Higher-order corrections to the laws of motion and precession for black holes and other bodies

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In a recent paper, Thorne and Hartle have derived laws of motion and precession for black holes and other bodies. Using two different methods, higher-order corrections to those laws are derived here: a time change of the body's mass-energy due to coupling of the time derivatives of the body's quadrupole moments to the external curvature; a force due to coupling of the time derivatives of the body's quadrupole moments to the external curvature and coupling of the body's quadrupole moments to the gradient of the external curvature; and a torque due to coupling of the body's octopole moments to the gradient of the external curvature.

I. INTRODUCTION AND SUMMARY

Recently, Thorne and Hartle[1] have derived laws of motion and precession for black holes and other bodies. Their analysis characterizes the body of interest by three parameters: $M=$(mass), $L=$(size), and $T=$(time scale for changes of multipole moments); and it characterizes the external universe, through which the body moves, by three other parameters: $\mathscr{R}=$(radius of curvature along body's world line), $\mathscr{L}=$(inhomogeneity scale of curvature), and $\mathscr{T}=$(time scale for changes in curvature). Their analysis relies on the approximations that the body's moments change slowly and the body is well isolated:

$$T \gg L \gtrsim M\ , \quad \mathscr{R} \gg L\ , \quad \mathscr{L} \gg L\ , \quad \mathscr{T} \gg L\ , \tag{1}$$

and on the assumption that there is negligible gravitating matter in a "buffer region" $L \ll r \ll \mathscr{L}$ surrounding the body. The external universe's curvature in the buffer region is then nearly constant and satisfies the vacuum Einstein equation. Thorne and Hartle set up in this buffer region a coordinate system which is as nearly Lorentz as the spacetime curvature permits, and in which the body is at rest at time $t=0$; and in this "local asymptotic rest frame of the body" they compute the effects of the external universe on the body's motion.

The resulting Thorne-Hartle laws of motion and precession, written in terms of components in the body's local asymptotic rest frame, take the following form to leading order in the small dimensionless parameters L/T, $L/\mathscr{R}$, $L/\mathscr{L}$, and $L/\mathscr{T}$, and a similar set of parameters with L replaced by M:

$$\frac{dM}{dt} \ll \frac{ML}{\mathscr{R}^2}\ , \tag{2a}$$

$$\frac{dP^i}{dt} = -\mathscr{S}^a \mathscr{B}_a{}^i = O\left[\frac{ML}{\mathscr{R}^2}\right]\ , \tag{2b}$$

$$\frac{d\mathscr{S}^i}{dt} = -\epsilon^i{}_{ab}\mathscr{I}^a{}_c\mathscr{E}^{cb} - \tfrac{4}{3}\epsilon^i{}_{ab}\mathscr{S}^a{}_c\mathscr{B}^{bc} = O\left[\frac{ML^2}{\mathscr{R}^2}\right] \tag{2c}$$

[Eqs. (1.9) of Thorne and Hartle,[1] denoted henceforth as Eqs. (TH,1.9)]. Here P^j, $\mathscr{S}^j$, $\mathscr{I}^{jk}$, and $\mathscr{S}^{jk}$ are the body's momentum, spin, and mass and current quadrupole moments; $\epsilon^i{}_{ab}$ is the flat-space Levi-Cività tensor (used to form vector cross products); and $\mathscr{E}^{jk}$ and $\mathscr{B}^{jk}$ are the electric and magnetic parts of the Riemann curvature tensor of the external universe. Summation over repeated indices (which are always spatial, $i=1,2,3$) is assumed; and indices on the multipole moments are raised and lowered with the three-dimensional flat metric (Kronecker delta). Thorne and Hartle also discuss at some length the procedure for converting these laws of motion into equations of motion for any given situation satisfying Eqs. (1).

