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Radiative corrections to classical particle motion in a magnetic field

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The classical behavior of a charged particle moving in a magnetic field is derived by the WKB approximation and wave-packet method from the Klein-Gordon equation with the Schwinger radiative term. The lifetime of the wave-packet state is calculated for a constant magnetic field.

I. INTRODUCTION

The problem of the influence of bremsstrahlung on a particle moving in an electromagnetic field has been a subject of interest for many years.[1–7] The natural approach to solve this problem is to derive an equation of motion involving the bremsstrahlung, and the Lorentz-Dirac equation is generally considered as the most acceptable solution of the problem. It was obtained on the basis of classical electrodynamics by decomposing the energy-momentum tensor of the retarded self-field into a sum that renormalizes mass and a term that gives radiation reaction.[8]

A theoretical rederivation of this equation based on an absorber mechanism was provided by Wheeler and Feynman.[9] Nevertheless, the Lorentz-Dirac equation derived by the different methods has certain imperfections which need special discussion and approaches not involved in theory. The difficulties are as follows: (a) The Lorentz-Dirac equation involves the derivative of acceleration and it needs an extra condition in addition to the Newtonian initial condition to determine the motion. (b) It gives runaway solutions which can be avoided only by introducing a preacceleration. (c) In certain cases it implies that the external energy supplied to the particle goes only into kinetic energy and radiation is created from an acceleration self-energy which becomes more and more negative.[10]

The purpose of the present paper is not the rederivation of the Lorentz-Dirac equation without imperfections but to derive by combination of the WKB approximation and wave-packet method the classical behavior of a charged radiating particle from the Klein-Gordon equation with the Schwinger radiative term. Our problem is related to the method of Censor who has considered the quantum-mechanical problem of motion of particles in a dissipative system using a wave-packet and eikonal representation of the wave function.[11]

The natural idea which is presented in Censor's article and which is accepted in our article is the stipulation that the group velocity must be a real quantity in a dissipative system using a wave-packet and eikonal representation of the wave function.[11]

II. FORMULATION OF THE PROBLEM AND SOLUTION

Our starting point is the Klein-Gordon equation with the Schwinger mass operator or, in other words, with a radiative term:[12,13]

$$(\Pi^\mu\Pi_\mu+m^2+\delta m^2)\phi=0\ , \tag{1}$$

where

$$\Pi_\mu=\frac{1}{i}\partial_\mu-eqA_\mu \tag{2}$$

with

$$\partial_\mu=\frac{\partial}{\partial x^\mu}\ ,\quad x^\mu=(ct,\vec{x})\equiv(x^0,\vec{x})=(-x_0,\vec{x}) \tag{3}$$

and

$$q=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}. \tag{4}$$

A_μ is the electromagnetic potential, q is the charge matrix, ϕ is the two-component wave function, and the radiative term δm^2 is the following mathematical object:[12,13]

$$\delta m^2=ie^2\int\frac{(dk)}{(2\pi)^4}(2\Pi-k)^\mu\frac{1}{k^2}\frac{1}{(\Pi-k)^2+m^2}\times(2\Pi-k)_\mu+\text{C.T.}\ , \tag{5}$$

where C.T. are the contact terms defined in Refs. 12 and 13.

The eigenvalues of the operator (5) have been calculated for a constant magnetic field by Tsai[14] to give (the C.T. are involved)

$$\delta m^2=\kappa+i\lambda\ , \tag{6}$$

where

$$\begin{aligned}\kappa&=\text{Re}\,\delta m^2[\vec{E}=0,\vec{H}=(0,0,H=\text{const})]\\&=\frac{\alpha}{\pi}m^2\left[\left(\frac{eH}{m^2}\right)\left(\tfrac{4}{3}\ln\frac{m^2}{2eH}-\frac{7}{72}\right)\right.\\&\qquad+(2n+1)\left(\frac{eH}{m^2}\right)\left(\tfrac{8}{3}\ln\frac{m^2}{2eH}\right.\\&\qquad\qquad\left.\left.-\frac{32}{5}\ln2+\tfrac{343}{90}\right)\right],\\&\qquad h=1\ ;n=0,1,2,\ldots\end{aligned} \tag{7}$$

and

$$\lambda = \mathrm{Im}\delta m^2[\vec{E}=0,\vec{H}=(0,0,H=\text{const})]$$
$$= -\alpha m^2\left[\frac{4n}{3}\left[\frac{eH}{m^2}\right]+\left[\frac{eH}{m^2}\right]^3[-\tfrac{2}{15}(2n+1)^2 - \tfrac{4}{3}(2n+1)+\tfrac{22}{15}]\right],$$
$$h=1\ ; n=0,1,2,\ldots . \quad (8)$$

