# Particle in an external electromagnetic field. II. The exact velocity in a constant and uniform field

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The purpose of this paper is to use Clifford algebraic techniques to solve for the relativistic velocity of a charged particle in constant electric and magnetic fields of arbitrary orientation. This problem already has standard methods of solution, but the application of the present method provides a means of critically comparing previous work on this topic.

## I. INTRODUCTION

This paper utilizes the Clifford algebra of differential forms as developed by Kähler[1,2] and the author,[3–5] and which has very recently become the subject of considerable interest in physics.[6–14] The present application is entirely distinct from the work which deals with fermions on a lattice, and illustrates a separate facet of this elegant and powerful mathematical formalism.

We give an exact description for the relativistic motion of a charged particle in constant, homogeneous electric and magnetic fields of arbitrary orientation.

An explicit vector expression is derived for the case when the particle is initially at rest. Any possible radiation reaction is ignored. There are two standard, though distinct, textbook solutions of the problem.[15,16] More general solutions which, as a group, follow more or less the same philosophy are Refs. 17—22. These authors obtain the relativistic velocity as a function of the proper time. Some related, but distinct approaches are to be found in Refs. 23—26. What has been lacking up until now in the literature is a critical comparison of the various results obtained for the same problem. We provide such a comparison here, and point out several discrepancies among the solutions.

## II. THE CLIFFORD ALGEBRA OF DIFFERENTIAL FORMS

The geometrical basis of Minkowski spacetime consists of 16 basis differential forms, written in terms of the basis one-forms $\sigma^\mu = dx^\mu$ and the antisymmetric Grassmann product as

$$\{1,\sigma^\mu,\sigma^\mu\wedge\sigma^\nu,\sigma^\mu\wedge\sigma^\nu\wedge\sigma^\lambda,\omega=\sigma^1\wedge\sigma^2\wedge\sigma^3\wedge\sigma^4\}$$
$$\mu\neq\nu\neq\lambda,\ \mu,\nu,\lambda=1,2,3,4\ . \quad (1)$$

For convenience, we label the spatial three-volume form by $\eta=\sigma^1\wedge\sigma^2\wedge\sigma^3$. In general, one has tensors of type zero (scalar), one (vector), two, three, and four, which are linear combinations of the basis forms (1). For example, the relativistic velocity $u$ is a vector type defined as follows (sum over repeated indices):

$$u=u^\mu\sigma^\mu=\mathbf{u}+u^4\sigma^4,\ \mathbf{u}=u^i\sigma^i,\ \ i=1,2,3$$
$$u^\mu=\frac{dx^\mu}{d\tau},\ \ \tau=(x_\mu x^\mu)^{1/2},\ \ \mathbf{u}=\gamma\mathbf{V}\ ,$$
$$u^4=\gamma=(1-|\mathbf{V}|^2)^{-1/2}\ . \quad (2)$$

The electromagnetic field $f$ is a tensor of type two, which can be decomposed uniquely into its magnetic and electric components. In the decomposition, the duality in the three-dimensional spatial subspace of Minkowski spacetime plays an important role, and is denoted by a star with an index (3 for purely spatial components, 4 for all spacetime components),

$$f=\tfrac{1}{2}f^{\mu\nu}\sigma^\mu\wedge\sigma^\nu=\mathbf{E}\wedge\sigma^4-\underset{3}{*}\,\mathbf{B}\ ,$$
$$\mathbf{E}=E^i\sigma^i,\ \ E^i=f^{i4},\ \ B^i=-\tfrac{1}{2}\epsilon^{ijk}f^{jk}\ . \quad (3)$$

By introducing the "vee product" between the basis forms one defines an algebra of tensor types which permits associative multiplication, division, exponentiation, etc. Moreover, in this geometrical setting, representation matrices are not needed hence never appear in the calculations. We define the vee product v among the basis one-forms as follows:[1–5]

$$\sigma^\mu \mathrm{v}\sigma^\nu=\sigma^\mu\wedge\sigma^\nu=-\sigma^\nu \mathrm{v}\sigma^\mu,\ \ \mu\neq\nu$$
$$\sigma^i \mathrm{v}\sigma^i=-1,\ \ \sigma^4 \mathrm{v}\sigma^4=1\ (\text{no sum})\ . \quad (4)$$

The general vee product between a basis $p$-form and a basis $q$-form is defined in terms of the permutation group, and is given in Refs. 4 and 5. For the purposes of this paper, we need only a few specific vee products which are recalled here. For example, some products between basis forms of higher rank are

$$\eta \mathrm{v}\eta=1,\ \ \omega \mathrm{v}\omega=-1,\ \ \eta \mathrm{v}\sigma^4=\omega=-\sigma^4 \mathrm{v}\eta\ ,$$
$$\sigma^1 \mathrm{v}(\sigma^1\wedge\sigma^2)=-\sigma^2,\ \ \sigma^1 \mathrm{v}(\sigma^2\wedge\sigma^3)=\eta\ , \quad (5)$$
$$(\sigma^1\wedge\sigma^2)\mathrm{v}(\sigma^1\wedge\sigma^4)=\sigma^2\wedge\sigma^4,\ \ \text{etc.}$$

