Relativistic Doppler-shift effects

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The lack of knowledge of relativistic distribution functions for interacting particles introduces uncertainty into the cosmological interpretation of Doppler shifts, particularly for systems such as quasars. Here we define an average Doppler shift, reducible to the nonrelativistic form in the appropriate limit. However, even for the relativistic ideal situation (Jüttner distribution), second-order effects yield shift corrections after averaging. Such effects should be of interest for nuclear x-ray and γ-ray lines.

I. INTRODUCTION

In the majority of cases involving the Doppler shift in astrophysics, the nonrelativistic formula is sufficient. However, with systems such as quasistellar objects and putative distant galaxies, effects at least of order $(v/c)^2$ are brought into play. While in the nonrelativistic case the shift gives the center-of-mass velocity directly, this is no longer the case relativistically. And, regardless of the magnitude of the center-of-mass velocity in terms of $z=v/c$, it is also important to consider the effect of a relativistic distribution in the "moving" frame of reference.

In regard to the interpretation of the red-shift data there are discordant views.[1] For systems that approach nonrelativistic ideal situations there appears to be no problem in extracting velocities for use in interpretation in Hubble plots.[2,3] For compact, relativistic systems it is not clear what the appropriate distribution functions are, especially in view of the lack of a relativistic statistical mechanics of interacting particles.[4,5] Although kinetic equations that treat $(v/c)^2$ interaction effects are known,[6-8] the solutions to these highly nonlinear equations are unknown. Further, in regard to stability questions, there are Čerenkov poles appearing to this order.[9]

Therefore, in this paper we restrict attention to cases where the Jüttner (relativistic Maxwell-Boltzmann) distribution and the completely degenerate Fermi-Dirac distribution are used. This is sufficient to introduce modifications, which are herein described. That there is current interest in ideal relativistic cases is evidenced, for example, by the work of Karsch and Miller;[10] also, the exotic "fireball" system consisting of quarks is presumed to be asymptotically free, in the limit of small separations. Although interacting relativistic systems are ultimately of interest, it is known that even from the analysis of model systems such as the Einstein-Hopf oscillator there are marked differences between relativistic and nonrelativistic situations.[11,12]

Since broad spectral lines are possibly indicative of relativistic distributions, and since at least some quasars are felt to have such distributions, we have included a calculation of the relativistic linewidth in the paper. Thus a broad line occurring with a large Doppler shift might be diagnostic.

In the present noninteracting case the differences between the relativistic and nonrelativistic examples are intuitively due to the nonlinearity involving particles with velocity components toward and away from the observer. However, in the general, relativistic interacting situation, where, for example, the center-of-mass definition is uncertain due to a lack of knowledge of a relativistic statistical mechanics of interacting particles,[4,5] intuitive reasons may fail us.

II. DOPPLER-SHIFT AVERAGE

For a source moving with $\boldsymbol{\beta}_s=\mathbf{u}/c$ relative to the observer the relativistic Doppler shift is

$$\nu=\frac{(1-\beta_s{}^2)^{1/2}}{1-\boldsymbol{\beta}_s\cdot\hat{\mathbf{k}}}\nu_0\,, \tag{1}$$

where $\hat{\mathbf{k}}$ is a unit vector defining the line of sight from the source to the observer and ν_0 is the rest frequency. If the source is itself a constituent of a larger body whose center-of-mass motion is described by $\boldsymbol{\beta}=\mathbf{v}/c$, and if relative to this system the source has $\boldsymbol{\beta}_0=\mathbf{u}_0/c$, and $\gamma^{-2}=1-\beta^2$, then relative to the observer,

$$\boldsymbol{\beta}_s=\frac{\boldsymbol{\beta}_0+(\gamma-1)(\boldsymbol{\beta}_0\cdot\boldsymbol{\beta})/\beta^2+\gamma\boldsymbol{\beta}}{\gamma(1+\boldsymbol{\beta}_s\cdot\boldsymbol{\beta})}\,. \tag{2}$$

