely over the collective fields

$$Z=\int_{M,M^\dagger,L_\mu,R_\mu}\int_{\psi,\bar\psi}\exp(NS_{\rm eff})\ , \tag{5}$$

$$\begin{aligned} S_{\rm eff}&=\ln\det D-\frac{1}{Ng_1{}^2}\int d^4x\ {\rm tr}M^\dagger M\\ &\quad-\frac{1}{Ng_2{}^2}\int d^4x\ {\rm tr}(L_\mu{}^2+R_\mu{}^2)\ ,\\ D_{x,y}&=(i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+iMP_L+iM^\dagger P_R)\delta^4(x-y)\ . \end{aligned} \tag{6}$$

In the limit of large N, (5) can be evaluated at the minimum of $S_{\rm eff}$. We look for translation-invariant solutions

$$\begin{aligned} &M(x)={\rm diag}(\lambda_1,\lambda_2,\ldots,\lambda_n),\quad \lambda_i\,\text{real}\\ &L_\mu=R_\mu=0\ . \end{aligned} \tag{7}$$

For these the value of the effective action is

$$S_{\rm eff}(\lambda_1,\ldots,\lambda_n)=\sum_k\left[{\rm Tr}\ln(i\partial\!\!\!/+i\lambda_k)-\frac{1}{Ng_1{}^2}\int d^4x\ \lambda_k{}^2\right]. \tag{8}$$

The minimum is reached at $\partial S_{\rm eff}/\partial\lambda=0$, when all the eigenvalues are equal to $\overline{H}$, say. The gap equation is

$$i\ {\rm tr}_{\rm Dirac}\left\langle x\left|\frac{1}{i\partial\!\!\!/+i\overline{H}}\right|x\right\rangle=\frac{2\overline{H}}{Ng_1{}^2}\ . \tag{9}$$

Since (1) is a cutoff field theory,

$${\rm tr}_{\rm Dirac}\left\langle x\left|\frac{1}{i\partial\!\!\!/+i\overline{H}}\right|y\right\rangle=-4i\overline{H}\int^\Lambda\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+\overline{H}^2}\ . \tag{10}$$

Evaluating (10) the gap equation takes the form

$$\lambda\left\{1-\frac{Ng_1{}^2}{8\pi^2}\left[\Lambda^2-\overline{H}^2\ln\left[1+\frac{\Lambda^2}{\overline{H}^2}\right]\right]\right\}=0\ . \tag{11}$$

Equation (11) has two mutually exclusive solutions:

$$\lambda=0 \tag{12}$$

or

$$1-\frac{\overline{H}^2}{\Lambda^2}\ln\left[1+\frac{\Lambda^2}{\overline{H}^2}\right]=\frac{8\pi^2}{N\Lambda^2g_1{}^2}\ . \tag{13}$$

The first solution (12) corresponds to the case of unbroken chiral symmetry. The second solution corresponding to broken chiral symmetry exists only if the coupling is greater than a critical value

$$\Lambda^2g_1{}^2>\Lambda^2g_{1c}{}^2=\frac{8\pi^2}{N}\ . \tag{14}$$

Further since all the eigenvalues of M are equal, the chiral symmetry is broken from U(n)$\times$U(n) to diagonal U(n). By considering Gaussian fluctuations around $M=0$ and $M=\overline{H}$ in (5), it can be shown that for $g_1>g_{1c}$, the solution $M=0$ is unstable, whereas the solution $M=\overline{H}$ is stable.

The above conclusion remains unchanged for a large class of cutoff procedures, for example, one could use a smooth cutoff in (10),

$${\rm tr}_{\rm Dirac}\left\langle x\left|\frac{1}{i\partial\!\!\!/+\overline{H}}\right|x\right\rangle=-4i\overline{H}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-k^2/\Lambda^2}}{k^2+\overline{H}^2}\ . \tag{15}$$

The detailed form of the gap equation (9) and the value of H are now different. However, the value of the critical coupling remains unchanged.

IV. CURRENT-ALGEBRA ANOMALIES AND THE WESS-ZUMINO TERM

We now focus attention on the fermion part of the functional integral (5) or equivalently on the determinant of D in (6). Denoting the fermion integral by

$$Z_\psi=\int\prod_x d\psi\,d\bar\psi\exp\left[\int d^4x\ \bar\psi(i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+iM^\dagger P_L+iMP_R)\psi\right]. \tag{16}$$

We note that Z is formally invariant under local U(n)$\times$U(n) gauge transformations:

$$\begin{aligned} &\psi_L\to\Omega_L\psi_L,\ \ L_\mu\to\Omega_LL_\mu\Omega_L^\dagger+i\partial_\mu\Omega_L\Omega_L^\dagger,\ \ M\to\Omega_LM\ ,\\ &\psi_R\to\Omega_R\psi_R,\ \ R_\mu\to\Omega_RR_\mu\Omega_R^\dagger+i\partial_\mu\Omega_R\Omega_R^\dagger,\ \ M\to M\Omega_R^\dagger\ . \end{aligned} \tag{17}$$

It is important to emphasize that local symmetry considerations only apply to the fermion part of (5). The entire integral (5) has only a global U(n)$\times$U(n) symmetry which as we have seen is spontaneously broken to diagonal U(n).

The local invariance of (18) is only formal. It is well known that the currents corresponding to this symmetry are not conserved due to the presence of anomalies. More recently Fujikawa has shown that in general local phase rotations of left and right chiral fermions do not leave the fermion measure invariant,[16] i.e., det$D$ is multiplied by a phase on making a local chiral rotation on the collective fields unless fermion representations conspire to cancel phases among themselves.

To proceed further we take up the important question of the definition of the path integral (16), or equivalently the definition of det$D$. In Euclidean space the operator D is elliptic in the space of functions with the scalar product $(\chi,\psi)=\int\chi^\dagger(x)\psi(x)\,d^4x$. The presence of chiral couplings imply that D is not self-adjoint. Hence its eigenvalues are complex. Further if we make the reasonable assumption that Euclidean space is compactified to S_4 by identifying points at infinity, the operator D has discrete eigenvalues,

$$\epsilon_n=e^{i\Delta_n}\,|\,\epsilon_n\,|\ .$$

The determinant is then formally defined by the formula

$$\ln\det D=\sum_n\ln\epsilon_n=i\sum_n\Delta_n+\tfrac{1}{2}\sum_n\ln|\,\epsilon_n\,|^2\ . \tag{18}$$

Since $|\epsilon_n|^2$ are eigenvalues of the non-negative operator $D^\dagger D$, we have

$$\frac{1}{2}\sum_n \ln|\epsilon_n|^2=\tfrac{1}{2}\mathrm{Tr}\ln D^\dagger D \ . \tag{19}$$

