Spinor matter fields in SL(2, C) gauge theories of gravity: Lagrangian and Hamiltonian approaches

Marek Antonowicz
Institute of Theoretical Physics, Warsaw University, Hoża 69, 00-681 Warsaw, Poland

Wiktor Szczyrba
Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, 00-950 Warsaw, Poland
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We consider the SL(2, C)-covariant Lagrangian formulation of gravitational theories with the presence of spinor matter fields. The invariance properties of such theories give rise to the conservation laws (the contracted Bianchi identities) having in the presence of matter fields a more complicated form than those known in the literature previously. A general SL(2, C) gauge theory of gravity is cast into an SL(2, C)-covariant Hamiltonian formulation. Breaking the SL(2, C) symmetry of the system to the SU(2) symmetry, by introducing a spacelike slicing of spacetime, we get an SU(2)-covariant Hamiltonian picture. The qualitative analysis of SL(2, C) gauge theories of gravity in the SU(2)-covariant formulation enables us to define the dynamical symplectic variables and the gauge variables of the theory under consideration as well as to divide the set of field equations into the dynamical equations and the constraints. In the SU(2)-covariant Hamiltonian formulation the primary constraints, which are generic for first-order matter Lagrangians (Dirac, Weyl, Fierz-Pauli), can be reduced. The effective matter symplectic variables are given by SU(2)-spinor-valued half-forms on three-dimensional slices of spacetime. The coupled Einstein-Cartan-Dirac (Weyl, Fierz-Pauli) system is analyzed from the $(3+1)$ point of view. This analysis is complete; the field equations of the Einstein-Cartan-Dirac theory split into 18 gravitational dynamical equations, 8 dynamical Dirac equations, and 7 first-class constraints. The system has $4+8=12$ independent degrees of freedom in the phase space.

I. INTRODUCTION

The rapid development of gauge theories of elementary particles in the last fifteen years stimulated a growing interest in unified theories of fundamental interactions. There is a strong belief among physicists that such a unified theory should also describe gravitational phenomena.[1] This is why several attempts were made to incorporate the classical Einstein theory into the gauge scheme. Some of these approaches revive the old Kaluza-Klein ideas,[2] and some others going beyond classical geometry are based on supersymmetric methods.[3] In the present paper, however, we deal with much more standard constructions. Remaining within the framework of four-dimensional spacetimes, we equip them with metric and affine structures. That is to say, we fix a global tetrad one-form $e^{(\alpha)}=e^{(\alpha)}{}_{\mu}dx^{\mu}$ on spacetime M as well as an so(3,1)-valued connection one-form $\Gamma^{(\alpha)}{}_{(\beta)}=\Gamma^{(\alpha)}{}_{(\beta_{\mu})}dx^{\mu}$. The global tetrad field $e^{(\alpha)}$ determines a spinor structure of spacetime and we employ the spin representation of tetrad and connection one-forms. In such a representation the tetrad one-form $e^{A\dot{B}}=e^{A\dot{B}}{}_{\mu}dx^{\mu}$ takes its values in the space $H(2)$ of complex, Hermitian 2×2 matrices and values of the connection one-form $\Gamma^{A}{}_{B}=\Gamma^{A}{}_{B\mu}dx^{\mu}$ lie in the Lie algebra sl$(2,C)$, i.e., in the space of complex, traceless 2×2 matrices. The quantities $e^{A\dot{B}}$ and $\Gamma^{A}{}_{B}$ are the gravitational potentials. Matter coupled to gravity is described by its potentials being differential k-forms ϕ^{Σ} on spacetime with values in the space of an appropriate spinor representation of SL$(2,C)$. Such a description of matter is widely accepted today.[4] It enables us to discuss not only tensor and Dirac fields but also the electromagnetic field as well as the Rarita-Schwinger field playing an important role in supergravity. The field potentials $e^{A\dot{B}},\Gamma^{A}{}_{B},\phi^{\Sigma}$ are subject to the natural action of the local SL$(2,C)$ group. Field Lagrangians are SL$(2,C)$ invariant, real four-forms on spacetime; gravitational Lagrangians K depend on the tetrad field, curvature, and torsion of the connection; matter Lagrangians L are functions of the tetrad field, matter potentials, and their covariant exterior derivatives. The dynamics of the system is given by the Euler-Lagrange (EL) equations:

$$\begin{aligned} &\delta(K+L)/\delta e^{A\dot{B}}=0\ ,\\ &\delta(K+L)/\delta\Gamma^{A}{}_{B}=0\ ,\\ &\delta(K+L)/\delta\phi^{\Sigma}=0\ . \end{aligned} \tag{1.1}$$