The purpose of this paper is to derive all corrections to the laws of motion and precession (2) with magnitudes

$$\left[\frac{dM}{dt}\right]_{\rm corr} \gtrsim \frac{ML^2}{\mathscr{R}^2(\mathscr{L} \text{ or } \mathscr{T} \text{ or } T)}\ ,$$
$$\left[\frac{dP^i}{dt}\right]_{\rm corr} \gtrsim \frac{ML^2}{\mathscr{R}^2(\mathscr{L} \text{ or } \mathscr{T} \text{ or } T)}\ , \tag{3}$$
$$\left[\frac{d\mathscr{S}^i}{dt}\right]_{\rm corr} \gtrsim \frac{ML^3}{\mathscr{R}^2(\mathscr{L} \text{ or } \mathscr{T} \text{ or } T)}\ .$$

Those corrections turn out to be

$$\left[\frac{dM}{dt}\right]_{\rm corr} = -\tfrac{1}{2}\mathscr{E}_{ab}\dot{\mathscr{I}}^{ab} - \tfrac{2}{3}\mathscr{B}_{ab}\dot{\mathscr{S}}^{ab}\ , \tag{4a}$$

$$\left[\frac{dP^i}{dt}\right]_{\rm corr} = -\tfrac{1}{2}\mathscr{E}^i{}_{ab}\mathscr{I}^{ab} - \tfrac{8}{9}\mathscr{B}^i{}_{ab}\mathscr{S}^{ab} + \tfrac{4}{9}\epsilon^i{}_{ab}\dot{\mathscr{E}}^a{}_c\mathscr{S}^{bc} + \tfrac{1}{3}\epsilon^i{}_{ab}\mathscr{B}^a{}_c\dot{\mathscr{I}}^{bc}\ , \tag{4b}$$

$$\left[\frac{d\mathscr{S}^i}{dt}\right]_{\rm corr} = \tfrac{1}{2}\epsilon^i{}_{ab}\mathscr{E}^{acd}\mathscr{I}^b{}_{cd} + \epsilon^i{}_{ab}\mathscr{B}^{acd}\mathscr{S}^b{}_{cd}\ . \tag{4c}$$

Here overdots denote time derivatives. $\dot{\mathscr{I}}^{ab} \equiv d\mathscr{I}^{ab}/dt$; $\mathscr{I}_{abc}$ and $\mathscr{S}_{abc}$ are the body's mass octopole and current octopole moments;[2] and $\mathscr{E}_{abc}$ and $\mathscr{B}_{abc}$ are the electric-type and magnetic-type octopole moments of the external universe's curvature, i.e., they characterize the spatial gradient of that curvature.[1,3] Thorne and Hartle[1] discuss, but do not derive, the most important of the above corrections [both terms in (4a), which are the leading nonzero contributions to dM/dt; and the first term in (4b), which in many realistic situations will dominate over the lower-order term (2b) and thus will be the leading contribution to dP^i/dt].

The interpretation of the laws of motion (2) and (4) involves a number of subtleties which are discussed by Thorne and Hartle. One of these subtleties is crucial for our derivation, so we review it here: The nonlinear interaction of the body's curvature and the external curvature cause the body's mass M, momentum P^i, and spin $\mathcal{S}^i$ to be slightly ambiguous, i.e., to be uncertain by amounts

$$\Delta M \sim ML^2/\mathcal{R}^2 , \quad \Delta P^i \sim ML^2/\mathcal{R}^2 , \quad \Delta \mathcal{S}^i \sim M^3 L/\mathcal{R}^2 \tag{5}$$