We will denote the phase of $\det D$ by

$$\Delta=\sum_n \Delta_n=\mathrm{Im}(\ln\det D) \ . \tag{20}$$

The sums in (19) and (20) are over an infinite number of eigenvalues. In a cutoff field theory, these sums must be cut off at $n\sim\Lambda$. A smooth cutoff procedure which maintains certain symmetry principles (depending on theory) is desirable. Since we are modeling the strong interactions and our effective Lagrangian evaluates gauge-invariant fermion correlations of QCD at long distances, we have no choice but to define $\det D$ to ensure the conservation of vector currents. This is phenomenologically correct, and recently Witten and Vafa have proved the conservation of vector currents in QCD-type theories.[17] For these reasons we define (19) and (20) using the proper-time formulas of Schwinger:

$$\Delta=\mathrm{Im}(\ln\det D)=-\tfrac{1}{2}\mathrm{Im}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\mathrm{Tr}e^{-s\hat{D}^2} \ , \tag{21}$$

$$\tfrac{1}{2}\mathrm{Tr}\ln D^\dagger D=\tfrac{1}{2}\mathrm{Tr}\ln\hat{D}^\dagger\hat{D}=-\tfrac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\mathrm{Tr}e^{-s\hat{D}^\dagger\hat{D}} \ . \tag{22}$$

The choice of the operator $\hat{D}=i\gamma_5 D$ is dictated by the fact that $\ln\det D=(\ln\det\gamma_5 D)$ and that $\ln\det D=(\frac{1}{2}\ln\det\gamma_5 D\gamma_5 D)$ leads to the correct expression for the free energy of the Dirac particle in the absence of external fields. The factor i in $\hat{D}$, ensures the convergence of (22). Δ is related to the Wess-Zumino term.

Our main aim in this section is to extract the gauge dependence of $\ln Z$ in (16). Since $\mathrm{Tr}\ln\hat{D}^\dagger\hat{D}$ in (22) is chiral gauge invariant this is equivalent to extracting the gauge dependence of phase Δ in (21). We begin by separating the invariant and gauge degrees of freedom from the fields L_μ, R_μ, M by fixing an unitary-type gauge on the scalar field $M(x)$:

$$M(x)=\bar{\lambda}(x)=\mathrm{diag}(\lambda_1(x),\lambda_2(x),\ldots,\lambda_n(x)) \ , \quad \lambda_i\geq 0 \ . \tag{23}$$

It is always possible to go to this gauge by a local transformation in $\mathrm{U}(n)\times\mathrm{U}(n)$. Following the Faddeev-Popov (FP) method[19] we introduce the identity

$$\Delta_{\mathrm{FP}}(\bar{\lambda})\int\prod_x d\Omega_L d\Omega_R\,\delta(\Omega_L^\dagger M\Omega_R-\bar{\lambda})=1 \tag{24}$$

into the integral in (5) and perform the change of variables $M\to\Omega_L M\Omega_R{}^{-1}$ to get

$$\begin{aligned} Z&=\int dL\,dR\,d\Omega_L d\Omega_R d\bar{\lambda}\,\Delta_{\mathrm{FP}}(\bar{\lambda})\exp(-NS_{\mathrm{eff}}) \ , \\ S_{\mathrm{eff}}&=-\ln\det D+\frac{1}{g_1{}^2N}\sum_i\int d^4x\,\lambda_i{}^2 \\ &\quad+\frac{1}{g_2{}^2N}\int d^4x\,\mathrm{tr}(L_\mu{}^2+R_\mu{}^2) \ , \end{aligned} \tag{25}$$

$$\begin{aligned} D&=i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+i\Omega_R\bar{\lambda}\Omega_L^\dagger P_L+i\Omega_L\bar{\lambda}\Omega_R^\dagger P_R \ , \\ \Delta_{\mathrm{FP}}(\bar{\lambda})&=\prod_{i<j}[\lambda_i(x)-\lambda_j(x)]^2 \ . \end{aligned} \tag{26}$$

The amount of gauge that is fixed depends on the structure of the matrix $\bar{\lambda}$ which is a dynamical question. In the last section we saw that the large-N saddle point corresponds to $\bar{\lambda}=\bar{H}$ in (13). (The $\ln\Delta_{\mathrm{FP}}$ does not contribute to the saddle point since it is subleading order in $1/N$.) This means that the gauge condition (23) is left invariant by gauge transformations $[(V_L,V_R)\in\mathrm{U}(n)\times\mathrm{U}(n)]$ for which $V_L=V_R$. We denote this set by $\mathrm{diagU}(n)$. The gauge is fixed only up to $\mathrm{diagU}(n)$ and we have separated the gauge degree of freedom belonging to the coset $\mathrm{U}(n)\times\mathrm{U}(n)/\mathrm{diagU}(n)$. L_μ and R_μ are invariants under gauge transformations.

An important consequence of the above is that in the broken-symmetry phase, we can write (25) in terms of a single unitary matrix $U=\Omega_L\Omega_R{}^{-1}$, which represents the pionic collective mode. In particular the differential operator (26) becomes

$$D=i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+iH(U^\dagger P_L+UP_R) \ . \tag{27}$$

To calculate the phase (21) we consider the rotated operator

$$\begin{aligned} D^{[\Omega]}&=i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/^{[\Omega]}P_R+iH(\Omega U^\dagger P_L+U\Omega^\dagger P_R) \ , \\ R^{[\Omega]}&=\Omega R\Omega^\dagger+i\partial\Omega\Omega^\dagger \ , \quad \Omega(x)=e^{i\eta(x)} \end{aligned} \tag{28}$$

and establish a differential equation for the phase of $D^{[\Omega]}$. Noting that $D^{[\Omega]}=(P_R+P_L\Omega)D^{[\Omega=1]}(P_L+P_R\Omega^{-1})$, under small variations $\Omega\to\Omega+\delta\Omega$, we have

$$\delta D^{[\Omega]}=\tfrac{1}{2}[\delta\Omega\Omega^{-1},D^{[\Omega]}]-\tfrac{1}{2}\{\gamma_5\delta\Omega\Omega^{-1},D^{[\Omega]}\} \ , \tag{29}$$

the first is a vector variation, the second an axial-vector variation. Our regularization preserves vector symmetries, hence the first variation does not contribute to the variation of the phase Δ and we have

$$\delta[\Delta(\Omega)]=-\mathrm{Im}\,\mathrm{Tr}(\gamma_5\delta\Omega\Omega^{-1}e^{-\hat{D}[\Omega]^2/\Lambda^2}) \ . \tag{30}$$

The evaluation of the trace in (30) is long and tedious. The main steps are relegated to Appendix A. Here we present the result which agrees with Bardeen's calculation,[20,21]

$$\begin{aligned} \delta[i\Delta(\Omega)]&=\int d^4x\,\mathrm{tr}_f(i\delta\Omega\Omega^{-1}B) \ , \\ B&=-\frac{1}{8\pi^2}[\tfrac{1}{4}F_V{}^2+\tfrac{1}{12}F_A{}^2 \\ &\quad-\tfrac{2}{3}i(F_VA^2+AF_VA+A^2F_V)-\tfrac{8}{3}A^4] \ . \end{aligned} \tag{31}$$

In (31) we have used the notation of differential forms

$$\begin{aligned} &F=F_{\mu\nu}dx_\mu\wedge dx_\nu, \quad F^2=F\wedge F \ , \\ &F_V=dV+iV^2+iA^2, \quad F_A=dA+iAV+iVA \ , \\ &V=\tfrac{1}{2}(L+R^{[\Omega]}), \quad A=\tfrac{1}{2}(L-R^{[\Omega]}) \ . \end{aligned}$$

V and A are vector and axial-vector fields, respectively.