The vee products between the tensor types follow as a consequence of the product on the basis forms. The following will be used in this paper; details may be obtained from Refs. 4 and 5 and references therein:

$$u\mathrm{v}u=(u^4)^2-(\mathbf{u}\cdot\mathbf{u})=1\ , \tag{6a}$$

$$\eta\mathrm{v}\mathbf{B}=-\underset{3}{*}\mathbf{B}\Longrightarrow f=\mathbf{E}\mathrm{v}\sigma^4+\eta\mathrm{v}\mathbf{B}\ , \tag{6b}$$

$$\omega\mathrm{v}f=\underset{4}{*}f=\mathbf{B}\mathrm{v}\sigma^4-\eta\mathrm{v}\mathbf{E}\ , \tag{6c}$$

$$u\mathrm{v}f=-\mathbf{u}\times\mathbf{B}-u^4\mathbf{E}-(\mathbf{u}\cdot\mathbf{E})\sigma^4 +\omega\mathrm{v}[\mathbf{u}\times\mathbf{E}-u^4\mathbf{B}-(\mathbf{u}\cdot\mathbf{B})\sigma^4]\ , \tag{6d}$$

$$f\mathrm{v}u=\mathbf{u}\times\mathbf{B}+u^4\mathbf{E}+(\mathbf{u}\cdot\mathbf{E})\sigma^4 +\omega\mathrm{v}[\mathbf{u}\times\mathbf{E}-u^4\mathbf{B}-(\mathbf{u}\cdot\mathbf{B})\sigma^4]\ , \tag{6e}$$

$$f\mathrm{v}f=(\mathbf{E}\cdot\mathbf{E})-(\mathbf{B}\cdot\mathbf{B})-2\omega(\mathbf{E}\cdot\mathbf{B})\ . \tag{6f}$$

These rules should give an idea of how the algebra works in practice. There is a point, however, which is likely to lead to confusion. For purely spatial vector types, the vee product is still defined in the Minkowski metric, so that, for example ($\mathbf{E}=E^i\sigma^i,\mathbf{B}=B^i\sigma^i$),

$$\begin{aligned}\mathbf{E}\mathrm{v}\mathbf{B}&=-(\mathbf{E}\cdot\mathbf{B})-\eta\mathrm{v}(\mathbf{E}\times\mathbf{B})\ ,\\ \mathbf{E}\mathrm{v}\mathbf{E}&=-(\mathbf{E}\cdot\mathbf{E})=-E^iE^i\ .\end{aligned} \tag{7}$$

This is necessary in order to have the proper embedding of the three-dimensional space into four-dimensional space-time. One has the same multiplication rules for a purely spatial vector type as for the spatial part of a space-time vector type. The key point to remember is that there is a unique product in the Clifford algebra, and that is v. What are traditionally thought of as "products" such as $(\mathbf{a}\cdot\mathbf{b})$, $(\mathbf{a}\times\mathbf{b})^i$, $(a,b)$ are merely particular combinations of scalar components in this context.[5]

In the description of relativistic motion, the automorphisms of tensor types are of importance. They are spatial rotations, Lorentz boosts, and duality rotations. The first two are described in a way similar to the standard matrix realization. For instance, the spatial rotation of any tensor type $\alpha$ counterclockwise about a direction $\hat{\boldsymbol{\theta}}$ by an angle $|\boldsymbol{\theta}|$ is (a caret denotes a unit vector)

$$\begin{aligned}\alpha'&=\mathbb{R}(\boldsymbol{\theta})\mathrm{v}\alpha\mathrm{v}\mathbb{R}(-\boldsymbol{\theta})\ ,\\ \mathbb{R}(\boldsymbol{\theta})&=\exp(-\tfrac{1}{2}\eta\mathrm{v}\boldsymbol{\theta})\\ &=\cos\frac{|\boldsymbol{\theta}|}{2}-\eta\mathrm{v}\hat{\boldsymbol{\theta}}\sin\frac{|\boldsymbol{\theta}|}{2}\ .\end{aligned} \tag{8}$$

The Lorentz boost of a tensor type $\alpha$ in the direction $\hat{\mathbf{b}}$ by boost parameter $|\mathbf{b}|$, where the frame velocity $\mathbf{V}$ satisfies $|\mathbf{V}|=\tanh|\mathbf{b}|$, is

$$\begin{aligned}\alpha''&=\mathrm{L}(\mathbf{b})\mathrm{v}\alpha\mathrm{v}\mathrm{L}(-\mathbf{b})\ ,\\ \mathrm{L}(\mathbf{b})&=\exp(-\tfrac{1}{2}\mathbf{b}\mathrm{v}\sigma^4)\\ &=\cosh\frac{|\mathbf{b}|}{2}-\hat{\mathbf{b}}\mathrm{v}\sigma^4\sinh\frac{|\mathbf{b}|}{2}\ .\end{aligned} \tag{9}$$

The transformations in the algebra described by Eqs. (8) and (9) are transformations of the basis forms (1). The process defines a rearrangement of the basis forms which can be written as a combination of the original forms. This gives the transformation in either of two possible forms for each distinct tensor type:

$$\begin{aligned}a'&=(a^\mu\sigma^\mu)'=a^\mu(\sigma^\mu)'=(a'^\mu)\sigma^\mu\ ,\\ f'&=\tfrac{1}{2}(f^{\mu\nu}\sigma^\mu\wedge\sigma^\nu)'=\tfrac{1}{2}f^{\mu\nu}(\sigma^\mu\wedge\sigma^\nu)'\\ &=\tfrac{1}{2}(f'^{\mu\nu})\sigma^\mu\wedge\sigma^\nu\ .\end{aligned} \tag{10}$$