[Eq. (TH,1.8)]. This means that we can obtain physically meaningful changes of M, P^i, and $\mathcal{S}^i$ by integrating the laws of motion (2) and (4) only if we integrate over a time long enough that the changes exceed the uncertainties (5). Correspondingly, dM/dt, dP^i/dt, and $d\mathcal{S}^i/dt$ are actually ambiguous. Any set of formulas that give the same time-integrated changes to within the uncertainties of Eq. (5) are just as good as Eqs. (2) and (4). This means, in particular, that one can add to dM/dt [Eqs. (2a) and (4a)] any multiple of $(d/dt)(\mathcal{E}_{ab}\mathcal{I}^{ab})$ or $(d/dt)(\mathcal{B}_{ab}\mathcal{S}^{ab})$, and can add to dP^i/dt [Eqs. (2b) and (4b)] any multiple of $(d/dt)(\epsilon^i{}_{ab}\mathcal{E}^a{}_c\mathcal{S}^{bc})$ or $(d/dt)(\epsilon^i{}_{ab}\mathcal{B}^a{}_c\mathcal{I}^{bc})$—or equivalently, one can replace the terms in Eqs. (4a) and (4b) involving time derivatives by averages, $\langle\ \rangle$, of those terms over a few internal time scales, $(\text{few})\times T$ [cf. Eq. (TH,1.15)].

The remainder of this paper consists of two derivations of the corrections (4) to the laws of motion and precession. The first derivation, based on the techniques of Thorne and Hartle,[1] is given in Sec. II. This derivation is valid for any body, including a black hole, that satisfies the constraints of Eqs. (1). The second derivation, given in Sec. III, is restricted to the special case of an arbitrary body with absolutely negligible self-gravity. It serves as a check of the first derivation and provides additional insight into the physical origins of the energy changes (4a), forces (4b), and torques (4c).

II. DERIVATION FOR AN ARBITRARY BODY

We begin our derivation from the standard formulas

$$\dot{M} = -\oint (-g)t^{0j}d^2S_j , \tag{6a}$$

$$\dot{P}^i = -\oint (-g)t^{ij}d^2S_j , \tag{6b}$$

$$\dot{\mathcal{S}}^i = -\oint (-g)\epsilon^i{}_{jk}x^j t^{kl}d^2S_l , \tag{6c}$$

where $t^{\mu\nu}$ is the Landau-Lifshitz pseudotensor and the surface integral is over a closed two-surface in the buffer region. (See Thorne and Hartle[1] for a discussion of the applicability of these formulas to this problem.) In these formulas, one could equally well use some other energy-momentum pseudotensor, provided it participates in conservation laws of the type discussed in Sec. 20.3 of MTW.[4] In that case, the time changes could be different from those in Eqs. (4), but the differences could only be as great as the uncertainties discussed in Sec. I.

Since we are only interested in the nonlinear couplings between the external universe and the body moving through it, the metric we will use is the sum of that for a single body and that for the external universe, with the spatial origin attached at time $t=0$ to the world line of the center of the body. To further simplify the calculation, we notice that only those coupling terms with the same temporal and spatial transformation properties as the left-hand side of Eqs. (6) can appear and survive the surface integral. Take $\dot{M}$ as an example. Under time reversal, $\dot{M}\to-\dot{M}$; under a spatial reflection, $\dot{M}\to\dot{M}$. At the order $ML^2/\mathcal{R}^2T$ and $ML^2/\mathcal{R}^2\mathcal{T}$ [no scalar can be constructed if T (or $\mathcal{T}$) is replaced by $\mathcal{L}$ or $\mathcal{R}$], the transformation properties of the only possible terms are

	$\dot{M}$	$\mathcal{E}_{ab}\dot{\mathcal{I}}^{ab}$ or $\dot{\mathcal{E}}_{ab}\mathcal{I}^{ab}$	$\mathcal{B}_{ab}\dot{\mathcal{S}}^{ab}$ or $\dot{\mathcal{B}}_{ab}\mathcal{S}^{ab}$	$\mathcal{B}_{ab}\dot{\mathcal{I}}^{ab}$ or $\dot{\mathcal{B}}_{ab}\mathcal{I}^{ab}$	$\mathcal{E}_{ab}\dot{\mathcal{S}}^{ab}$ or $\dot{\mathcal{E}}_{ab}\mathcal{S}^{ab}$
time reversal	$-$	$-$	$-$	$+$	$+$
spatial reflection	$+$	$+$	$+$	$-$	$-$