It is clear that the set of solutions of field equations (1.1) is invariant with respect to the action of the local SL$(2,C)$ group. In the literature, such theories are usually called SL$(2,C)$ gauge theories of gravity. Of course, such a definition essentially generalizes the standard notion of Yang-Mills gauge fields. Therefore, several authors tried to reformulate the tetrad-connection gauge approach and to replace it with the purely internal Yang-Mills scheme in an appropriately chosen principal bundle. The inhomogeneous SO(3,1) group (the Poincaré group), the inhomogeneous GL$(4,R)$ group or de Sitter group were mostly taken as the structure groups of those bundles. A profound discussion of the problem and the relevant bibliography can be found in Refs. 5, 6, and 7.

Any Lagrangian SL(2, C) gauge theory of gravity based on variational potentials $e^{A\dot{B}}$, $\Gamma^{A}{}_{B}$, ϕ^{Σ}, and field equations (1.1) can be cast into a Hamiltonian form. The most natural is the SL(2, C)-covariant Hamiltonian formulation presented in Sec. V. The set $\mathscr{G}$ of all conceivable geometric configurations of the system under consideration carries a natural symplectic structure given by a closed two-form Ω on $\mathscr{G}$. The invariance of the theory with respect to the diffeomorphism group of spacetime leads to a natural definition of the energy-momentum function $\mathscr{E}$ on $\mathscr{G}$. Infinitesimal developments of symplectic variables are generated by the Hamilton equation

$$d\mathscr{E}(V) = -2\Omega(Y,V) . \qquad (1.2)$$

Here Y is the vector of evolution and V is an arbitrary (sample) vector tangent to $\mathscr{G}$.

Infinitesimal variations of symplectic variables induced by the vector Y coincide with the corresponding covariant Lie derivatives of these variables. In such an approach the full gravitational gauge group G of the theory has a rather complicated structure. It is given by the bundle product of the local SL(2, C) group and the diffeomorphism group of spacetime (see Appendix C).

The field equations formulated in the SL(2, C)-covariant Hamiltonian picture are strictly equivalent to the EL equations and they do not help us to give a deeper analysis of the system. We are able, however, to elucidate the structure of field equations more profoundly if we break the original SL(2, C) symmetry by means of a spacelike slicing of spacetime.[8] Then we naturally obtain an SU(2)-covariant Hamiltonian formulation. Such a formulation enables us to separate the dynamical symplectic variables and gauge variables and helps to analyze the dynamical structure of field equations. Even for general Lagrangians we are able to give a partial analysis of the dynamics. In particular, we clarify the structure of ten Hamiltonian constraints and prove their maintenance in the process of evolution. A general scheme of how to analyze the SU(2)-covariant Hamiltonian equations is discussed in Sec. VII. The complete dynamical analysis of field equations, however, is possible only if the Lagrangian is specified. As an example, we investigate the coupled Einstein-Cartan-Dirac (ECD) system and give a complete analysis of its dynamics. Our approach to the ECD equations eliminates all matter constraints by means of an appropriate redefinition of the matter symplectic variables. The genuine dynamical matter variables for the Dirac field are given by half-forms on the initial surface σ. For the ECD theory, the gravitational field equations can be essentially reduced and, eventually, we are left with 18 gravitational dynamical equations and 7 gravitational initial-value constraints. Moreover, it follows from our symplectic analysis that in the ECD theory we have 12 independent degrees of freedom in the phase space and 10 (gravitational) gauge variables.

The symplectic analysis of the Hamiltonian field equations can be undoubtedly applied to more complicated gravitational Lagrangians than the Einstein-Hilbert Lagrangian. The most interesting case presents the class of gravitational Lagrangians quadratic in curvature and torsion. The linearlized version of theories with such Lagrangians was investigated in Ref. 9 although mainly from the quantum point of view. In subsequent papers we intend to investigate the dynamics of gravitational theories with quadratic Lagrangians, to pose the initial-value problem, and to determine independent degrees of freedom.