In this paper we will always use a prime to denote the second part of (10). For example, for an electric field vector, $\mathbf{E}'=(E'^i)\sigma^i$, where $E'^i$ are the usual Lorentz-transformed components of the electric field.[22] The primed electromagnetic field $f'$ is decomposed as in (3) and (6b), using the Lorentz-transformed field components

$$\begin{aligned}&f=\mathbf{E}\mathrm{v}\sigma^4+\eta\mathrm{v}\mathbf{B}\ ,\\ \Longrightarrow\ &f'=\mathbf{E}'\mathrm{v}\sigma^4+\eta\mathrm{v}\mathbf{B}'\\ &\quad=(E'^i)\sigma^i\mathrm{v}\sigma^4+\eta\mathrm{v}\sigma^j(B'^j)\ .\end{aligned} \tag{11}$$

We now describe the physical setting of the problem. The equation of motion of a charged particle in an electromagnetic field can be written as a commutator in the vee algebra in terms of the proper time $\tau$ as follows ($m$ is the mass, and $q$ is the charge):[21,22]

$$\frac{du}{d\tau}=\frac{q}{2m}[f,u]\ . \tag{12}$$

The scalar component equation derived from Eq. (12) is what is commonly used as a starting point for the solution, namely, $du^\lambda/d\tau=(q/2m)f^{\lambda\mu}u_\mu$. In that case, however, the geometrical information of (12) is lost, and the solution is constrained along each component of $u$. In contrast, a general solution can be obtained directly from (12) which maintains the vectorial generality since it is an intrinsic solution in the vee algebra. For an electromagnetic field with no space or time dependence, one has the following general solution[22] [see also Ref. 20, Eq. (2.3a); note, however, that the corresponding expression has a sign misprint in the exponential]:

$$u(\tau)=\exp\left[\frac{q\tau}{2m}f\right]\mathrm{v}u(0)\mathrm{v}\exp\left[-\frac{q\tau}{2m}f\right]\ . \tag{13}$$

This is the complete solution, written as an automorphism of the initial velocity $u(0)$. Expression (13) is intrinsically Lorentz covariant, and contains a Lorentz boost coupled with a spatial rotation which act on the initial velocity. It is the object of this paper to separate the rotation from the boost—and doing this in a way which does not lose the interaction terms is a mathematically nontrivial matter.[21,22]

## III. THE METHOD OF SOLUTION

The key to obtaining an exact solution to the problem is to Lorentz transform the arbitrary electromagnetic field $f$

to a frame where the electric and magnetic fields are parallel. This is an elementary exercise. One may boost an electromagnetic field $f$ in the $\mathbf{E}\times\mathbf{B}$ direction using a particular boost vector $\mathbf{a}$ to obtain

$$\mathbf{E}'=\mathbf{E}\left[\cosh|\mathbf{a}|-\frac{|\mathbf{B}|^2}{|\mathbf{E}\times\mathbf{B}|}\sinh|\mathbf{a}|\right]+\frac{\mathbf{B}(\mathbf{E}\cdot\mathbf{B})}{|\mathbf{E}\times\mathbf{B}|}\sinh|\mathbf{a}|\ , \tag{14a}$$

$$\mathbf{B}'=\mathbf{B}\left[\cosh|\mathbf{a}|-\frac{|\mathbf{E}|^2}{|\mathbf{E}\times\mathbf{B}|}\sinh|\mathbf{a}|\right]+\frac{\mathbf{E}(\mathbf{E}\cdot\mathbf{B})}{|\mathbf{E}\times\mathbf{B}|}\sinh|\mathbf{a}|\ , \tag{14b}$$

$$\mathbf{a}=\tfrac{1}{2}\operatorname{arctanh}\left[\frac{2|\mathbf{E}\times\mathbf{B}|}{|\mathbf{E}|^2+|\mathbf{B}|^2}\right]\frac{\mathbf{E}\times\mathbf{B}}{|\mathbf{E}\times\mathbf{B}|}\ . \tag{14c}$$

The whole point of performing this transformation is that the separation of the exponential of the primed field $f'$ is trivial, in sharp contrast to the separation of the original (unprimed) field (see Refs. 21 and 22). From this point on, we adopt the normalized proper time $\zeta=q\tau/m$,

$$\begin{aligned}\exp\left[\frac{\zeta}{2}f'\right]&=\exp\left[\frac{\zeta}{2}\mathbf{E}'\mathrm{v}\sigma^4+\frac{\zeta}{2}\eta\mathrm{v}\mathbf{B}'\right]\\&=\exp\left[\frac{\zeta}{2}\mathbf{E}'\mathrm{v}\sigma^4\right]\mathrm{v}\exp\left[\frac{\zeta}{2}\eta\mathrm{v}\mathbf{B}'\right]\\&=\mathbb{L}(-\zeta\mathbf{E}')\mathrm{v}\mathbb{R}(-\zeta\mathbf{B}'),\quad \zeta=q\tau/m\ .\end{aligned} \tag{15}$$

The separation is exact because the two terms on the right-hand side of (15) commute (this follows since $\mathbf{E}'$ is parallel to $\mathbf{B}'$). Now recall a well-known result for the exponential of an element $A$, where $\mathbb{U}$ is any operator.[27,28] This result is usually given for a matrix product, but is equally valid for tensor types in the vee product.