Therefore only $\mathcal{E}_{ab}\dot{\mathcal{I}}^{ab}$, $\dot{\mathcal{E}}_{ab}\mathcal{I}^{ab}$, $\mathcal{B}_{ab}\dot{\mathcal{S}}^{ab}$, and $\dot{\mathcal{B}}_{ab}\mathcal{S}^{ab}$ will appear in the final answer (for a more detailed discussion, see Thorne and Hartle,[1] Sec. III E):

$$\begin{aligned}\dot{M} &= \mu_1\mathcal{E}_{ab}\dot{\mathcal{I}}^{ab} + \mu_2\dot{\mathcal{E}}_{ab}\mathcal{I}^{ab} + \mu_3\mathcal{B}_{ab}\dot{\mathcal{S}}^{ab} + \mu_4\dot{\mathcal{B}}_{ab}\mathcal{S}^{ab} \\ &= (\mu_1-\mu_2)\mathcal{E}_{ab}\dot{\mathcal{I}}^{ab} + (\mu_3-\mu_4)\mathcal{B}_{ab}\dot{\mathcal{S}}^{ab} + (\text{negligible total time derivatives}) .\end{aligned} \tag{7a}$$

Similarly,

$$\dot{P}^i = \mu_5\mathcal{E}^i{}_{ab}\mathcal{I}^{ab} + \mu_6\mathcal{B}^i{}_{ab}\mathcal{S}^{ab} + (\mu_7-\mu_8)\epsilon^i{}_{ab}\mathcal{E}^a{}_c\dot{\mathcal{S}}^{bc} + (\mu_9-\mu_{10})\epsilon^i{}_{ab}\mathcal{B}^a{}_c\dot{\mathcal{I}}^{bc} , \tag{7b}$$

$$\dot{\mathcal{S}}^i = \mu_{11}\epsilon^i{}_{ab}\mathcal{E}^{acd}\mathcal{I}^b{}_{cd} + \mu_{12}\epsilon^i{}_{ab}\mathcal{B}^{acd}\mathcal{S}^b{}_{cd} . \tag{7c}$$

Our task now reduces to the calculation of these μ's. The linearized, truncated, and trace-reversed metric perturbation needed for this calculation is (see Thorne[2] for the metric of the central body; see Zhang[3] for the metric of the external universe)

$$\bar{h}^{00} \equiv -4\psi = 6\frac{\mathscr{I}_{ab}x^a x^b}{r^5} + 10\frac{\mathscr{I}_{abc}x^a x^b x^c}{r^7} - 2\mathscr{E}_{ab}x^a x^b - \tfrac{2}{3}\mathscr{E}_{abc}x^a x^b x^c , \tag{8a}$$

$$\bar{h}^{0j} \equiv -A^j = 4\frac{\epsilon^j{}_{pq}x^q \mathscr{S}^p{}_l x^l}{r^5} + \frac{15}{2}\frac{\epsilon^j{}_{pq}x^q \mathscr{S}^p{}_{bc}x^b x^c}{r^7} + 2\frac{\dot{\mathscr{I}}^j{}_a x^a}{r^3}$$
$$+ \tfrac{2}{3}\epsilon^j{}_{pq}x^q \mathscr{B}^p{}_l x^l + \tfrac{1}{3}\epsilon^j{}_{pq}x^q \mathscr{B}^p{}_{bc}x^b x^c + \tfrac{10}{21}(\dot{\mathscr{E}}_{ab}x^a x^b x^j - \tfrac{2}{5}r^2 \dot{\mathscr{E}}^j{}_a x^a) , \tag{8b}$$

$$\bar{h}^{ij} = \frac{8}{3}\frac{\epsilon_{pq}{}^{(i}\dot{\mathscr{S}}^{j)p}x^q}{r^3} + \tfrac{5}{21}(x^{(i}\epsilon^{j)}{}_{pq}\dot{\mathscr{B}}^q{}_l x^p x^l - \tfrac{1}{5}r^2 \epsilon_{pq}{}^{(i}\dot{\mathscr{B}}^{j)q}x^p) . \tag{8c}$$