In order to find some information about the classical motion of the particle we replace the operator δm^2 in Eq. (1) by its eigenvalues and use the well-known fact that the classical limit of the quantum-mechanical equations can be obtained by the WKB approximation of the wave function:[15]

$$\phi_{\mathrm{WKB}} = e^{(i/h)S}(a_0 + h a_1 + h^2 a_2 + \cdots) . \quad (9)$$

It is known that the zero-order approximation

$$\phi_{(0)\mathrm{WKB}} = a_0 \exp[(i/h)S]$$

generates the Hamilton-Jacobi equation

$$(P^\mu + eA^\mu)(P_\mu + eA_\mu) + m^2 + \delta m^2 = 0$$

with $(eq)' = -e$ where the latter incorporates the charge assignment of particle with charge e, $P^\mu = \partial^\mu S$ is the generalized momentum. However, the expression $m^2 + \delta m^2$ in the Hamilton-Jacobi equation is a complex number and it therefore makes the Hamiltonian-Jacobi equation meaningless in the classical sense.

To overcome this obstacle in order to get the classical information about the particle motion we will combine the zero-order WKB approximation

$$\phi_{(0)\mathrm{WKB}} = a_0 e^{(i/h)S} \quad (10)$$

with the wave-packet method.

In the sufficiently small space-time interval we obviously can write[16]

$$S \approx S_{(0)} + \frac{\partial S}{\partial \vec{x}}\cdot\vec{x} + \frac{\partial S}{\partial t} t \quad (11)$$

or using the Hamilton-Jacobi equations $\vec{P} = \partial S/\partial \vec{x}$, $-H = \partial S/\partial t$, we have

$$S \approx S_{(0)} + \vec{P}\cdot\vec{x} - Ht , \quad (12)$$

where $\vec{P}$ is the generalized momentum of the particle, H is its energy, and obviously for $|\delta m^2| \ll m^2$ and $A^0 = 0$ it is $(\vec{p} = \vec{P} + e\vec{A})$

$$H(A^0=0) = (\vec{p}^{\,2} + m^2 + \delta m^2)^{1/2} \approx E + \frac{\kappa}{2E} + \frac{i}{2}\frac{\lambda}{E} , \quad (13)$$

where $E = -(p^2 + m^2)^{1/2}$ for bound states.

After insertion of Eq. (13) into Eq. (12) and then Eq. (12) into Eq. (10) we get

$$\phi_{(0)\mathrm{WKB}} \approx a_0 e^{(i/h)[S_{(0)} + \vec{P}\cdot\vec{x} - Et]} e^{(\lambda/2Eh)t} e^{i\kappa/2Eh} \quad (14)$$

which means that the particle with the complex mass is in the quasistationary state which decays according to decaying law $\exp[(\lambda/2Eh)t]$ with the decay rate

$$\gamma = -\frac{\lambda}{hE}\ ; E<0\ ,\ \lambda \geqq 0\ , \quad (15)$$

where the formula (15) is in agreement with the formula (69) in Ref. 14.

To get further information we use the obvious approximation

$$H \approx H_{(0)} + \frac{\partial H}{\partial \vec{P}}\cdot(\vec{P} - \vec{P}_{(0)}) + \cdots$$
$$= H_{(0)} + \vec{v}\cdot(\vec{P} - \vec{P}_{(0)}) + \cdots , \quad (16)$$

where we have used the Hamilton equation $\vec{v} = \partial H/\partial \vec{P}$, $\vec{v}$ being the velocity of the particle with momentum $\vec{P}$. Then, after insertion of Eq. (16) into Eq. (12) we have

$$S \approx S_{(0)} - H_{(0)}t + \vec{P}_{(0)}\cdot\vec{x} + (\vec{x} - \vec{v}t)\cdot(\vec{P} - \vec{P}_{(0)}) . \quad (17)$$

Now we construct the wave-packet solution of Eq. (1) by $\vec{P}$ integration of Eq. (10) with the exponent (17). We find

$$\phi = a_0 e^{(i/h)[S_{(0)} - H_{(0)}t + \vec{P}_{(0)}\cdot\vec{x}]} \times \int d\vec{P}\, g(\vec{P}) e^{(i/h)(\vec{x} - \vec{v}t)\cdot(\vec{P} - \vec{P}_{(0)})} , \quad (18)$$

where $g(\vec{P})$ is the suitable weight function which forms the envelope of the wave packet $G(\vec{P}_{(0)}, \vec{x} - \vec{v}t)$,

$$G(P_{(0)}, \vec{x} - \vec{v}t) = \int d\vec{P}\, g(\vec{P}) e^{(i/h)(\vec{x} - \vec{v}t)\cdot(\vec{P} - \vec{P}_{(0)})} . \quad (19)$$

The function ϕ in Eq. (18) describes a wave packet with a carrier wave

$$a_0 \exp[(i/h)(S_{(0)} - H_{(0)}t + \vec{P}_{(0)}\cdot\vec{x})]$$

and an envelope $G(\vec{P}_{(0)}, \vec{x} - \vec{v}t)$ which moves at constant velocity according to the law $\vec{x} = \vec{v}t$ in the small space-time interval.