In order to integrate the anomaly equation (31) it is convenient to write (31) in a form that expresses the right-hand side of (31) as the divergence of a current.[22] This is done using Bardeen's identity:[20]

$$\int d^4x \, \mathrm{tr}_f(i\delta\Omega\Omega^{-1}B) = -\delta[iC_1(L,R^{[\Omega]})] + \int d^4x \, \mathrm{tr}_f[i\delta\Omega\Omega^{-1}G(R^{[\Omega]})] \, , \tag{32}$$

where

$$C_1(L,R) = \frac{1}{48\pi^2}\int d^4x \, \mathrm{tr}\{ i[R,L](dL+dR) + RL^3 + R^3L - \tfrac{1}{2}LRLR \} \tag{33}$$

and

$$G(R) = \frac{1}{24\pi^2} d[R\,dR + (i/2)R^3] \, .$$

The differential equation (31) can now be recast in the form

$$\delta(i\Delta + iC_1) = \int d^4x \, \mathrm{tr}_f[i\delta\Omega\Omega^{-1}G(R^{[\Omega]})] \, . \tag{34}$$

Using the parametrization $\Omega = e^{i\eta}$, we can write $\delta\Omega\Omega^{-1} = e^{i(\eta+\delta\eta)}e^{-i\eta}$. The group law is defined by the equation $e^{i\alpha}e^{i\beta} = e^{i\chi(\alpha,\beta)}$. Hence we have to first order in $\delta\eta$,

$$i\delta\Omega\Omega^{-1} = -t_a u_b^a(\eta)\delta\eta^b \, , \quad u_b^a = \left[\frac{\partial}{\partial\alpha}\chi(\alpha,-\eta)\right]_{\alpha=\eta} \, . \tag{35}$$

We define the differential operator $\mathscr{D}_a = iu_b^a \delta/\delta\eta_b$ in the configuration space of the collective field $U(x)$. Then (34) reads

$$\mathscr{D}_a(\Delta + C_1) = \frac{1}{24\pi^2}\mathrm{tr}t_a d[R^{[\Omega]}dR^{[\Omega]} + \tfrac{1}{2}i(R^{[\Omega]})^3] = \mathrm{tr}t_a G(R^{[\Omega]}) \, . \tag{36}$$

Equation (36) says that in the presence of fermions "Gauss's law" is modified by the anomaly term in (36).

The crucial point in integrating (36) is that in the right-hand side of (36) the collective mode Ω makes explicit appearance separated from the invariant coordinates R. This enables us to separate the purely longitudinal part of (36):

$$\mathscr{D}_a(\Delta + C_1) = \mathrm{tr}t_a[G(R^{[\Omega]}) - G(0^{[\Omega]})] + \mathrm{tr}t_a G(0^{[\Omega]}) \, . \tag{37}$$

$G(0^{\Omega})$ stands for the anomaly term (33) evaluated when $R_\mu = 0$:

$$G(0^{[\Omega]}) = -\frac{1}{48\pi^2}(i\,d\Omega\,\Omega^{-1})^4 \, . \tag{38}$$

The remaining part can be rewritten as

$$\mathrm{tr}t_a[G(R^{[\Omega]}) - G(0^{[\Omega]})] = -\mathscr{D}_a C_2(R^{[\Omega]}; i\,d\Omega\,\Omega^{-1}) \, . \tag{39}$$

Under the variation $\Omega \to e^{i\delta\eta}\Omega$, $\delta(i\Omega^{-1}d\Omega) = -\Omega^{-1}d\delta\eta\Omega$. Using this (39) can easily be integrated to get[23]

$$C_2(R^{[\Omega]}; i\,d\Omega\,\Omega^{-1}) = -\frac{1}{48\pi^2}\int d^4x \, \mathrm{tr}\{ (R^{[\Omega]})^3 i\,d\Omega\,\Omega^{-1} + (i\,d\Omega\,\Omega^{-1})^3 R^{[\Omega]} + \tfrac{1}{2}(i\,d\Omega\,\Omega^{-1})R^{[\Omega]}(i\,d\Omega\,\Omega^{-1})R^{[\Omega]} + i(i\,d\Omega\,\Omega^{-1})(R^{[\Omega]}dR^{[\Omega]} + dR^{[\Omega]}R^{[\Omega]})\} \, . \tag{40}$$

So finally we come to the nontrivial differential equation

$$\mathscr{D}_a(\Delta + C_1 + C_2) = -\frac{1}{48\pi^2}\mathrm{tr}t_a(i\,d\Omega\,\Omega^{-1})^4 \, . \tag{41}$$

As was originally pointed out by Witten[6] (41) cannot be integrated in four dimensions. Following our method[10] we will integrate it along a path in the configuration space of the collective field Ω. The Wess-Zumino consistency condition which is a zero-curvature condition assures us that the solution is path independent. We parametrize the path by its length. The line element in configuration space is given by

$$(ds)^2 = \int d^4x \, \mathrm{tr}(i\,d\Omega\,\Omega^{-1})^2 \, . \tag{42}$$

The tangent vector at the point s along the path is given by

$$\tau_a(s) = i\,\mathrm{tr}t_a\left[\frac{d\Omega}{ds}\Omega^{-1}\right] \, ,$$

hence the projection of the derivative $\mathscr{D}_a$ along the tangent $\tau_a(s)$ at s is given by

$$\mathscr{D}_a = i\,\mathrm{tr}t_a\left[\frac{d\Omega}{ds}\Omega^{-1}\right]\frac{d}{ds} \, . \tag{43}$$

The differential equation becomes

$$\frac{d}{ds}(\Delta + C_1 + C_2) = \frac{1}{48\pi^2}\int d^4x \left[\frac{d}{ds}\Omega\,\Omega^{-1}\right](d\Omega\,\Omega^{-1})^4 \, . \tag{44}$$

Let the point $s=0$ correspond to the configuration $\Omega(x,0)=1$ and the end point s correspond to $\Omega(x,s)=U(x)$. Now (44) can be integrated between these two points:

$$\Gamma(s) - \Gamma(0) = \frac{1}{48\pi^2}\int_0^s ds' \int d^4x \, \mathrm{tr}\left[\frac{d\Omega}{ds'}\Omega^{-1}\right](d\Omega\,\Omega^{-1})^4 \, . \tag{45}$$

A simple rescaling of s enables us to write (45) as

$$\Gamma(x_5=1)-\Gamma(x_5=0)=\frac{1}{48\pi^2}\int_0^1 dx_5\int_{S^4}d^4x\ \mathrm{tr}\left[\frac{d}{ds}\Omega\,\Omega^{-1}\right](d\Omega\,\Omega^{-1})^4\ , \tag{46}$$

where $\Omega(x,x_5=0)=1$ and $\Omega(x,x_5=1)=U(x)$. A more symmetrical form of (46) is

$$\Gamma(x_5=1)-\Gamma(x_5=0)=\frac{1}{240\pi^2}\int_{D^5}d\Sigma_{ijklm}\mathrm{tr}(\partial_i\Omega\Omega^{-1}\partial_j\Omega\Omega^{-1}\partial_k\Omega\Omega^{-1}\partial_l\Omega\Omega^{-1}\partial_m\Omega\Omega^{-1})\ , \tag{47}$$

where $d\Sigma_{ijklm}$ is the volume element of a five-dimensional disk with space-time S^4 as its boundary.