Inserting Eqs. (8) into MTW (Ref. 4) Eq. (20.22) with $\tilde{g}^{\mu\nu} = \eta^{\mu\nu} - \bar{h}^{\mu\nu}$, and keeping only those terms that will contribute to the final answer, we obtain

$$16\pi(-g)t^{0k} = -A^k{}_{,j}\dot{A}^j - 12\dot{\psi}g^k + 4\epsilon^k{}_{ab}H^a g^b$$
$$- 2A_{a,b}\bar{h}^{a[k,b]} , \tag{9a}$$

$$16\pi(-g)t^{ij} = 4g^i g^j + H^i H^j - 8\dot{A}^{(i}g^{j)}$$
$$+ \tfrac{1}{2}\delta^{ij}(8\mathbf{g}\cdot\dot{\mathbf{A}} - 4\mathbf{g}^2 - \mathbf{H}^2) , \tag{9b}$$

where $\mathbf{g}$ is the gravitational acceleration and $\mathbf{H}$ is the gravitational analog of the magnetic field (three-dimensional notation is used here):

$$\mathbf{g} = -\nabla\psi , \quad \mathbf{H} = \nabla\times\mathbf{A} . \tag{10}$$

In Eq. (9b) the $\dot{\mathbf{A}}$ terms are needed only for the calculation of $\dot{P}^i$.

Now we can insert the necessary parts of Eqs. (8) into Eqs. (9a) and (9b) separately, and perform the surface integrations in Eqs. (6). The results, after dropping negligible total time derivative terms, are Eqs. (4) above. Because in this derivation we only need the metric in the buffer region where it is linearized, the results are valid for any body, including a black hole, provided that the conditions of Eq. (1) are satisfied.

III. DERIVATION FOR A TEST BODY

In this section, another derivation of Eqs. (4) will be presented. The basic idea is local energy-momentum conservation. The purpose of doing this is twofold. First, it can serve as a check of the previous derivation. Second, in the derivation of Sec. II the physical meanings were almost buried in the complicated, though straightforward, algebra; and the physics may be much clearer in this calculation.

Our derivation will be restricted to a body that has totally negligible self-gravity and is made of "normal" material, for which $|T^{ab}| \ll T^{00}$ in the body's center-of-mass frame. The mass, momentum, and spin of this body are

$$M = \int T^{00}(-g)d^3x , \tag{11a}$$

$$P^i = \int T^{0i}(-g)d^3x , \tag{11b}$$

$$\mathscr{S}^i = \int \epsilon^i{}_{pq}x^p T^{0q}(-g)d^3x , \tag{11c}$$

where $T^{\mu\nu}$ is the energy-momentum tensor for the body alone, and g is the determinant of the metric for the external universe alone. The factor $(-g)$ could equally well be $(-g)^n$ for any n of order unity; by changing the power of $(-g)$ one changes M, P^i, and $\mathscr{S}^i$ by amounts of order their uncertainties [Eqs. (5)]. As in Sec. II our calculation is performed in a coordinate system that is as nearly Lorentz and mass centered as possible, and in which the body is momentarily (at $t=0$) at rest: $P^i=0$ and $\mathscr{I}^i = $ (mass dipole moment) $=0$ at $t=0$.