Substituting (2) into (1) leads to

$$\nu=\frac{[(1-\beta_0{}^2)(1-\beta^2)]^{1/2}\nu_0}{1+\boldsymbol{\beta}_0\cdot\boldsymbol{\beta}-\cos\theta_{u\hat{\mathbf{k}}}[(1+\boldsymbol{\beta}_0\cdot\boldsymbol{\beta})^2-(1-\beta_0{}^2)(1-\beta^2)]^{1/2}}\,, \tag{3}$$

where

$$\cos\theta_{u\hat{\mathbf{k}}}=\frac{\mathbf{u}\cdot\hat{\mathbf{k}}}{u}=\frac{\beta_0\cos\theta_{u_0\hat{\mathbf{k}}}+(\gamma-1)\beta_0\cos\theta_{u_0v}\cos\theta_{v\hat{\mathbf{k}}}+\gamma\beta\cos\theta_{v\hat{\mathbf{k}}}}{\gamma[(1+\boldsymbol{\beta}_0\cdot\boldsymbol{\beta})^2-(1-\beta_0{}^2)(1-\beta^2)]^{1/2}}\,, \tag{4}$$

and $\theta_{u_0\hat{\mathbf{k}}}$ indicates, for example, the angle between u_0 and $\hat{\mathbf{k}}$, etc. Assuming that the cases of interest here correspond to motion such that v is parallel or antiparallel to the line of sight, then

$$\nu=\frac{[(1-\beta^2)(1-\beta_0^2)]^{1/2}\nu_0}{1+\boldsymbol{\beta}_0\cdot\boldsymbol{\beta}\mp\gamma^{-1}[\boldsymbol{\beta}_0\cdot\boldsymbol{\beta}+(\gamma-1)(\boldsymbol{\beta}_0\cdot\boldsymbol{\beta}+\gamma\beta)]} . \quad (5)$$

Defining $a=1\pm\beta$, $b=\beta_0(1\pm\beta)$, the angle averaged result

$$\langle\nu\rangle=\frac{[(1-\beta_0^2)(1-\beta^2)]^{1/2}}{2b}\ln\left[\frac{a+b}{a-b}\right]\nu_0 \quad (6)$$

follows. The limit

$$\lim_{\beta_0\to 0}\langle\nu\rangle=\left[\frac{1\pm\beta}{1\mp\beta}\right]^{1/2}\nu_0=\nu_{b,r} \quad (7)$$

retrieves the red- and blue-shift forms of (1), as required. Since $\beta_0=u_0/c\equiv cp/E$, (6) may be written as

$$\langle\nu\rangle=\frac{(1-\beta_0^2)^{1/2}}{2\beta_0}\left[\frac{1\pm\beta}{1\mp\beta}\right]^{1/2}\ln\left[\frac{1+\beta_0}{1-\beta_0}\right]\nu_0$$
$$=\frac{m_0c}{2p}\ln\left[\frac{[1+(p/m_0c)^2]^{1/2}+p/m_0c}{[1+(p/m_0c)^2]^{1/2}-p/m_0c}\right]\nu_{b,r} , \quad (8)$$

where m_0 is the source rest mass. Next we must average over the momentum magnitude. Since the phase average is

$$\langle\bar{\nu}\rangle=\int \nu f(p)d^3p\,d^3x\Big/\int f(p)d^3p\,d^3x , \quad (9)$$

we may continue with the averaging over the momentum magnitude, where $f(p)$ is given by the Jüttner distribution

$$f(p)=K\exp\{-\alpha[1+(p/m_0c)^2]^{1/2}\} , \quad (10)$$

and where $\alpha=m_0c^2/kT$; defining $\sinh\Theta=p/m_0c$ and normalizing,

$$\langle\bar{\nu}\rangle=\frac{\alpha}{(m_0c)^3K_2(\alpha)}\times\int_0^\infty\langle\nu\rangle\exp\{-\alpha[1+(p/m_0c)^2]^{1/2}\}p^2dp$$
$$=\frac{\alpha}{2K_2(\alpha)}\int_0^\infty\ln\left[\frac{\cosh\Theta+\sinh\Theta}{\cosh\Theta-\sinh\Theta}\right]\times e^{-\alpha\cosh\Theta}\sinh\Theta\cosh\Theta\,d\Theta\,\nu_{b,r}$$
$$=\frac{\alpha}{K_2(\alpha)}\int_0^\infty\Theta e^{-\alpha\cosh\Theta}\sinh\Theta\cosh\Theta\,d\Theta\,\nu_{b,r}$$
$$=\frac{1}{\alpha K_2(\alpha)}[\alpha K_1(\alpha)+K_0(\alpha)]\nu_{b,r} . \quad (11)$$

In the expressions above, $K_n(\alpha)$ is a modified Bessel function. In the high-temperature regime for $\alpha\ll 1$, it is apparent that

$$\langle\bar{\nu}\rangle/\nu_{b,r}\to\frac{\alpha}{2}+\frac{\alpha}{2}\ln\frac{2}{\alpha} . \quad (12)$$

This follows since $K_n(\alpha)\sim\frac{1}{2}(2/\alpha)^n$, $K_0(\alpha)\sim\ln(2/\alpha)$ for $\alpha\ll 1$.