*Theorem 1:*

$$\exp(\mathbb{U}^{-1}\mathrm{v}A\mathrm{v}\mathbb{U})=\mathbb{U}^{-1}\mathrm{v}\exp(A)\mathrm{v}\mathbb{U}\ . \tag{16}$$

Using Theorem 1 gives a decomposition of the exponential of the original electromagnetic field by applying the inverse Lorentz transformation taking $f'$ into $f$,

$$\begin{aligned}&f=\mathbb{L}^{-1}(\mathbf{a})\mathrm{v}f'\mathrm{v}\mathbb{L}(\mathbf{a}),\quad \mathbb{L}(-\mathbf{a})=\mathbb{L}^{-1}(\mathbf{a})\ ,\\&\Longrightarrow\exp\left[\frac{\zeta}{2}f\right]=\mathbb{L}(-\mathbf{a})\mathrm{v}\exp\left[\frac{\zeta}{2}f'\right]\mathrm{v}\mathbb{L}(\mathbf{a})\ .\end{aligned} \tag{17}$$

The relativistic velocity can now be written as a multiple transformation of the initial velocity of the particle $u(0)$, after using (13), (15), (17), and the commutation of the two terms in (15),

$$\begin{aligned}u(\zeta)&=\exp\left[\frac{\zeta}{2}f\right]\mathrm{v}u(0)\mathrm{v}\exp\left[-\frac{\zeta}{2}f\right]\\&=\mathbb{L}(-\mathbf{a})\mathrm{v}\mathbb{L}(-\zeta\mathbf{E}')\mathrm{v}\mathbb{R}(-\zeta\mathbf{B}')\mathrm{v}\mathbb{L}(\mathbf{a})\mathrm{v}u(0)\mathrm{v}\mathbb{L}(-\mathbf{a})\mathrm{v}\mathbb{R}(\zeta\mathbf{B}')\mathrm{v}\mathbb{L}(\zeta\mathbf{E}')\mathrm{v}\mathbb{L}(\mathbf{a})\ .\end{aligned} \tag{18}$$

This expression (18) may be evaluated directly to give a vector form for the relativistic velocity of a particle with arbitrary initial velocity. It is, however, rather tedious to do so explicitly, and so we restrict ourselves to discussing an important special case—the relativistic velocity of a particle starting from rest. The initial velocity is then simply $u(0)=\sigma^4$, and we may utilize the algebraic rules in order to simplify the result. Recall some identities from Ref. 22; the objects $\alpha$ and $\beta$ are any tensor types.

*Theorem 2:*

$$\alpha\mathrm{v}\beta=\beta\mathrm{v}\alpha\Longleftrightarrow\exp(\beta)\mathrm{v}\alpha=\alpha\mathrm{v}\exp(\beta)\ , \tag{19a}$$

$$\alpha\mathrm{v}\beta=-\beta\mathrm{v}\alpha\Longleftrightarrow\exp(\beta)\mathrm{v}\alpha=\alpha\mathrm{v}\exp(-\beta)\ , \tag{19b}$$

$$\exp(\beta)\mathrm{v}\exp(-\beta)=\exp(-\beta)\mathrm{v}\exp(\beta)=1\ . \tag{19c}$$

One should note that $\sigma^4$ commutes with $\eta\mathrm{v}\boldsymbol{\theta}$ in the rotation operator (8) for any vector $\boldsymbol{\theta}$, and that $\sigma^4$ anticommutes with $\mathbf{a}\mathrm{v}\sigma^4$ in the boost operator (9) for any vector $\mathbf{a}$. Therefore, moving $\sigma^4$ to the right of expression (18) by applying Theorem 2 results in

$$u(\zeta)=\mathbb{L}(-\mathbf{a})\mathrm{v}\mathbb{L}(-\zeta\mathbf{E}')\mathrm{v}\mathbb{R}(-\zeta\mathbf{B}')\mathrm{v}\mathbb{L}(2\mathbf{a})\mathrm{v}\mathbb{R}(\zeta\mathbf{B}')\mathrm{v}\mathbb{L}(-\zeta\mathbf{E}')\mathrm{v}\mathbb{L}(-\mathbf{a})\mathrm{v}\sigma^4\ . \tag{20}$$

The core of this expression is a spatial rotation by $\zeta\mathbf{B}'$ of the boost operator $\mathbb{L}(2\mathbf{a})$. This is easily written down from the formulas given in Ref. 22 and by moving another $\sigma^4$ which comes from $\mathbb{L}(2\mathbf{a})$ to the right. Since $\mathbf{a}$ is orthogonal to $\mathbf{B}'$, this rotation is in a plane normal to $\mathbf{E}\times\mathbf{B}$,

$$\mathbb{R}(-\zeta\mathbf{B}')\mathrm{v}\mathbb{L}(2\mathbf{a})\mathrm{v}\mathbb{R}(\zeta\mathbf{B}')=\cosh|\mathbf{a}|-\sinh|\mathbf{a}|(\hat{\mathbf{a}}\cos\zeta B'-\hat{\mathbf{B}}'\times\hat{\mathbf{a}}\sin\zeta B')\mathrm{v}\sigma^4\ . \tag{21}$$