By taking the time derivatives of Eqs. (11) and using the fact that $T^{\mu\nu}=0$ outside the body, we obtain

$$\dot{M} = \int [T^{0\nu}(-g)]_{,\nu}d^3x , \quad \dot{P}^i = \int [T^{i\nu}(-g)]_{,\nu}d^3x ,$$
$$\dot{\mathscr{S}}^i = \int \epsilon^i{}_{pq}x^p[T^{\nu q}(-g)]_{,\nu}d^3x .$$

By combining with the local law of energy-momentum conservation

$$T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}(T^{\mu\nu}\sqrt{-g})_{,\nu} + \Gamma^\mu_{\nu\alpha}T^{\nu\alpha} = 0$$

or equivalently,

$$[T^{\mu\nu}(-g)]_{,\nu} = -\Gamma^\mu_{\alpha\beta}T^{\alpha\beta} + 2(\psi T^{\mu\alpha})_{,\alpha}$$

and removing negligible time derivatives, vanishing surface integrals, and other terms that are negligible, we bring these equations into the form

$$\dot{M} = \int (g_a - \dot{A}_a)T^{0a}d^3x , \tag{12a}$$

$$\dot{P}^i = \int [(g^i - \dot{A}^i)T^{00} + \epsilon^i{}_{ab}T^{0a}H^b + g^i T^a{}_a]d^3x , \tag{12b}$$

$$\dot{\mathscr{S}}^i = \int \epsilon^i{}_{ab}x^a[(g^b - \dot{A}^b)T^{00} + \epsilon^b{}_{pq}T^{0p}H^q]d^3x . \tag{12c}$$

The $g^i T^a{}_a$ term in Eq. (12b) is not discarded because the $g^i T^{00}$ integral, which one might have thought to be dominant, gives as its formally leading piece $\mathscr{E}^i{}_a\mathscr{I}^a$, which vanishes in our mass-centered coordinates ($\mathscr{I}^a=0$).

The physical meanings are rather clear here. Take $\dot{M}$

as an example. In the Newtonian limit, $T^{0i}=\rho v^i$, Eq. (12a) reduces to

$$\dot{M}=\int (g_a-\dot{A}_a)\rho v^a d^3x=\int F_a v^a d^3x \ ,$$

where $F_a=\rho(g_a-\dot{A}_a)$ is the gravitational force per unit volume exerted on the body by the external universe. Thus in the Newtonian limit, the total mass-energy of an object changes because the external universe does "$\mathbf{F}\cdot\mathbf{v}$" work on various parts of it. This is just what we would naively expect. We can understand $\dot{P}^i$ and $\dot{\mathscr{S}}^i$ in a similar fashion.

In order to express the $\dot{M}$, $\dot{P}^i$, and $\dot{\mathscr{S}}^i$ of Eqs. (12) in terms of the multipole moments of the body and the universe, we can express g_i, A_i, and H_i in terms of the external universe's moments [Eqs. (8) and (10)] and then perform the integrations using the definitions of the body's multipole moments[2]

$$\mathscr{I}_{A_l}\equiv\left[\int T^{00}x_{a_1}x_{a_2}\cdots x_{a_l}d^3x\right]^{\rm STF} ,$$

$$\mathscr{S}_{A_l}\equiv\left[\int \epsilon_{a_1pq}x^pT^{0q}x_{a_2}x_{a_3}\cdots x_{a_l}d^3x\right]^{\rm STF} .$$

Here $(\cdots)^{\rm STF}$ means take the symmetric, trace-free part. The result is the laws of motion (2) and (4). Because the actual calculation is not so straightforward as in the last section, we give a few of its details in an appendix.