The nondegenerate case has $K_n(\alpha)\sim(\pi/2)^{1/2}e^{-\alpha}/\alpha^{1/2}$, for $\alpha\gg 1$. Thus we have the first-order result [note that α^{-1} corresponds in order to $(v/c)^2$]

$$\langle\bar{\nu}\rangle\sim(1+\alpha^{-1})\nu_{b,r} . \quad (13)$$

So, in the event that $kT\ll m_0c^2$ and $v/c\ll 1$, the usual result is obtained, namely, the system has a Doppler effect given by $\nu=\nu_0(1\pm z)$, where $z=v/c$ [Eq. (13) also is written explicitly to show effects of order α^{-1}]. On the other hand, Eq. (12) indicates that if $kT\gg m_0c^2$, the Doppler shift is altered, with $\langle\bar{\nu}\rangle\sim(\alpha/2)\ln(2/\alpha)\nu_{b,r}$ where we have used the relativistic form (7).

Looking at the completely degenerate state for the Fermi-Dirac case, replace (11) by

$$\frac{\langle\bar{\nu}\rangle_{T=0}}{\nu_{b,r}}=\frac{3}{\sinh^3\Theta_F}\int_0^{\Theta_F}\Theta\sinh\Theta\cosh\Theta\,d\Theta$$
$$=\frac{3}{4\sinh^3\Theta_F}(2\Theta_F\cosh 2\Theta_F-\sinh 2\Theta_F) , \quad (14)$$

where Θ_F defines the Fermi-limit parameter. In (14) the normalization factor follows at $T=0$ since

$$\int_0^{\Theta_F}\sinh^2\Theta\cosh\Theta\,d\Theta=\sinh^3\Theta_F/3 . \quad (15)$$

Note that $\sinh\Theta_F=p_F/m_0c=(h/m_0c)(3n/8\pi)^{1/3}$, and $\sinh\Theta_F$ is of order unity for hydrogen with $n\sim 10^{39}$ cm^{-3} density. The criterion for applying the completely degenerate case requires that the Fermi temperature is much greater than stellar temperatures. Thus, typically, the argument may be applied to white dwarf stars, neutron stars, or even denser media. Of course, the treatment assumes the ideal relativistic case and so, as discussed earlier, the lack of a quantum or classical relativistic statistical mechanics of interacting particles may severely delimit the application. In addition the assumption of the general form (1) does not necessarily lead uniquely to relativistic transformations for the thermodynamic functions.[13–15] These general questions are not considered here, however.

III. LINEWIDTH CALCULATION

Further, we investigate the relativistic implications for the linewidth. Defining the function $g(\nu)$, the number of atoms emitting radiation in the center-of-mass frame in a spectral range is

$$\int_{\nu_0(1+\beta\cos\theta_0)/(1-\beta^2)^{1/2}}^{\nu_0}g(\nu)d\nu=\frac{\alpha}{2K_2(\alpha)}\int_0^{\Theta(\beta)}e^{-\alpha\cosh\Theta}\sinh^2\Theta\cosh\Theta\,d\Theta , \quad (16)$$

where we have used (10) and a velocity "cone" is defined (see the analogous nonrelativistic derivation in Gill[16]). Note that, as in (1), the transformation

$$\cos\theta_0=\frac{\cos\theta-\beta}{1-\beta\cos\theta} \quad (17)$$

holds, where θ is the angle between the line of sight and velocity in the observer frame. Note also that the relativistic situation requires that transverse as well as line-of-sight motion contributes to the Doppler shift. [Recalling the classic Ives-Stillwell[17] experiment in which transverse $(v/c)^2$ effects first were measured, there were problems with the diffuseness of lines, which were overcome by experimental design. In the event of a relativistic astrophysical distribution, discrete line effects may be masked by overlapping relativistic line broadening.]

Since $\cosh\Theta = 1/(1-\beta_0{}^2)^{1/2}$ in (16), and differentiating[16] with respect to β, we get

$$g\left[\nu_0\left[\frac{1+\beta\cos\theta_0}{(1-\beta^2)^{1/2}}\right]\right]=\frac{\alpha\beta^2}{2\nu_0\cos\theta_0 K_2(\alpha)}e^{-\alpha/(1-\beta^2)^{1/2}} . \tag{18}$$