Putting (21) back into (20) and again using the vee algebra rules gives the relativistic velocity as two parts, in which we label $A_1$ and $A_2$ for convenience,

$$u(\zeta)=A_1\cosh|\mathbf{a}|-A_2\sinh|\mathbf{a}| \ , \tag{22a}$$

$$\begin{aligned} A_1&=\mathbb{L}(-\mathbf{a})\mathrm{v}\mathbb{L}(-2\zeta\mathbf{E}')\mathrm{v}\mathbb{L}(-\mathbf{a})\mathrm{v}\sigma^4 \ , \\ A_2&=\mathbb{L}(-\mathbf{a})\mathrm{v}\mathbb{L}(-\zeta\mathbf{E}')\mathrm{v}(\hat{\mathbf{a}}\cos\zeta B'-\hat{\mathbf{B}}'\times\hat{\mathbf{a}}\sin\zeta B') \\ &\qquad \mathrm{v}\mathbb{L}(\zeta\mathbf{E}')\mathrm{v}\mathbb{L}(\mathbf{a}) \ . \end{aligned} \tag{22b}$$

The two terms $A_1$ and $A_2$ can be straightforwardly evaluated using Lorentz transformations and Theorem 2, with the identity $[\mathbf{a}\mathrm{v}\sigma^4,\mathbf{b}]=2(\mathbf{a}\cdot\mathbf{b})\sigma^4$. The results are as follows:

$$\begin{aligned} A_1&=\hat{\mathbf{E}}'\sinh\zeta E'+\hat{\mathbf{a}}\sinh|\mathbf{a}|\cosh\zeta E' \\ &\quad+\sigma^4\cosh|\mathbf{a}|\cosh\zeta E' \ , \\ A_2&=\hat{\mathbf{a}}\cosh|\mathbf{a}|\cos\zeta B'-\hat{\mathbf{B}}'\times\hat{\mathbf{a}}\sin\zeta B' \\ &\quad+\sigma^4\sinh|\mathbf{a}|\cos\zeta B' \ . \end{aligned} \tag{23}$$

The relativistic velocity $u(\zeta)$ (22a) can therefore be written in vector form, using (23) and separating the spatial and time parts:

$$\begin{aligned} \mathbf{u}(\zeta)&=\mathbf{E}'\frac{\sinh\zeta E'}{E'}\cosh|\mathbf{a}| \\ &\quad+\tfrac{1}{2}\hat{\mathbf{a}}(\cosh\zeta E'-\cos\zeta B')\sinh 2|\mathbf{a}| \\ &\quad+\mathbf{B}'\times\hat{\mathbf{a}}\frac{\sin\zeta B'}{B'}\sinh|\mathbf{a}| \ , \end{aligned} \tag{24a}$$

$$\begin{aligned} u^4(\zeta)&=\tfrac{1}{2}\cosh\zeta E'(\cosh 2|\mathbf{a}|+1) \\ &\quad-\tfrac{1}{2}\cos\zeta B'(\cosh 2|\mathbf{a}|-1) \ . \end{aligned} \tag{24b}$$

The final step is to eliminate the direction of the parallel vector fields $\mathbf{E}'$ and $\mathbf{B}'$ from (24a) in order to write the solution in terms of the directions of the original fields $\mathbf{E}$ and $\mathbf{B}$. For this, the following algebraic identities involving the Lorentz invariants $\kappa_1$ and $\kappa_2$, and the duality rotation invariant $\kappa$, are useful:

$$\begin{aligned} \kappa_1&=|\mathbf{E}|^2-|\mathbf{B}|^2, \quad \kappa_2=2(\mathbf{E}\cdot\mathbf{B}) \ , \\ \kappa&=[(\kappa_1)^2+(\kappa_2)^2]^{1/2} \ , \end{aligned} \tag{25a}$$

$$\begin{aligned} \sinh|\mathbf{a}|&=\left[\frac{|\mathbf{E}|^2+|\mathbf{B}|^2-\kappa}{2\kappa}\right]^{1/2} \ , \\ \cosh|\mathbf{a}|&=\left[\frac{|\mathbf{E}|^2+|\mathbf{B}|^2+\kappa}{2\kappa}\right]^{1/2} \ , \\ \sinh 2|\mathbf{a}|&=2\frac{|\mathbf{E}\times\mathbf{B}|}{\kappa}, \quad \cosh 2|\mathbf{a}|=\frac{|\mathbf{E}|^2+|\mathbf{B}|^2}{\kappa} \ . \end{aligned} \tag{25b}$$

Using the identities (25) in (14), one sees that the squares of the primed vector fields are simply

$$\begin{aligned} (E')^2&=|\mathbf{E}'|^2=\frac{\kappa+\kappa_1}{2}\geq 0 \ , \\ (B')^2&=|\mathbf{B}'|^2=\frac{\kappa-\kappa_1}{2}\geq 0 \ , \\ &\Longrightarrow (E')^2+(B')^2=\kappa, \quad (E')^2-(B')^2=\kappa_1 \ , \\ E'B'&=\tfrac{1}{2}\kappa_2 \ . \end{aligned} \tag{26}$$