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APPENDIX

In this appendix we will carry out the calculations leading to Eqs. (4) from Eqs. (12). First we define a few quantities that will be used later:

$$I_{A_l}\equiv\int T^{00}x_{A_l}d^3x \ , \tag{A1a}$$

$$J^a{}_{A_l}\equiv\int T^{0a}x_{A_l}d^3x \ , \tag{A1b}$$

$$K^{ab}{}_{A_l}\equiv\int T^{ab}x_{A_l}d^3x \ , \tag{A1c}$$

where $x_{A_l}\equiv x_{a_1}x_{a_2}\cdots x_{a_l}$. From these definitions, the finiteness of the body, and the law of energy-momentum conservation $T^{\alpha\beta}{}_{,\beta}=0$ (with gravitational effects here neglected), we can find a few very useful relations:

$$J^{ab}=\tfrac{1}{2}\dot{I}^{ab}+J^{[ab]} \ , \tag{A2a}$$

$$J^{abc}=\tfrac{1}{3}\dot{I}^{abc}+\tfrac{2}{3}J^{[ab]c}+\tfrac{2}{3}J^{[ac]b} \ , \tag{A2b}$$

$$K^{cca}=\dot{J}^{cca}-\tfrac{1}{2}\dot{J}^{acc} \ . \tag{A2c}$$

Here square brackets denote antisymmetrization, and indices are raised and lowered with the three-dimensional flat metric (Kronecker delta).

Using Eqs. (A1) and (A2) we can rewrite Eqs. (12) as follows:

$$\dot{M}=\tfrac{1}{2}\dot{\mathscr{E}}_{ab}I^{ab}+\tfrac{2}{3}\dot{\mathscr{B}}_{ab}\mathscr{S}^{ab} \ ,$$

$$\begin{aligned}\dot{P}^i&=-\tfrac{1}{2}\mathscr{E}^i{}_{ab}I^{ab}+\tfrac{2}{3}\epsilon^i{}_{pq}\dot{\mathscr{B}}^p{}_lI^{ql}-2\epsilon^i{}_{jk}\mathscr{B}^k{}_aJ^{ja}-\tfrac{4}{3}\epsilon^i{}_{jk}\mathscr{B}^k{}_{ab}J^{jab}+\tfrac{4}{3}\delta^{ij}_{pq}\dot{\mathscr{E}}^q{}_aJ_j{}^{pa}-\mathscr{E}^i{}_aK^{jja}\\&=-\tfrac{1}{2}\mathscr{E}^i{}_{ab}I^{ab}+\tfrac{2}{3}\epsilon^i{}_{pq}\dot{\mathscr{B}}^p{}_lI^{ql}-2\epsilon^i{}_{jk}\mathscr{B}^k{}_aJ^{[ja]}-\epsilon^i{}_{jk}\mathscr{B}^k{}_a\dot{I}^{ja}\\&\quad-\tfrac{4}{9}\epsilon^i{}_{jk}\mathscr{B}^k{}_{ab}\dot{I}^{jab}-\tfrac{16}{9}\epsilon^i{}_{jk}\mathscr{B}^k{}_{ab}J^{[ja]b}+\tfrac{8}{9}\dot{\mathscr{E}}_{ab}J^{[bi]a}+\tfrac{4}{9}\dot{\mathscr{E}}^i{}_aJ^{[ba]}{}_b+\tfrac{4}{9}\dot{\mathscr{E}}_{ab}\dot{I}^{abi}-\tfrac{5}{18}\dot{\mathscr{E}}^i{}_a\dot{I}^{abb} \ ,\end{aligned}$$

and

$$\begin{aligned}\dot{\mathscr{S}}^i&=-\tfrac{1}{2}\epsilon^i{}_{pq}\mathscr{E}^q{}_{ab}I^{pab}+\mathscr{B}_{abc}(J^{abci}+\tfrac{1}{3}J^{iabc})-\tfrac{2}{3}\mathscr{B}^i{}_{ab}(J^{ccab}+J^{abcc})-\tfrac{4}{3}\mathscr{B}_{abc}J^{iabc}+\tfrac{4}{3}\mathscr{B}^i{}_{ab}J^{ccab}\\&=-\tfrac{1}{2}\epsilon^i{}_{pq}\mathscr{E}^q{}_{ab}I^{pab}-2\mathscr{B}_{abc}J^{[ia]bc}-\tfrac{4}{3}\mathscr{B}^i{}_{ab}J^{[ac]b}{}_c \ .\end{aligned}$$