Now $\Gamma=\Delta+C_1+C_2$. Hence using the definitions (28), (32), and (38) and noting that $C_2(U=1)=0$ we get the result

$$\begin{aligned}(\mathrm{Im}\ln\det D^{[\Omega=1]}&-\mathrm{Im}\ln\det D^{[\Omega=U]})\\ &=i[C_1(L;R^{[U]})-C_1(L;R)]+iC_2(R^{[U]};i\,dU\,U^{-1})\\ &\quad+\frac{i}{240\pi^2}\int_{D_5}d\Sigma(d\Omega\,\Omega^{-1})^5\ .\end{aligned} \tag{48}$$

Equation (48) gives a formula for the difference between the phase of two determinants, one corresponding to $\Omega(x)=1$ and the other $\Omega(x)=U(x)$.

We now note that the vector field $R^{[U]}$ is invariant under the local right transformation $R\to R^{[\alpha]}$ and $U\to\alpha^{-1}U$. Hence we would expect that $\mathrm{Im}\ln\det D[\Omega=U]=\mathrm{Im}\ln\det(i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/^{[U]}+i\bar{H})$ is also invariant under local right transformations and hence cannot contribute to all but the first term on the right-hand side since these vary under $R\to R^{[\alpha]}$ and $U\to\alpha^{-1}U$. In Appendix B we sketch the proof that in the long-wavelength approximation in fact $\mathrm{Im}\ln\det D[\Omega=U]=0$. By the long-wavelength limit we mean for wavelengths much larger than $1/\bar{H}$, a calculation which assumes that we are in the broken phase where $\bar{H}\sim\langle\bar{\psi}\psi\rangle$. Hence in the long-wavelength approximation the phase of $\ln\det[i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+iH(U^\dagger P_L+UP_R)]$ is given by

$$\begin{aligned}\mathrm{Im}\ln\det D^{[\Omega=1]}=&\ i[C_1(L;R^{[U]})-C_1(L;R)]\\ &+iC_2(R^{[U]};i\,dU\,U^{-1})\\ &+\frac{i}{240\pi^2}\int_{D_5}d\Sigma(d\Omega\,\Omega^{-1})^5\ .\end{aligned} \tag{49}$$

Equation (49) can be written symmetrically in L and R by noting that the entire calculation could have been repeated for the left rotated operator $\tilde{D}=i\partial\!\!\!/-L\!\!\!/^{[\Omega]}P_R-R\!\!\!/P_R+iH(U^\dagger V^\dagger P_L+VUP_R)$. Then the analog of (49) is

$$\begin{aligned}\mathrm{Im}\ln\det\tilde{D}^{[\Omega=1]}=&\ i[C_1(L^{[U^{-1}]};R)-C_1(L;R)]\\ &+iC_2(L^{[U^{-1}]};i\,dU\,U^{-1})\\ &+\frac{i}{240\pi^2}\int_{D_5}d\Sigma(\Omega^{-1}d\Omega)^5\ .\end{aligned}$$

Hence

$$\begin{aligned}i\Delta(L,R,U)&=\tfrac{1}{2}(\mathrm{Im}\ln\det D^{[\Omega=1]}+\mathrm{Im}\ln\det\tilde{D}^{[\Omega=1]})\\ &=\frac{i}{2}[C_1(L^{[U^{-1}]};R)+C_1(L;R^{[U]})-2C_1(L;R)]+\frac{i}{2}[C_2(L^{[U^{-1}]};i\,dU\,U^{-1})+C_2(R^{[U]};i\,dU\,U^{-1})]\\ &\quad+\frac{i}{240\pi^2}\int d\Sigma(d\Omega\,\Omega^{-1})^5\ .\end{aligned} \tag{50}$$

We note that if $L=R=0$, (50) agrees with our previous result.

V. LONG-WAVELENGTH EXPANSION OF THE MODULUS OF THE DETERMINANT

Now that we have extracted the phase of the determinant we proceed to evaluate $\ln|\det D|$ which was defined in Eq. (22),

$$\ln|\det D|=\tfrac{1}{2}\mathrm{Tr}\ln D^\dagger D=-\tfrac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\mathrm{Tr}e^{-sD^\dagger D}\ . \tag{51}$$

In (51) $D=i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/P_R+i\bar{H}(U^\dagger P_L+UP_R)$. However since $DD^\dagger$ is anomaly free, we work with the gauge-rotated operator:

$$\begin{aligned}D^{[U]}&=(P_R+P_L U)D(P_L+P_R U^\dagger)\\ &=i\partial\!\!\!/-L\!\!\!/P_L-R\!\!\!/^{[U]}P_R+iH\ ,\\ R^{[U]}&=URU^{-1}+i\partial UU^{-1}\ .\end{aligned} \tag{52}$$

Introducing the vector and axial-vector fields

$$\begin{aligned}&V=\frac{L+R^{[U]}}{2},\quad A=\frac{L-R^{[U]}}{2}\ ,\\ &D^{[U]}=i\partial\!\!\!/-V\!\!\!/-A\!\!\!/\gamma_5+iH\ .\end{aligned} \tag{53}$$

Then with definition $\hat{O}=(D^{[U]^\dagger})D^{[U]}$, (51) becomes

$$\ln\det|D|=-\tfrac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\mathrm{Tr}e^{-s\hat{O}}\ . \tag{54}$$

The operator $\hat{O}$ can be treated as a quantum-mechanical operator $\hat{O}\equiv\hat{O}(\hat{P}_\mu,\hat{X}_\mu)$ with $\hat{P}_\mu=i\partial_\mu$ and $\hat{X}_\mu=x_\mu$. In the plane-wave basis the trace in (54) can be written as

$$\mathrm{Tr}e^{-s\hat{O}(\hat{P},\hat{x})}=\int d^4x\,d^4y\frac{d^4k}{(2\pi)^4}\langle x\,|\,e^{-s\hat{O}(\hat{P}+k,\hat{x})}\,|\,y\rangle\ . \tag{55}$$