The vector product which appears in (24a) is rewritten along $\mathbf{E}$ and $\mathbf{B}$ as follows, from (14):

$$\begin{aligned} \mathbf{B}'\times\hat{\mathbf{a}}&=\mathbf{E}\left[\frac{|\mathbf{B}|^2\cosh|\mathbf{a}|}{|\mathbf{E}\times\mathbf{B}|}-\sinh|\mathbf{a}|\right] \\ &\quad-\mathbf{B}\frac{(\mathbf{E}\cdot\mathbf{B})}{|\mathbf{E}\times\mathbf{B}|}\cosh|\mathbf{a}| \ . \end{aligned} \tag{27}$$

Substituting everything back into the solution (24) and simplifying gives the most convenient form for the relativistic velocity,

$$\begin{aligned} \mathbf{u}(\zeta)&=\frac{\mathbf{E}}{\kappa}(E'\sinh\zeta E'+B'\sin\zeta B') \\ &\quad+\frac{\mathbf{B}}{\kappa}(B'\sinh\zeta E'-E'\sin\zeta B') \\ &\quad+\frac{\mathbf{E}\times\mathbf{B}}{\kappa}(\cosh\zeta E'-\cos\zeta B'), \quad \kappa\neq 0 \ , \end{aligned} \tag{28a}$$

$$\begin{aligned} u^4(\zeta)&=\tfrac{1}{2}(\cosh\zeta E'+\cos\zeta B') \\ &\quad+\left[\frac{|\mathbf{E}|^2+|\mathbf{B}|^2}{2\kappa}\right](\cosh\zeta E'-\cos\zeta B') \ , \\ &\qquad \kappa\neq 0 \ . \end{aligned} \tag{28b}$$

The various constants in the solution are given by Eqs. (25a) and (26). The variable $\zeta$ is a normalized proper time $\zeta=q\tau/m$, where $\tau$ is the proper time, $q$ is the charge, and $m$ is the mass.

In the above form, the solution is an infinite series of even powers of $E'$ and $B'$. This is an exact expression for the relativistic velocity of a charged particle in a constant, homogeneous electromagnetic field, starting from rest.

## IV. COMPARISON WITH OTHER WORK

In this section we look at other solutions of this problem and compare them both to our result and to each other. There are important differences arising from the inequivalence of the methods used which have never, to our knowledge, been pointed out. For convenience in comparing the results of Refs. 17–20, we have listed the relevant parameters which figure in the solution in Table I.

One of the standard methods of obtaining the result is presented in Refs. 18 and 19. It is a solution of the $4\times4$ system of linear ordinary differential equations which is the Minkowski force in component form. The four eigenvalues of the electromagnetic field turn out to be exactly the scalar combinations $E'$ and $B'$ defined in the previous section—and this is the point of contact between the various methods,

TABLE I. Different notations for the solution parameters.

| This paper | Taub (Ref. 17) | Hellwig (Ref. 18) | Bacry *et al.* (Ref. 19) | Hestenes (Ref. 20) | Itzykson and Zuber (Ref. 16) |
|---|---|---|---|---|---|
| $\kappa_1=\lvert\mathbf{E}\rvert^2-\lvert\mathbf{B}\rvert^2$ | $a$ | | $-\Delta$ | $\alpha^2-\beta^2$ | $\mathbf{n}^2$ |
| $\kappa_2=2(\mathbf{E}\cdot\mathbf{B})$ | $2b$ | | $\Phi$ | $2\alpha\beta$ | 0 |
| $\kappa=[(\kappa_1)^2+(\kappa_2)^2]^{1/2}$ | $(a^2+4b^2)^{1/2}$ | | $(\Delta^2+\Phi^2)^{1/2}$ | $\lvert z\rvert^2$ | $\mathbf{n}^2$ |
| $E'=\left[\frac{\kappa+\kappa_1}{2}\right]^{1/2}$ | $\nu=\alpha\kappa$ | $\bar{\nu}$ | $\epsilon$ | $\alpha$ | $(\mathbf{n}^2)^{1/2}$ |
| $B'=\left[\frac{\kappa-\kappa_1}{2}\right]^{1/2}$ | $\theta=-\beta\kappa$ | $\bar{\omega}$ | $\beta$ | $\beta,\kappa_2>0$; $-\beta,\kappa_2<0$ | 0 |
| $\zeta E'=\frac{q\tau E'}{m}$ | $\nu\lambda s$ | $\nu\tau$ | $-\mu\tau$ | $\alpha\lambda\tau$ | $2a\tau$ |
| $\zeta B'=\frac{q\tau B'}{m}$ | $\theta\lambda s$ | $\omega\tau$ | $-\omega\tau$ | $\beta\lambda\tau,\kappa_2>0$; $-\beta\lambda\tau,\kappa_2<0$ | 0 |

$$\det(f^{\mu\nu}-\lambda g^{\mu\nu})=0 \Longrightarrow \lambda=\pm E',\pm iB' . \tag{29}$$

The general solution follows as a linear combination of exponentials of the eigenvalues, with parameter $\zeta=q\tau/m$ (Refs. 18 and 19),

$$u(\zeta)=C_1\cosh\zeta E'+C_2\sinh\zeta E' +C_3\cos\zeta B'+C_4\sin\zeta B' . \tag{30}$$

Here $C_1$, $C_2$, $C_3$, and $C_4$ follow from the four eigenvectors of the matrix $f$, which are derivable as combinations of the electric and magnetic field components, and the components of the initial velocity. These are not explicitly evaluated in Refs. 18 and 19. Expression (30) is in complete agreement with our result—the difference being that we obtained the vector components directly from the algebraic formalism.