In writing these three equations in terms of multipole moments, we notice that

$$\mathscr{I}_{A_l}=(I_{A_l})^{\rm STF} \ ,$$

$$\epsilon^i{}_{pq}\dot{\mathscr{E}}^{pa}\mathscr{S}^q{}_a=-\epsilon^i{}_{pq}\dot{\mathscr{E}}^{pa}(\epsilon^q{}_{mn}J^{mn}{}_a+\epsilon_{[a\,|\,mn}J^{mn\,|\,q]})=-2\dot{\mathscr{E}}_{pa}J^{[ip]a}-\dot{\mathscr{E}}^i{}_aJ^{[ab]}{}_b \ ,$$

where $\epsilon_{[a\,|\,mn}J^{mn\,|\,q]}$ means that we are only antisymmetrizing indices a, q; and

$$\begin{aligned}\epsilon^i{}_{jk}\mathscr{S}^j{}_{ab}\mathscr{B}^{kab}&=\tfrac{1}{3}\epsilon^i{}_{jk}\mathscr{B}^k{}_{ab}(\epsilon^a{}_{pq}J^{qpbj}+\epsilon^b{}_{pq}J^{qpaj}+\epsilon^j{}_{pq}J^{qpab})\\&=\tfrac{1}{3}\epsilon^i{}_{jk}\mathscr{B}^k{}_{ab}(2\epsilon^{[a}{}_{pq}J^{\,|\,qpb\,|\,j]}+2\epsilon^{[b}{}_{pq}J^{\,|\,qpa\,|\,j]}+3\epsilon^i{}_{pq}J^{qpab})\\&=\tfrac{4}{3}\mathscr{B}^i{}_{ab}J^{[ad]b}{}_d+2\mathscr{B}_{abc}J^{[ia]bc} \ .\end{aligned}$$

Using these relations, we can further simplify Eqs. (12) into

$$\dot{M} = -\tfrac{1}{2}\mathcal{E}_{ab}\dot{\mathcal{I}}^{ab} - \tfrac{2}{3}\mathcal{B}_{ab}\dot{\mathcal{S}}^{ab} , \tag{A3a}$$

$$\dot{P}^{i} = -\tfrac{1}{2}\mathcal{E}^{i}{}_{ab}\mathcal{I}^{ab} - \tfrac{8}{9}\mathcal{B}^{i}{}_{ab}\mathcal{S}^{ab} + \tfrac{1}{3}\epsilon^{i}{}_{ab}\mathcal{B}^{a}{}_{c}\dot{\mathcal{I}}^{bc} + \tfrac{4}{9}\epsilon^{i}{}_{ab}\dot{\mathcal{E}}^{a}{}_{c}\mathcal{S}^{bc} - \tfrac{4}{9}\epsilon^{i}{}_{pq}\mathcal{B}^{q}{}_{ab}\dot{\mathcal{I}}^{pab} - \tfrac{8}{15}\dot{\mathcal{E}}^{i}{}_{a}\dot{I}^{abb} + \tfrac{4}{9}\dot{\mathcal{E}}_{ab}\dot{I}^{abi} , \tag{A3b}$$

$$\dot{\mathcal{S}}^{i} = -\tfrac{1}{2}\epsilon^{i}{}_{pq}\mathcal{E}^{q}{}_{ab}\mathcal{I}^{pab} - \epsilon^{i}{}_{pq}\mathcal{B}^{q}{}_{ab}\mathcal{S}^{pab} . \tag{A3c}$$

The last three terms in Eq. (A3b) are part of the next-higher-order contributions $[\sim(ML^3/\mathcal{R}^2T)(1/\mathcal{T}$ or $1/\mathcal{L})]$ and thus can be dropped. After this, Eqs. (A3) are the same as Eqs. (4).

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