Solving for β in terms of ν, using (17),

$$\beta=\frac{-\nu_0{}^2\cos\theta_0-\nu^2[1-(\nu_0/\nu)^2\sin^2\theta_0]^{1/2}}{\nu^2+\nu_0{}^2\cos^2\theta_0} ,$$

also, then

$$\frac{1}{(1-\beta^2)^{1/2}}=\frac{R+R^{-1}\cos^2\theta_0}{[1+\cos^2\theta_0-2\cos\theta_0(1-R^{-1}\sin^2\theta_0)^{1/2}]^{1/2}} ,$$

where $R=(\nu/\nu_0)^2$. From (18) it then follows that

$$g(\nu)=\frac{-\alpha}{2\nu_0 K_2(\alpha)\cos\theta_0}\left[\frac{R(1-R^{-1}\sin^2\theta_0)^{1/2}+\cos\theta_0}{R^2+\cos^2\theta_0}\right]^2\exp\left[-\frac{(R+R^{-1}\cos^2\theta_0)}{[1+\cos^2\theta_0-2\cos\theta_0(1-R^{-1}\sin^2\theta_0)^{1/2}]^{1/2}}\right] \tag{19}$$

and for $\theta=\theta_0=\pi$,

$$g(\nu)=\frac{\alpha}{2\nu_0 K_2(\alpha)}\left[\frac{\nu^2-\nu_0{}^2}{\nu^2+\nu_0{}^2}\right]\exp\left[-\frac{\alpha(\nu^2+\nu_0{}^2)}{2\nu\nu_0}\right] . \tag{20}$$

In the case for which $\alpha \gg 1$ (nonrelativistic limit)

$$g(\nu)=\frac{\alpha^{3/2}}{2\nu_0}\left[\frac{2}{\pi}\right]^{1/2}e^{\alpha}\left[\frac{\nu^2-\nu_0{}^2}{\nu^2+\nu_0{}^2}\right]^2 e^{-\alpha(\nu-\nu_a)^2/2\nu\nu_0}e^{-\alpha}$$

$$\simeq\frac{2\pi c}{\nu_0}\left[\frac{m_0}{2\pi kT}\right]^{3/2}v^2 e^{-m_0v^2/kT} , \tag{21}$$

where $\nu=\nu_0(1+v/c)$ has been used.

Then, for the longitudinal example, the exponential in (20) leads to a half-width given by

$$\frac{\Delta\nu}{\nu_0}=\frac{2\ln 2}{\alpha}\left[1+\left[1+\frac{2\alpha}{\ln 2}\right]^{1/2}\right] . \tag{22}$$

For the limiting cases

$$\frac{\Delta\nu}{\nu_0}=\frac{2^{3/2}(\ln 2)^{1/2}}{\alpha^{1/2}} , \quad \alpha \gg 1 , \tag{23a}$$

$$\frac{\Delta\nu}{\nu_0}=2+\frac{4\ln 2}{\alpha} , \quad \alpha \ll 1 . \tag{23b}$$

These results indicate a narrowing of the Doppler width in the nonrelativistic case and a broadening tendency in the extreme relativistic case. Of course, the broadening is dictated, in the general case, by the exponential in (19). Other pressure broadening effects are not addressed here; they may be more important in extremely condensed matter. The expressions above assume that the Doppler broadening predominates.

If it is true, as Bahcall[18] points out, that quasar absorption lines that are broad originate in the quasar material, then (22) may lead to non-negligible corrections to (7). Recalling the opinion that "not all red-shifts are of cosmological origin,"[19] it may be that better knowledge of the relativistic distribution (interacting particles) can help explain the discrepant effects.

CONCLUSION

Among the consequences of these relations it is clear that, within the limitations outlined, a broad line, with typical width[20] given by (22), might not have its correct relativistic shift of origin completely reflected by use of (7), since, according to (11), there is a modification of the longitudinal shift. For quasirelativistic situations, the narrower width would be compared with the shift formula (13), where a correction of order α^{-1} $(\alpha \gg 1)$ would follow. In addition a completely degenerate system would have a corrected shift given by (14);[21] for the width it is possible that (22) may not be applicable in extremely condensed matter.

In the general case given by (11), it is clear that the Doppler shift will be modified to the extent that the longitudinal form (7) no longer can be expected to strictly hold. This illustrates the potential importance of the distribution function in changing the Doppler-shift expressions.

Nuclear spectral lines have apparently not yet been observed to any extent—but it follows that the relativistic distributions with temperatures exceeding 10^{13} K could be important here. Also, as noted earlier, although the Jüttner distribution is an idealized case, it may be (somewhat paradoxically) important in an extremely dense quark gas.

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[21]In the limit as the Fermi parameter goes to zero, the correct nonrelativistic limit is obtained as $\langle \bar{\nu} \rangle_{T=0} \rightarrow \nu$, where ν is the usual nonrelativistic form for the Doppler shift.