More in the same line with our method is the method of Ref. 16. There, a smaller Clifford algebra is used, and the explicit solution is written down for the case of orthogonal fields $\mathbf{E}\cdot\mathbf{B}=0$ when a particle starts from rest [Ref. 16, Eq. (1-61)]. There is complete agreement between that result and (28) with $\kappa_2=0$, $\kappa=\kappa_1$, $B'=0$, $E'=\sqrt{\kappa_1}$.

Our solution also agrees with the standard solution for orthogonal fields obtained via a Lorentz transformation to a frame where the electric field vanishes. See, for example, Ref. 29, p. 469, where the solution discussed in Ref. 15 is worked out explicitly. The case of orthogonal fields is also the subject of Ref. 30, where another exact method is developed in order to calculate the instantaneous gyrofrequency.

The solution of Taub[17] is obtained in a slightly different manner from that of other authors.[18,19] He uses an algebraic method similar to ours, although he does not employ the Clifford algebra. The eigenvalues (29) arise purely algebraically from separating the electromagnetic field into two parts. Reference 17 gives an explicit expression for the velocity which is, however, distinct from ours. We recall it here, translated into our language, and using zero initial velocity for simplicity [Ref. 17, Eq. (2.19)]

$$\begin{aligned}\mathbf{u}_{\rm Taub}(\zeta)&=\frac{\mathbf{E}}{\kappa}(E'\sinh\zeta E'+B'\sin\zeta B')\\ &\quad-\frac{i\mathbf{B}}{\kappa}(B'\sinh\zeta E'-E'\sin\zeta B')\\ &\quad+\frac{\mathbf{E}\times\mathbf{B}}{\kappa}(\cosh\zeta E'-\cos\zeta B') ,\\ u_{\rm Taub}{}^4(\zeta)&=\frac{{\kappa_1}^2}{2\kappa^2}(\cosh\zeta E'+\cos\zeta B')+\frac{{\kappa_2}^2}{\kappa^2}\\ &\quad+\left[\frac{\lvert\mathbf{E}\rvert^2+\lvert\mathbf{B}\rvert^2}{2\kappa}\right](\cosh\zeta E'-\cos\zeta B') .\end{aligned} \tag{31}$$

This expression (31) is distinct from ours (28). The discrepancy is due to a serious algebraic error in going from Eq. (2.18) to Eq. (2.19) in Ref. 17. Otherwise, the method of Ref. 17 indeed gives the correct result.

The method followed by Hestenes[20] is extremely close to ours, since he uses the Clifford algebra in four dimensions. The procedure of Ref. 20 is also very similar to that of Ref. 17. There are, however, some crucial sign differences in the equations as printed in Ref. 20, giving a different result. We recall $u(\zeta)$ for a particle starting from rest, from Eq. (2.27) of Ref. 20,

$$\begin{aligned}\mathbf{u}_{\rm Hest}(\zeta)&=\frac{\mathbf{E}}{\kappa}(E'\sinh\zeta E'-B'\sin\zeta B')\\ &\quad+\frac{\mathbf{B}}{\kappa}(B'\sinh\zeta E'+E'\sin\zeta B')\\ &\quad+\frac{\mathbf{E}\times\mathbf{B}}{\kappa}(\cosh\zeta E'-\cos\zeta B') ,\\ u_{\rm Hest}{}^4(\zeta)&=\tfrac{1}{2}(\cosh\zeta E'+\cos\zeta B')\\ &\quad+\left[\frac{\lvert\mathbf{E}\rvert^2+\lvert\mathbf{B}\rvert^2}{2\kappa}\right](\cosh\zeta E'-\cos\zeta B') .\end{aligned} \tag{32}$$

This expression is almost identical to Eq. (28), except for two signs in the $\mathbf{E}$ and $\mathbf{B}$ terms. These signs do not affect the initial conditions, nor the correct reduction to the orthogonal and parallel field cases. Nevertheless, these signs are misprints, as the result of Hestenes should in fact agree entirely with ours.

There remains the comparison of our exact result (28) with the exponential expansion given in Refs. 21 and 22. We expand (28) to fourth order in $\zeta$ to obtain

$$\mathbf{u}(\zeta)\approx\zeta\mathbf{E}+\frac{\zeta^2}{2}\mathbf{E}\times\mathbf{B}+\frac{\zeta^3}{3!}\left[\kappa_1\mathbf{E}+\frac{\kappa_2}{2}\mathbf{B}\right]+\frac{\zeta^4}{4!}\kappa_1\mathbf{E}\times\mathbf{B}\ ,$$
$$u^4(\zeta)\approx 1+\frac{\zeta^2}{2}|\mathbf{E}|^2+\frac{\zeta^4}{4!}(|\mathbf{E}|^4-|\mathbf{E}\times\mathbf{B}|^2)\ . \tag{33}$$

This is precisely the result of Refs. 21 and 22, except that the fourth-order term in $\mathbf{u}(\zeta)$ is new here. Therefore, the exact expression (28) indeed sums the infinite series obtained for the solution in Refs. 21 and 22. This is a confirmation of the general method introduced in Refs. 21 and 22 for calculating the relativistic velocity. A purely heuristic derivation of most of the terms in (33) is discussed in the Appendix, which is independent of any particular method of solution. Expansion of (32) implies acceleration along, instead of around, a magnetic field line.