We identify the motion of the envelope with the classical motion of the particle moving at velocity $\vec{v}$. But at this stage of the investigation, $\vec{v}$ is the complex quantity and therefore it does not mean it is the physical velocity. To avoid this obstacle in order to get the physically meaningful description of reality we stipulate the transformation

$$\vec{P}_{(0)} \to \vec{P}_{(0)} + i\vec{\epsilon} , \quad (20)$$

where $\vec{\epsilon}$ is to be determined to be valid,

$$\mathrm{Im}\,\vec{v} = 0 . \quad (21)$$

Using

$$\vec{v}(\vec{P}_{(0)} + i\vec{\epsilon}, m^2 + \delta m^2)$$
$$\approx \vec{v}(\vec{P}_{(0)}, m^2) + i\vec{\epsilon}\cdot\frac{\partial \vec{v}}{\partial \vec{P}} + \kappa\frac{\partial \vec{v}}{\partial m^2} + i\lambda\frac{\partial \vec{v}}{\partial m^2} , \quad (22)$$

for $|\delta m^2| \ll m^2$, $|\vec{\epsilon}| \ll |\vec{P}_{(0)}|$, we get, after applying the requirement (21), the following equation for $\vec{\epsilon}$:

$$\frac{\partial v_i}{\partial P_j}\epsilon_j+\lambda\frac{\partial v_i}{\partial m^2}=0 . \tag{23}$$

Then, instead of Eq. (22) we have

$$\vec{v}=\vec{v}(\vec{P}_{(0)},(m^*)^2) , \tag{24}$$

$$(m^*)^2=m^2+\kappa . \tag{25}$$

The last formula can be easily interpreted in such a way that the radiation of a charged particle moving in an electromagnetic field changes its mass not only in the quantum theory but also in the classical limit.

The transformation (20) leads to the transformation

$$\frac{i}{h}\vec{v}\cdot\vec{P}_{(0)}t\rightarrow\frac{i}{h}\vec{v}\cdot\vec{P}_{(0)}t-\frac{2\vec{v}\cdot\vec{\epsilon}}{2h}t \tag{26}$$

which necessitates the time dependence of the wave function in the form

$$\phi\sim e^{-(\Gamma/2)t} , \tag{27}$$

where

$$\Gamma=\gamma+\frac{2\vec{v}\cdot\vec{\epsilon}}{h} , \tag{28}$$

and it may be easy to specify the quantity Γ using the obvious relations

$$(\delta_{ij}-v_iv_j)\epsilon_j=\tfrac{1}{2}\lambda(\vec{p}^{\,2}+m^2)^{-1/2}v_i , \tag{29}$$

$$\vec{\epsilon}\cdot\vec{v}=\frac{\lambda}{2}\frac{v^2}{1-v^2}(\vec{p}^{\,2}+m^2)^{-1/2} . \tag{30}$$

Then $[\vec{v}=\vec{p}(\vec{p}^{\,2}+m^2)^{-1/2}]$,

$$\Gamma=\gamma+\frac{2\vec{v}\cdot\vec{\epsilon}}{h}=\frac{\lambda}{h}\frac{(\vec{p}^{\,2}+m^2)^{1/2}}{m^2} . \tag{31}$$

The quantity Γ is here interpreted as a lifetime of a wave packet moving at velocity $\vec{v}$ in a magnetic field on the orbit corresponding to the quantum number n.

III. DISCUSSION

The radiation of the accelerated particle of nonzero charge is the key to understanding certain phenomena in modern astrophysics, e.g., pulsars, particle physics, and the key for tuning particle accelerators.

We have seen that by representing the motion of the particle by the wave packet, which corresponds to the solution of the Klein-Gordon equation with the Schwinger radiative term, the particle state is quasistationary with the decay rate (31). The result (31) is not in contradiction with the relation (15) because the decay rate (15) corresponds to the quasistationary state of the wave function $\phi_{(0)\mathrm{WKB}}=a_0\exp[(i/h)S]$, while the decay rate (31) corresponds to the quasistationary state of the wave packet, or in other words to another real situation.

The derived results in the present paper suggest the classical picture of motion of the particle undergoing radiation reaction. In the very small space-time interval a particle with mass $(m^2+\kappa)^{1/2}$ is moving at velocity $\vec{v}$ and it remains at this velocity only for time Γ, Γ being the decay rate of wave packet (31), and after time Γ the particle changes its velocity. The change of the velocity is caused by the complex mass of the particle which corresponds to the influence of radiation of the particle on the particle motion.

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