Equation (54) can now be expanded as

$$\ln|\det D| = -\tfrac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}e^{-\bar{H}^2 s}\int d^4x\, d^4y\int\frac{d^4k}{(2\pi)^4}e^{-sk^2}\langle x|e^{-s[\hat{O}(\hat{p}+k,\hat{x})-k^2-\bar{H}^2]}|y\rangle\ . \tag{56}$$

This formula gives a systematic expansion in powers of $1/\bar{H}$. Details are given in Appendix C. The final expression involving operators up to dimension four is

$$\begin{aligned} N\ln|\det D| = -N\int d^4x\, \mathrm{tr}_f\{ & d_1[\tfrac{1}{3}(F^V_{\mu\nu})^2+\tfrac{1}{3}(F^A_{\mu\nu})^2+4\bar{H}^2A_\mu{}^2+(\partial_\mu H)^2+(H^2-\bar{H}^2)^2] \\ & +d_2 i[A_\mu,A_\nu]F^V_{\mu\nu}+d_3A_\mu{}^2(H^2-\bar{H}^2)+d_4(\partial_\mu A_\nu+i[V_\mu,A_\nu])^2 \\ & +d_5(A_\mu{}^2)^2+d_6[A_\mu,A_\nu]^2\}\ . \end{aligned} \tag{57}$$

The fields are defined as

$$F^V_{\mu\nu}=\partial_\mu V_\nu-\partial_\nu V_\mu+i[V_\mu,V_\nu]+i[A_\mu,A_\nu],\quad F^A_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i[V_\mu,A_\nu]+i[A_\mu,V_\nu]\ . \tag{58}$$

The coefficients are given by

$$d_1=F_{-1},\quad d_2=-4F_0,\quad d_3=4F_{-1}-8F_0,\quad d_4=-\tfrac{4}{3}F_0,\quad d_5=-\tfrac{8}{3}(2F_0-F_1),\quad d_6=-\tfrac{4}{3}(F_0+F_1)\ ,$$

where

$$F_{-1}=\frac{1}{16\pi^2}\int_x^\infty\frac{ds}{s}e^{-s},\quad F_0=\frac{1}{16\pi^2}e^{-x},\quad F_1=\frac{1}{16\pi^2}\int_x^\infty ds\, se^{-s}\ ,\quad x=\frac{\bar{H}^2}{\Lambda^2}\ .$$

As expected (58) is invariant under vector gauge transformations: $V\to\Omega V\Omega^{-1}+i\partial\Omega\Omega^{-1}$, $A\to\Omega A\Omega^{-1}$. Axial-vector transformations are not a symmetry of (58). This is because the long-wavelength expansion is performed in the broken-symmetry phase with respect to the mass $\bar{H}\sim\langle\bar{\psi}\psi\rangle$.

VI. THE PERTURBATIVE SPECTRUM

Now that we have completed our calculations let us collect our results. The effective action at long wavelengths ($k\ll 1/\bar{H}$) turns out to be

$$NS_{\mathrm{eff}}=iN\Delta(L,R,U)+N\ln|\det D|-\frac{N}{Ng_1{}^2}\int\mathrm{tr}H^2-\frac{N}{Ng_2{}^2}\int\mathrm{tr}(V_\mu{}^2+A_\mu{}^2)\ , \tag{59}$$

where Δ is the Wess-Zumino term given by (50) and $\ln|\det D|$ is given by (58).

The spectrum of this effective action can be analyzed in the large-N limit. Let us first consider the perturbative spectrum which consists of small oscillations around the classical broken symmetric configuration: $U=1$, $L_\mu=R_\mu=0$, $H=\bar{H}$. The Wess-Zumino term is irrelevant for this problem since it only provides the anomalous vertices. We analyze the quadratic part of the real part of the action:

$$\begin{aligned} -S^{(2)}_{\mathrm{eff}}=\int d^4x\,\mathrm{tr}\Bigg[& \frac{d_1}{3}(\partial_\mu V_\nu-\partial_\nu V_\mu)^2+\frac{d_1}{3}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+d_1(\partial_\mu\sigma)^2+4\bar{H}^2d_1\sigma^2 \\ & +4\bar{H}^2d_1(A_\mu+\tfrac{1}{2}\partial_\mu\pi)^2+\frac{2}{Ng_2{}^2}(A_\mu{}^2+V_\mu{}^2)\Bigg]\ . \end{aligned} \tag{60}$$

In (60) we have used the definitions $A=(L-R)/2$, $V=(L+R)/2$, $\sigma=H(x)-\bar{H}$, and $U=e^{i\pi}\approx 1+i\pi$. In (60) there is a mixing between the axial-vector field A_μ and $\partial_\mu\pi$. To diagonalize the quadratic form we define the linear combination

$$\tilde{A}_\mu=A_\mu+\frac{\bar{H}^2}{\alpha}\partial_\mu\pi,\quad \alpha=\frac{1+2\bar{H}^2d_1Ng_2{}^2}{d_1Ng_2{}^2}\ . \tag{61}$$

Then (60) becomes

$$\begin{aligned} -S^{(2)}_{\mathrm{eff}}=\int d^4x\,\mathrm{tr}\Bigg[& \frac{d_1}{3}(\partial_\mu V_\nu-\partial_\nu V_\mu)^2+\frac{2}{Ng_2{}^2}V_\mu{}^2+\frac{d_1}{3}(\partial_\mu\tilde{A}_\nu-\partial_\nu\tilde{A}_\mu)^2+2d_1\alpha\tilde{A}_\mu{}^2 \\ & +d_1\left[(\partial_\mu\sigma)^2+4\bar{H}^2\sigma^2+\frac{\bar{H}^2}{1+2\bar{H}^2d_1Ng_2{}^2}(\partial_\mu\pi)^2\right]\Bigg]\ . \end{aligned} \tag{62}$$

The fields V_μ and $\tilde{A}_\mu$ are to be identified with the vector and axial-vector mesons, with masses

$$m_V{}^2=\frac{3}{d_1Ng_2{}^2},\quad m_{\tilde{A}}{}^2=\frac{3}{d_1Ng_2{}^2}+6\bar{H}^2\,, \tag{63}$$

leading to the relation

$$m_{\tilde{A}}{}^2-m_V{}^2=6\bar{H}^2\,. \tag{64}$$

The field σ is that of a neutral meson with mass $m_\sigma{}^2=4\bar{H}^2$. π is the pion and the coefficient of $(\partial_\mu\pi)^2$ is to be identified with the pion-decay constant

$$F_\pi{}^2=\frac{N\bar{H}^2d_1}{1+2\bar{H}^2d_1Ng_2{}^2}\,. \tag{65}$$

VII. SOLITONS

We now comment on the possibility of soliton solutions of the effective Lagrangian (57). A simple possibility is to set the vector and axial-vector fields to zero: $V_\mu=A_\mu=0$, and the density field $H(x)=\bar{H}$. The resulting effective Lagrangian is that of the nonlinear σ model with higher-order terms:

$$-S_{\rm eff}=\int d^4x\;{\rm tr}\left[4\bar{H}^2d_1(l_\mu)^2+3d_4(\partial_\mu l_\mu)^2+\frac{d_5}{16}(l_\mu{}^2)^2+\frac{d_4-d_6}{16}[l_\mu,l_\nu][l_\nu,l_\mu]\right], \tag{66}$$

where

$$l_\mu=i\partial_\mu UU^{-1}\,.$$

The second term with the coefficient d_3 contributes to the inverse pion propagator: $4\bar{H}^2d_1k^2-3d_3k^4$. However, since we are restricted to slowly varying functions for which $k^2\ll\bar{H}^2$, the second term proportional to k^4 is neglected. This elementary perturbative argument could be carried over to the soliton sector by minimizing (66) in the class of slowly varying functions where we can neglect the term $d_3(\partial_\mu l_\mu)^2$ and the relevant effective action is

$$-S_{\rm eff}=\int d^4x\;{\rm tr}\left[4\bar{H}^2d_1(l_\mu{}^2)+\frac{d_4-d_6}{16}[l_\mu,l_\nu][l_\nu,l_\mu]+\frac{d_5}{16}(l_\mu{}^2)^2\right]. \tag{67}$$

The signs of the coefficients in (67) are important. First, the energy density should be positive, second, if a stable, time-independent soliton solution is to exist, the contributions to the energy from the four-derivative terms (G_4, say) should also be positive. The coefficients occurring in (67) are functions of $x=\bar{H}^2/\Lambda^2$. We find that $(d_4-d_6)/16$ is positive for all x, d_5 is negative for the range $x=0-1.1$ and positive for $x>1.1$. We have numerically checked that for a wide range of slowly varying configurations the energy density is positive for all x. G_4, however, can be negative when d_5 is negative. Thus there is a range of x where $d_5>0$ and hence G_4 is positive where stable time-independent soliton solutions can exist. The details will be presented elsewhere.

VIII. BARYONS ARE SOLITONS

In this section we would like to identify the large-N solitons of the effective action (66) with baryons. Consider the two-point function

$$B(x)=\langle J(x)J^\dagger(0)\rangle\,, \tag{68}$$

where

$$J(x)=\frac{1}{N!}\epsilon_{i_1\cdots i_n}\psi_{i_1}(x)\cdots\psi_{i_n}(x)$$

is a color-singlet baryon number operator. We have suppressed the flavor and Dirac indices. Since baryon number, being a vector symmetry, is conserved the state $J(x)\,|\,0\rangle$ has a nonzero overlap with a baryon state. Also, because ψ_i are anticommuting

$$J(x)=\psi_1(x)\cdots\psi_N(x)\,,$$

and

$$B(x)=\langle\psi_i(x)\cdots\psi_N(x)\bar\psi_N(0)\cdots\bar\psi_1(0)\rangle\,. \tag{69}$$

The QCD considerations which led us to the NJL-type effective Lagrangian apply as well for (69). Introducing auxiliary variables and integrating over the fermions, we get

$$B(x)=\frac{1}{z}\int_{M,L,R}[G(x;M,L,R)]^N\exp(NS_{\rm eff})\,, \tag{70}$$

where $G(x,0;\xi)$ is the Green's function corresponding to the operator D_{xy} in (6). $[G(x,0;\xi)]^N$ is color gauge invariant and a symbolic notation for the product of quark propagators in presence of the fields ξ with appropriate Dirac and flavor indices. For large x the ln of the Green's functions will go like $e^{-m[\xi]|x|}$. Let us now assume that in the limit of large N, $B(x;0)$ is evaluated by the saddle-point equation,

$$\delta S_{\rm eff}-\delta m\,|\,x\,|=0\,. \tag{71}$$

Since for large $|\,x\,|$ we expect $S_{\rm eff}$ and m to be bounded, the variational equation (71) implies $\delta S_{\rm eff}=0$ and $\delta m=0$. Hence baryons are solitons of the mesonic effective action S.

The baryon current is defined in terms of the left and right variations of $\ln Z_\psi$. Z_ψ was defined in (16):

$$B_\mu=\frac{1}{N}{\rm tr}(J^L_\mu+J^R_\mu)\,,$$

$$J^L_\mu=\langle\bar\psi\gamma_\mu P_L\psi\rangle=-\frac{\delta}{\delta L_\mu}\ln Z_\psi\,, \tag{72}$$

$$J^R_\mu=\langle\bar\psi\gamma_\mu P_R\psi\rangle=-\frac{\delta}{\delta R_\mu}\ln Z_\psi\,.$$

Therefore the baryon current can be written as

$$B_\mu=-\frac{1}{N}{\rm tr}\left[\frac{\delta}{\delta L_\mu}+\frac{\delta}{\delta R_\mu}\right]\ln Z_\psi\,. \tag{73}$$

Now $\ln Z_\psi = \ln \det D = N(\frac{1}{2}\mathrm{Tr}\ln D^\dagger D + i\Delta)$, and the real part defined by (22) is invariant under vector gauge transformations, hence (73) becomes

$$B_\mu = -i\,\mathrm{tr}\left[\frac{\delta}{\delta L_\mu} + \frac{\delta}{\delta R_\mu}\right]\Delta\ . \tag{74}$$

The baryon current depends only on the variations of the phase of the determinant and in general is a complicated function of the fields L_μ, R_μ, and U. However in the limit of very long wavelengths when the massive vector and axial-vector particles decouple from the effective Lagrangian, we can consider the fields L_μ and R_μ in (16) as infinitesimal sources and the formula for the baryon current becomes[24]

$$B_\mu = -i\left[\mathrm{tr}\left[\frac{\delta}{\delta L_\mu} + \frac{\delta}{\delta R_\mu}\right]\Delta\right]_{L=R=0}\ . \tag{75}$$

In this long-wavelength limit the only contribution to (75) is from the five-dimensional term in (50),

$$B_\mu = -\frac{1}{24\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{tr}(\partial_\nu UU^{-1}\partial_\rho UU^{-1}\partial_\sigma UU^{-1})\ , \tag{76}$$

B_μ, as is well known, is a topological current and its charge, the integral baryon number

$$n = \frac{1}{24\pi^2}\int d\mathbf{x}\,\epsilon_{ijk}\mathrm{tr}(\partial_i UU^{-1}\partial_j UU^{-1}\partial_k UU^{-1})$$

is a topological invariant.

IX. CONCLUSION

In this paper we have presented a qualitative picture of the emergence of the chiral model from QCD using the Nambu–Jona-Lasinio–type effective Lagrangian as an intermediate step. Our calculations involved a detailed derivation of the anomalous Wess-Zumino term of current algebra, including vector, axial-vector, and pseudoscalar mesons. We have also calculated the effective Lagrangian for these particles at low energies. The coefficients d_i (58), of the effective Lagrangians (57) and (66), unlike the anomaly coefficients, are not universal. However it is likely that the Lagrangian of the form (57) and (66) would emerge from a more detailed renormalization-group treatment of the underlying gauge theory.