## V. CONCLUSION

This paper has provided an explicit, mathematically rigorous solution for the velocity of a charged particle in a constant electromagnetic field. The solution is known in general form,[18,19] but the explicit expressions given in the literature[17,20] disagree with each other, and also with our solution. A critical comparison of these results has not, to the best of our knowledge, been made previously. Such a comparison has perhaps been felt unnecessary, since both results, Refs. 17 and 20, correctly reduce to the well-known special case of orthogonal fields. Yet the correct explicit generalization to nonorthogonal fields has remained problematic.

We were able to clarify the physical correspondence of each distinct general result by providing an independent solution. Our approach was logically complementary: On the one hand, we employed the powerful mathematical formalism of the Clifford algebra in order to obtain the explicit solution as a consequence of the Lorentz group. On the other hand, we also give a physically intuitive approximate series solution, in the Appendix. This second result agrees remarkably well with the formal result, even though the derivation is heuristic and highly nonrigorous.

A critical comparison between the results of Refs. 17–22 and our result underlined the agreement with the general case in Refs. 18, 19, 21, and 22 and the special orthogonal case in Refs. 15, 16, 29, and 30, and the disagreement with the distinct general results of Refs. 17 and 20. In conclusion, we have tried to clear up the confusion created by having several distinct published solutions to the same problem, and also to track down errors that may be responsible for these differences.

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## APPENDIX: AN APPROXIMATE ITERATIVE SOLUTION

We include here an iterative approximation, which, even though it is not rigorous, illustrates the physical solution while avoiding entirely the formal aspects of the above methods. We integrate the Lorentz force, first without a magnetic field, then repeatedly with a magnetic field, substituting each time the resulting velocity. For simplicity we treat factors of $\gamma=(1-|\mathbf{V}|^2)^{-1/2}$ as constants in the integration. To first order, assuming the particle starts from rest, one has

$$\frac{d\mathbf{u}}{d\zeta}=\gamma\mathbf{E}\Longrightarrow\mathbf{u}\approx\gamma\mathbf{E}\zeta,\ \ \mathbf{u}=\gamma\mathbf{V},\ \ \zeta=q\tau/m\ . \tag{A1}$$

Substitute $\mathbf{u}$ from (A1) into the full Lorentz force and integrate, again treating $\gamma$ as a constant in the integrand:

$$\frac{d\mathbf{u}}{d\zeta}=\gamma\mathbf{E}+\mathbf{u}\times\mathbf{B}\approx\gamma\mathbf{E}+\gamma\mathbf{E}\times\mathbf{B}\zeta$$
$$\Longrightarrow\mathbf{u}\approx\gamma\mathbf{E}\zeta+\gamma\mathbf{E}\times\mathbf{B}\zeta^2/2\ . \tag{A2}$$

In this way one may obtain higher-order terms. Of course, these terms are of decreasing value because of the approximation, yet they indicate the structure of the solution. Integrate once more:

$$\frac{d\mathbf{u}}{d\zeta}\approx\gamma\mathbf{E}+\gamma[\mathbf{E}\times\mathbf{B}\zeta+(\mathbf{E}\times\mathbf{B})\times\mathbf{B}\zeta^2/2]$$
$$\Longrightarrow\mathbf{u}\approx\gamma\mathbf{E}\zeta+\gamma\mathbf{E}\times\mathbf{B}\zeta^2/2+\gamma(\mathbf{E}\times\mathbf{B})\times\mathbf{B}\zeta^3/3!\ . \tag{A3}$$

A fourth integration gives $\mathbf{u}$ to fourth order, as follows:

$$\mathbf{u}\approx\gamma\{\mathbf{E}\zeta+\mathbf{E}\times\mathbf{B}\zeta^2/2+[(\mathbf{E}\cdot\mathbf{B})\mathbf{B}-|\mathbf{B}|^2\mathbf{E}]\zeta^3/3!$$
$$-|\mathbf{B}|^2\mathbf{E}\times\mathbf{B}\zeta^4/4!\}\ . \tag{A4}$$

The factor of $\gamma$ is handled by taking $\mathbf{V}=\mathbf{u}/\gamma$ from (A4) and calculating $\gamma$ by expanding

$$u^4=\gamma=(1-|\mathbf{V}|^2)^{-1/2}\approx 1+\tfrac{1}{2}|\mathbf{V}|^2+\tfrac{3}{8}|\mathbf{V}|^4$$
$$\approx 1+\frac{\zeta^2}{2}|\mathbf{E}|^2+\frac{\zeta^4}{4!}(9|\mathbf{E}|^4-|\mathbf{E}\times\mathbf{B}|^2)\ . \tag{A5}$$

Now we can substitute $\gamma$ from (A5) into (A4) to obtain the relativistic velocity explicitly:

$$\mathbf{u}\approx\zeta\mathbf{E}+\frac{\zeta^2}{2}\mathbf{E}\times\mathbf{B}+\frac{\zeta^3}{3!}[(3|\mathbf{E}|^2-|\mathbf{B}|^2)\mathbf{E}+(\mathbf{E}\cdot\mathbf{B})\mathbf{B}]$$
$$+\frac{\zeta^4}{4!}(6|\mathbf{E}|^2-|\mathbf{B}|^2)\mathbf{E}\times\mathbf{B}\ . \tag{A6}$$

These expressions (A6) and (A5) are in very good agreement with the expansion of the exact result (33), considering the heuristic nature of the above derivation.

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