In the present paper we discuss two possible Hamiltonian structures of field theories. The first, the general SL(2, C)-covariant Hamiltonian formulation is closely related to the Kijowski-Tulczyjew theory of symplectic spaces, Lagrangian submanifolds, and their generating functions. The main ideas of this theory were presented in Ref. 10 and applied to gravity in Ref. 11. The second approach, that is, the SU(2)-covariant Hamiltonian picture is fairly close to the Hamiltonian analysis of field theories presented in papers by Fischer, Marsden, Moncrief, and Arms[12-14] (FMMA) with applications to general relativity and to Yang-Mills fields. These authors proved that in field theories symmetric solutions of field equations played a singled-out role. Namely, they are singular points in the "manifold" of all solutions. In one of our subsequent papers we will show that methods of the Hamiltonian analysis enable us to extend the results of FMMA to SL(2, C) gauge theories of gravity.

In our Hamiltonian analysis we do not use the Dirac classification of constraints.[15,16] However, the main results of the present paper can also be formulated in that language.

We would like to point out some important features of our analysis that are different from the results which were presented in literature previously.

(a) If matter potentials are spinor-valued k-forms and $k \geq 1$, then the formula for the energy-momentum three-form $T_{A\dot{B}}$ depends on the matter current, thereby essentially differing from the case $k = 0$.

(b) If $k \geq 1$ then the right-hand sides of the contracted Bianchi identities contain σ-transversal derivatives of left-hand sides of matter field equations. Such terms may lead to additional initial-value constraints.

(c) In the Hamiltonian formulation the energy-momentum three-form E_Z on the initial surface is determined not only by the left-hand sides of gravitational equations but it also contains the left-hand sides of some matter equations.

In the present paper, SL(2, C)-covariant formulation of the theory is given in the Cartan-Trautman language of differential forms and their covariant exterior derivatives. If, however, we pass to the (3 + 1) picture, we need a (3 + 1) decomposition of differential forms and their covariant exterior derivatives as well as the definition of an SU(2)-covariant "time" derivative. This technique is developed in Sec. VI and Appendix E.

Our notation is as follows. Small greek (spacetime) indices α,β,γ, etc., run from 0 to 3; small latin (spatial) indices p,r,s, etc., run from 1 to 3. Capital latin (spinor) indices $A,B,\dot{A},\dot{B}$, etc., take the values 1 and 2. Capital greek indices Δ,Λ,Σ denote collections of spinor indices (multi-indices). The small latin letters d,t,h,a placed at the upper left-hand side of a matrix denote its diagonal, traceless, Hermitian and anti-Hermitian parts, respectively. The symbols $\perp$ and $||$ denote the σ-transversal and σ-tangential parts of a corresponding differential form or vector field on spacetime. The operation of complex conjugation (denoted by an asterisk) converts the undotted spinor indices into dotted ones and conversely.

II. GEOMETRY OF SPACETIME AND GEOMETRY OF MATTER

Spacetime M is a smooth, connected, time- and space-oriented, noncompact, four-dimensional Lorentzian manifold.[17] We denote by LF(M) the principal $SO_+(3,1)$ bundle of time- and space-oriented coframes (of covectors) over M. A spinor structure of M is a principal $SL(2,C)$ bundle SF(M) [the bundle of spin-(co)frames] and a double-covering projection f:SF(M)$\to$LF(M) such that for every $s\in$SF(M) and $S\in SL(2,C)$

$$f(sS)=f(s)\Lambda(S)\ , \tag{2.1}$$

where Λ is a fixed double-covering homomorphism of $SL(2,C)$ onto $SO_+(3,1)$.

It is well known (cf. Ref. 18) that a spinor structure on spacetime M exists if and only if one of the topological invariants of M, its second Stiefel-Whitney class, vanishes. For four-dimensional noncompact Lorentzian manifolds, however, this topological condition can be replaced by a simpler one. Noncompact spacetimes carry spinor structures if and only if they are parallelizable,[19] that is, their bundles of (co)frames admit global sections. In such a case LF(M) as well as SF(M) are trivial and the spinor structure (SF(M),f) is determined by a global section e of LF(M) over M. The bundle space of SF(M) is the Cartesian product $M\times SL(2,C)$ and the projection f is given by

$$M\times SL(2,C)\ni(x,S)\to f(x,S)$$
$$=(x,e(x)\Lambda(S))\in \mathrm{LF}(M)\ . \tag{2.2}$$

Remark. In the present paper we consider spacetimes diffeomorphic to the product $R\times\sigma$, where σ is a three-dimensional orientable manifold. If, moreover, σ is a compact manifold then by virtue of the Stiefel theorem[20] it is parallelizable. Thus, M is also parallelizable and carries a spinor structure.