We emphasize that (66) is a more realistic description of chiral dynamics than the Skyrme model.[25] Also the existence of soliton solutions in the chiral model (66) is by no means obvious (since all coefficients are not positive) and does depend upon the details of the coefficients (66). A soliton solution to (66) in an appropriate range of parameters has been borne out by numerical calculations.

There is another theory to which the type of analysis we have presented can be applied.[11,12] In the electroweak theory of Glashow, Salam, and Weinberg, if we integrate out the heavy top and bottom generation, then in the very-long-wavelength limit, when only the longitudinal modes of the vector bosons are dominant we can expect a nonlinear chiral model which may be able to support soliton solutions. We mention that these considerations are quite different from those of Gipson[26] who has considered such solitons in the electroweak theory resulting from a strongly coupled Higgs sector.

Note added. While this work was in progress, we received the following papers which have also discussed the non-Abelian anomaly with vector mesons: (a) O. Kaymakcalan, S. Rajeev, and J. Schechter, Phys. Rev. D **30**, 594 (1984); H. Gomm, O. Kaymakcalan, and J. Schechter, *ibid.* **30**, 2345 (1984). (b) N. K. Pak and P. Rossi, CERN Report No. Th. 3831 (unpublished). (c) H. Kawai and S. H. H. Tye, Cornell report (unpublished).

References (a) and (b) use the trial and error method of Witten. They have also emphasized the importance of the conservation of vector currents to obtain the correct form of the Wess-Zumino term. Reference (c) integrates the anomaly equation along a path in configuration space. We mention that this procedure was previously employed in Ref. 10.

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APPENDIX A

Here we outline the main steps in the calculation of the variation of the phase of the fermionic determinant:

$$\delta(\Delta) = -\mathrm{Im}\,\mathrm{Tr}\gamma_5\delta\Omega\Omega^{-1}e^{-(\hat{D}^{[\Omega]})^2/\Lambda^2}\ ,$$

where

$$\begin{aligned}\hat{D}^{[\Omega]}(\hat{P},\hat{x}) &= \gamma_5[\hat{\not{P}} - \not{L}P_L - \not{R}^{[\Omega]}P_R + iH(\Omega U^\dagger P_L + U\Omega^\dagger P_R)]\\ &= \gamma_5(\hat{\not{P}} - \not{V} - \not{A}\gamma_5 + i\widetilde{M}^\dagger P_L + i\widetilde{M}P_R)\ ,\\ V &= \tfrac{1}{2}(L + R^{[\Omega]}),\quad A = \tfrac{1}{2}(L - R^{[\Omega]}),\quad \widetilde{M} = U\Omega^\dagger\ .\end{aligned} \tag{A1}$$

P and X are operators in the space of spinors with the inner product $(\phi,\chi) = \int d^4x\,\phi^\dagger(x)\chi(x)$, $\hat{P}_\mu = i\partial/\partial X^\mu$, and $\hat{X}_\mu = x_\mu$. Following Fujikawa we evaluate the trace in the plane-wave basis:

$$-\mathrm{Im}\,\mathrm{Tr}\delta\Omega\Omega^{-1}\gamma_5 e^{-\hat{D}^2/\Lambda^2} = -\mathrm{Im}\int\frac{d^4R}{(2\pi)^4}\langle k\,|\,\mathrm{tr}\delta\Omega\Omega^{-1}\gamma_5 e^{-\hat{D}^2/\Lambda^2}\,|\,k\rangle$$

(tr is over Dirac and flavor indices)

$$=-\mathrm{Im}\int \frac{d^4k}{(2\pi)^4}\int d^4x\, d^4y \langle x \mid \mathrm{tr}\gamma_5\delta\Omega\Omega^{-1}e^{ik\cdot\hat{x}}e^{-\hat{D}^2/\Lambda^2}e^{-ik\cdot\hat{x}}\mid y\rangle$$

$$=-\mathrm{Im}\int \frac{d^4k}{(2\pi)^4}\int d^4x\, d^4y \langle x \mid \mathrm{tr}\gamma_5\delta\Omega\Omega^{-1}e^{-\hat{D}^2(\hat{P}+k,\hat{x})/\Lambda^2}\mid y\rangle$$

$$=-\mathrm{Im}\int d^4x\, d^4y \left\langle x \left| \mathrm{tr}\gamma_5\delta\Omega\Omega^{-1}\left[1-\frac{T_1}{\Lambda^2}+\frac{T_2}{2!\Lambda^4}-\cdots\right]\right| y\right\rangle , \tag{A2}$$

where

$$T_n=\int \frac{d^4k}{(2\pi)^4}e^{-k^2/\Lambda^2}[\hat{D}^2(\hat{P}+k,\hat{x})-k^2]^n .$$

Dropping terms $O(1/\Lambda^2)$ implies that we drop operators of dimension >4. Thus we can get contributions only from T_1, T_2, T_3, and T_4. After performing the trace operation and taking only the purely imaginary part, we are left with only the "minimal" terms proportional to $\epsilon_{\mu\nu\alpha\beta}$. It is convenient to work with the operators $d_\mu=P_\mu-V_\mu$ and A_μ, since these transform covariantly under vector gauge transformations. In terms of these

$$\begin{aligned}D^2(\hat{P}+k,\hat{x})-k^2= {} & d_\mu d_\mu+2k_\mu d_\mu+[A_\mu,d_\mu]\gamma_5-A_\mu A_\mu+i[\not{d}-\not{A}\gamma_5+\not{k},\widetilde{M}^{\dagger}P_L+\widetilde{M}P_R]\\ & +\widetilde{M}^{\dagger}\widetilde{M}^{\dagger}P_L+\widetilde{M}\widetilde{M}P_R+\tfrac{1}{2}[\gamma_\mu,\gamma_\nu]d_\mu d_\nu+\frac{\gamma_5}{2}[\gamma_\mu,\gamma_\nu]\{A_\mu,d_\nu\}+\gamma_5[\gamma_\mu,\gamma_\nu]A_\mu k_\nu\\ & -\tfrac{1}{2}[\gamma_\mu,\gamma_\nu]A_\mu A_\nu .\end{aligned} \tag{A3}$$

Computing the T_i's is tedious but straightforward algebra. The k integrals reduce to Gaussians and can be easily done. Finally we obtain

$$T_1=T_4=0 ,$$

$$\begin{aligned}\frac{1}{2!\Lambda^4}T_2-\frac{1}{3!\Lambda^6}T_3= {} & 2\mathrm{tr}_f(\,d_\mu d_\nu d_\alpha d_\beta+\tfrac{1}{3}[A_\mu d_\nu+d_\mu A_\nu][A_\alpha d_\beta+d_\alpha A_\beta]-\tfrac{4}{3}A_\mu d_\nu d_\alpha A_\beta-\tfrac{1}{3}d_\mu d_\nu A_\alpha A_\beta\\ & -\tfrac{1}{3}A_\mu A_\nu d_\alpha d_\beta-\tfrac{1}{3}A_\mu A_\nu A_\alpha A_\beta)\epsilon_{\mu\nu\alpha\beta} .\end{aligned} \tag{A4}$$

From this we can easily get our final answer

$$-\mathrm{Im}\,\mathrm{Tr}\gamma_5\delta\Omega\Omega^{-1}e^{-\hat{D}^2/\Lambda^2}=\int d^4x\, \mathrm{tr}_f[i\delta\Omega\Omega^{-1}B(x)] . \tag{A5}$$

$B(x)$ is Bardeen's anomaly.