Different global sections e and $'e$ of LF(M) may lead to nonequivalent spinor structures. As a matter of fact, even though e and $'e$ are always related by an (x dependent) $SO_+(3,1)$ rotation it cannot, in general, be lifted to $SL(2,C)$. The obstructions for such a lift are nontrivial elements of the cohomology group $H^1(M,Z_2)$, which correspond to nonequivalent spinor structures on M. In particular, we have a unique spinor structure on simple-connected spacetime.[18] The mathematical properties of spin-spacetimes imply that for physical applications we should start with a noncompact, four-dimensional, connected and orientable manifold M and a global field of coframes e on it. A coframe e is an R^4-valued one-form on M; in local coordinates $e=(e^{(\alpha)})=(e^{(\alpha)}{}_\mu dx^\mu)$. At fixed $x\in M$, the natural pairing between vectors and covectors defines the linear map $T_x(M)\ni v\to\langle v,e\rangle\in R^4$ and we assume that (for every x) it is an isomorphism. Moreover, we assume that the orientation of M is consistent with e, that is, $\det[e^{(\alpha)}{}_\mu]>0$. The diagonal Minkowski metric $\eta=(\eta_{(\alpha)(\beta)})$ (we use the signature $+2$) determines the metric tensor on M:$g=\eta_{(\alpha)(\beta)}e^{(\alpha)}\otimes e^{(\beta)}$. In the $SO_+(3,1)$-gauge formulation[4–6,21,22] the natural action of the local Lorentz group in the set of coframes

$$e^{(\alpha)}\to {}'e^{(\alpha)}=L^{-1(\alpha)}{}_{(\beta)}e^{(\beta)}\ , \tag{2.3}$$

where $L\in SO_+(3,1)$, preserves the metric structure of M. If, however, $H^1(M,Z_2)\neq 0$, this action can change the spinor structure of M. Therefore, we have to pass to an appropriate $SL(2,C)$-covariant picture. Let $H(2)$ be the space of complex Hermitian 2×2 matrices. If $v,w\in H(2)$ then their scalar product is given by

$$\langle v,w\rangle=-v^{A\dot B}w^{C\dot D}\epsilon_{AC}\epsilon_{\dot B\dot D}\ , \tag{2.4}$$

where

$$[\epsilon^{AB}]=[\epsilon_{AB}]=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}=[\epsilon^{\dot A\dot B}]=[\epsilon_{\dot A\dot B}]\ .$$

The Infeld-van der Waerden matrices

$$[\sigma^{A\dot B}_{(0)}]=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\ ,\quad [\sigma^{A\dot B}_{(s)}]=\frac{1}{\sqrt{2}}\sigma_s \tag{2.5}$$

(σ_s are Pauli matrices) give an isomorphism σ between the spaces (R^4,η) and $(H(2),\epsilon)$. We have

$$\sigma(e^{(\alpha)})=e^{A\dot B}=e^{(\alpha)}\sigma^{A\dot B}_{(\alpha)}\ . \tag{2.6}$$

The $H(2)$-valued one-form $e^{A\dot B}$ is the spin representation of the coframe one-form e. In local coordinates on M we write

$$e^{A\dot B}=e^{A\dot B}{}_\mu dx^\mu\ .$$

Remarks. (i) The first index of a matrix denotes the row, and the second denotes the column. (ii) The complex conjugation changes undotted indices into dotted ones and conversely, e.g., $(S^A{}_B)^*=S^{\dot A}{}_{\dot B}$.

We have the natural action of the local $SL(2,C)$ group in the space of coframes

$$e^{A\dot B}\to {}'e^{A\dot B}=S^{-1A}{}_C S^{-1\dot B}{}_{\dot D}e^{C\dot D} \tag{2.7}$$

and this action preserves the metric (2.4).

We recall that the Infeld-van der Waerden matrices realize the covering homomorphism Λ:$SL(2,C)\to SO_+(3,1)$ by means of the formula (Appendix B)

$$L^{(\alpha)}{}_{(\beta)}=-S^A{}_C S^{\dot B}{}_{\dot D}\sigma^{(\alpha)}_{A\dot B}\sigma^{C\dot D}_{(\beta)}\ . \tag{2.8}$$

The kernel of Λ consists of two elements I and $-I$. The actions of $SO_+(3,1)$ and $SL(2,C)$ in the space of coframes are Λ consistent, that is, the diagram commutes

$$\begin{array}{ccc} H(2) & \xrightarrow{S} & H(2)\\ \uparrow\downarrow\ \sigma & & \uparrow\downarrow\ \sigma\ ,\\ R^4 & \xrightarrow{\Lambda(S)} & R^4\end{array}$$

where the horizontal arrows correspond to the transformations (2.3) and (2.7), and the isomorphism σ is given by (2.6).

In the modern approach, gravity is related to the metric and affine structure of spacetime. The affine structure of M is defined by a connection one-form ω on the bundle of Lorentz (co)frames LF(M) with values in the Lie algebra so(3,1) (see Ref. 23). For spacetime admitting a spinor structure, the basic geometric object is the corresponding SL($2,C$) bundle of spin (co)frames SF(M). Connection one-forms on SF(M) take their values in the Lie algebra sl($2,C$). [sl($2,C$) is the algebra of all complex traceless 2×2 matrices.]

It follows from the nature of the covering f:SF(M)$\rightarrow$LF(M) that (for a chosen spinor structure) there exists a one-to-one correspondence between connection one-forms on SF(M) and LF(M). If $M \ni x \rightarrow s(x) \in$ SF(M) is a global section of a spin bundle and ω is a connection one-form on SF(M) then $\Gamma = s^*\omega$ is an sl($2,C$)-valued one-form on spacetime. We have

$$\Gamma = \lambda(s^*(f^*\bar{\omega})) = \lambda((f\circ s)^*\bar{\omega}) = \lambda(e^*\bar{\omega}) = \lambda(\bar{\Gamma})$$

where $\bar{\omega}$ is a connection one-form on LF(M) corresponding to ω and λ is a Lie algebra isomorphism generated by Λ:SL($2,C$)$\rightarrow$SO$_+$(3,1) [cf. (2.8)]. We write

$$\begin{aligned} \Gamma^A{}_B &= -\tfrac{1}{2}\bar{\Gamma}^{(\alpha)}{}_{(\beta)}\sigma^{A\dot{C}}_{(\alpha)}\sigma^{(\beta)}_{B\dot{C}} , \\ \bar{\Gamma}^{(\alpha)}{}_{(\beta)} &= -\Gamma^A{}_B\sigma^{(\alpha)}_{A\dot{C}}\sigma^{B\dot{C}}_{(\beta)} - \Gamma^{\dot{A}}{}_{\dot{B}}\sigma^{(\alpha)}_{C\dot{A}}\sigma^{C\dot{B}}_{(\beta)} . \end{aligned} \tag{2.9}$$

For the bundle of Lorentz coframes we have an additional intrinsically defined object—the soldering R^4-valued form θ on the bundle space.[23] The projection f and the isomorphism σ induce the $H(2)$-valued one-form $\sigma(f^*\theta)$ on SF(M). Its pullback by s onto M coincides with the one-form $e^{A\dot{B}}$ defining the spinor structure and the metric on M

$$\underline{g} = -e^{A\dot{B}} \otimes e^{C\dot{D}}\epsilon_{AC}\epsilon_{\dot{B}\dot{D}} .$$

We see that the differential-geometric structure of spacetime is completely determined by $\Gamma^A{}_B$ and $e^{A\dot{B}}$ and we take these quantities as the gravitational potentials. In theories of gravity, however, the essential role play matter fields coupled to the gravitational field and for present applications it is natural to take spinor-valued differential forms on spacetime as matter field potentials.[4] Let ρ be a representation of SL($2,C$) in the complex vector space C^N. We define the right action of SL($2,C$) in the bundle $\Lambda^k T^*(M)\otimes C^N$ of C^N-valued k covectors on M

$$\phi S = \rho(S^{-1})\phi , \tag{2.10}$$

where $\phi \in \Lambda^k T^*(M)\otimes C^N$ and $S \in$ SL($2,C$). The natural right action of SL($2,C$) in SF(M) and the action (2.10) define the equivalence relation in the Whitney sum SF(M)$\oplus_M$($\Lambda^k T^*(M)\otimes C^N$). The quotient bundle $\Lambda^k T^*(M)\otimes\rho$ is called the bundle of k-covectors of type ρ on M and sections of this bundle are k-forms of type ρ on M. In the matrix representation the action (2.10) can be written as

$$'\phi^\Sigma = \rho^\Sigma{}_\Lambda(S^{-1})\phi^\Lambda , \tag{2.10$'$}$$

where $\Sigma, \Lambda = 1,2,\ldots,N$, and the generators of ρ are given by

$$\delta(\rho^\Sigma{}_\Lambda(S)) = \rho^\Sigma{}_{\Lambda A}\delta S^A{}_B + \rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}\delta S^{\dot{A}}{}_{\dot{B}} . \tag{2.11}$$