It is important to note that at the level of phenomenological Lagrangians, where the ultraviolet cutoff is finite, it is justified to drop $O(1/\Lambda^2)$ terms in the anomaly Eq. (A2), because we are assuming that the background fields are slowly varying over this scale.

APPENDIX B

We will now proceed to carry out the long-wavelength expansion of the phase of the determinant of $\hat{D}^{[\Omega=U]}$ and show that it is indeed zero. The long-wavelength expansion was explained in the text. We have

$$\begin{aligned}\mathrm{Im}\ln\det\hat{D}^{[\Omega=U]}&=-\tfrac{1}{2}\mathrm{Im}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}e^{-s(\hat{D}^{[U]})^2/\Lambda^2}\\ &=-\tfrac{1}{2}\mathrm{Im}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}e^{-\overline{H}^2 s}\int d^4x\, d^4y\left\langle x\left|1-\frac{s}{\Lambda^2}\widetilde{T}_1+\frac{s^2}{2!\Lambda^4}\widetilde{T}_2-\cdots\right|y\right\rangle ,\end{aligned} \tag{B1}$$

where

$$\widetilde{T}_n=\int e^{-k^2 s}\frac{d^4k}{(2\pi)^4}[\hat{D}^{[U]}(\hat{p}+k,\hat{x})-k^2-\overline{H}^2]^n .$$

The operator $\hat{D}^2(\hat{p}+k,\hat{x})-k^2$ has been written down in Appendix A. After a lot of algebra on similar lines to the calculation in Appendix A, we get

$$\tilde{T}_1=0\ ,$$

$$\frac{s^2}{2!\Lambda^4}\tilde{T}_2=2\mathrm{tr}_f(ddAd-dddA-Addd-dAdd+dAAA-AdAA+AAdA-AAAd)\ ,$$

$$-\frac{s^3}{3!\Lambda^6}\tilde{T}_3=-\tfrac{4}{3}\mathrm{tr}_f(ddAd-2dddA+2Addd-dAdd+4dAAA-5AddA+5AAdA-4AAAd)\ , \tag{B2}$$

$$\frac{s^4}{4!\Lambda^8}\tilde{T}_4=4\mathrm{tr}_f(dAAA-AdAA+AAdA-AAAd)\ ,$$

where

$$X_1X_2X_3X_4\equiv\epsilon_{\mu\nu\alpha\beta}X_{1\mu}X_{2\nu}X_{3\alpha}X_{4\beta}\ .$$

It can now be easily verified that

$$\frac{s^2}{2!\Lambda^4}\tilde{T}_2-\frac{s^3}{3!\Lambda^6}\tilde{T}_3+\frac{s^4}{4!\Lambda^8}\tilde{T}_4=\mathrm{tr}_f\{[[d,d],[A,d]]+[[d,A],[A,A]]\}=0\ . \tag{B3}$$

APPENDIX C

The long-wavelength expansion of the real part of the ln of the fermionic determinant has already been described in the text. We now describe the salient features of the calculation. We have

$$\begin{aligned}\ln\det\hat{D}^\dagger\hat{D}&=-\tfrac{1}{2}\mathrm{Tr}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}e^{-s\hat{D}^\dagger\hat{D}}\\&=-\tfrac{1}{2}\mathrm{Tr}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}e^{-\bar{H}^2s}\int\frac{d^4k}{(2\pi)^4}e^{-sk^2}\mathrm{tr}\left\langle x\left|1-sT_1+\frac{s^2}{2!}T_2-\cdots\right|y\right\rangle\ ,\end{aligned} \tag{C1}$$

where

$$T_n=\hat{D}^\dagger\hat{D}(\hat{p}+k,\hat{x})-k^2-\bar{H}^2\ .$$

After the algebra we will be left with integrals of the following type:

$$I_{nl}=\frac{1}{n!}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}e^{-\bar{H}^2s}s^n\int\frac{d^4k}{(2\pi)^4}k^{2l}e^{-sk^2}\ . \tag{C2}$$

All the integrals we need ($n<5,l<3$) can be written up to a constant as one of the following three integrals:

$$F_{-1}=\frac{1}{16\pi^2}\int_x^{\infty}\frac{ds}{s}e^{-s},\quad F_0=\frac{1}{16\pi^2}e^{-x},\quad F_1=\frac{1}{16\pi^2}\int_x^{\infty}s\,e^{-s}ds\ ,\quad x=\bar{H}^2/\Lambda^2\ . \tag{C3}$$

Here we digress to remark that if we had regularized the determinant in the following way,

$$\ln\det\hat{D}^\dagger\hat{D}=-\tfrac{1}{2}\mathrm{Tr}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}g(s\hat{D}^\dagger\hat{D})\ , \tag{C4}$$

where $g(y)$ is any smooth function which vanishes at $y=\infty$ along with its derivatives, the only difference in the calculation is that F_{-1}, F_0, and F_1 are modified to

$$F_{-1}=\frac{1}{16\pi^2}\int_x^{\infty}\frac{ds}{s}g(s),\quad F_0=\frac{1}{16\pi^2}\int_x^{\infty}ds\,g'(s),\quad F_1=\frac{1}{16\pi^2}\int_x^{\infty}ds\,sg''(s)\ . \tag{C5}$$

The operator $\hat{D}^\dagger\hat{D}(\hat{P}+k,\hat{x})$ is given by

$$\begin{aligned}\hat{D}^\dagger\hat{D}(\hat{P}+k,\hat{x})=\ &k^2+\bar{H}^2+2d_\mu k_\mu-2A_\mu k_\mu\gamma_5+(d_\mu d_\mu+A_\mu A_\mu)-(A_\mu d_\mu+d_\mu A_\mu)\gamma_5-i\gamma_\mu\partial_\mu H\\&+2iHA_\mu\gamma_\mu\gamma_5+(H^2-\bar{H}^2)-\frac{i}{2}\gamma_\mu\gamma_\nu F^V_{\mu\nu}-\frac{i}{2}\gamma_\mu\gamma_\nu\gamma_5F^A_{\mu\nu}\ .\end{aligned} \tag{C6}$$

We have taken $H(x)$ to be a multiple of the unit matrix, $d_\mu=\hat{P}_\mu-V_\mu$. After a tedious calculation we finally get the result quoted in Sec. V. Also (C5) indicates that the coefficients d_i are not universal.

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