The quantities $\rho^\Sigma{}_{\Lambda A}{}^B$ and $\rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}$ are traceless, i.e., $\rho^\Sigma{}_{\Lambda A}{}^A = 0 = \rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{A}}$. In spinor spaces the spinor indices are raised and lowered by means of the metric tensor ϵ, e.g.,

$$\phi^A = \epsilon^{AB}\phi_B , \quad \phi_A = \phi^B\epsilon_{BA} , \quad \phi^{\dot{A}} = \epsilon^{\dot{A}\dot{B}}\phi_{\dot{B}} , \text{ etc.} \tag{2.12}$$

The covariant exterior derivative of a k-form $\phi = (\phi^\Sigma)$ of type ρ on M with respect to a connection $\Gamma^A{}_B$ is the following $(k+1)$-form of type ρ on M (cf. Ref. 4):

$$D\phi^\Sigma = d\phi^\Sigma + \rho^\Sigma{}_{\Lambda A}{}^B\Gamma^A{}_B \wedge \phi^\Lambda + \rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}\Gamma^{\dot{A}}{}_{\dot{B}} \wedge \phi^\Lambda . \tag{2.13}$$

If coframes transform according to (2.7) and the components of ϕ and $D\phi$ transform according to (2.10), then the connection one-form $\Gamma^A{}_B$ has to obey the transformation rule

$$'\Gamma^A{}_B = S^{-1A}{}_C\Gamma^C{}_D S^D{}_B + S^{-1A}{}_C dS^C{}_B . \tag{2.14}$$

The covariant exterior derivative of the coframe one-form $e^{A\dot{B}}$ is called torsion

$$\Theta^{A\dot{B}} = De^{A\dot{B}} = de^{A\dot{B}} + \Gamma^A{}_C \wedge e^{C\dot{B}} + \Gamma^{\dot{B}}{}_{\dot{D}} \wedge e^{A\dot{D}} \tag{2.15}$$

and the covariant exterior derivative of the connection one-form is called curvature

$$\Omega^A{}_B = D\Gamma^A{}_B = d\Gamma^A{}_B + \Gamma^A{}_C \wedge \Gamma^C{}_B \tag{2.16}$$

(the last formula follows from the transformation properties of $\Gamma^A{}_B$). The covariant exterior derivative of the matter form ϕ^Σ is called the field strength $F^\Sigma = D\phi^\Sigma$. Direct calculations give rise to the Ricci formula

$$DF^\Sigma = DD\phi^\Sigma = \rho^\Sigma{}_{\Lambda A}{}^B\Omega^A{}_B \wedge \phi^\Lambda + \rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}\Omega^{\dot{A}}{}_{\dot{B}} \wedge \phi^\Lambda \tag{2.17}$$

and to the first and second Bianchi identities

$$D\Theta^{A\dot{B}} = \Omega^A{}_C \wedge e^{C\dot{B}} + \Omega^{\dot{B}}{}_{\dot{D}} \wedge e^{A\dot{D}} , \quad D\Omega^A{}_B = 0 , \tag{2.17$'$}$$

respectively.

III. VARIATIONAL FIELD EQUATIONS AND THE CONTRACTED BIANCHI IDENTITIES

In the present paper the differential geometric structure of spacetime is determined by a Lorentz coframe e in the spin representation and by an sl($2,C$)-valued connection one-form Γ on spacetime. The distribution of matter is represented by a k-form ϕ of type ρ on M. ϕ is described by its components ϕ^Σ with respect to a global section s of the bundle SF(M). The interaction between gravity and matter and the dynamics of the coupled system are described by means of gravitational and matter Lagrangians K and L and the corresponding variational equations. The Lagrangians are R-valued four-forms on M; K depends on the gravitational potentials $e^{A\dot{B}}, \Gamma^A{}_B$, and their exterior derivatives $de^{A\dot{B}}, d\Gamma^A{}_B$. The matter Lagrangian L should, in principle, depend on fields of spin coframes s, on the components ϕ^Σ of spinor field, on their exterior derivatives $d\phi^\Sigma$, and on $\Gamma^A{}_B$. In all conceivable situations, however, matter Lagrangians depend on bilinear combinations of ϕ^Σ and $d\phi^\Sigma$ and therefore the transformation $s \to -s = s(-I)$, $\phi^\Sigma \to -\phi^\Sigma$ does not affect the value of L. Hence, we may assume that L depends on the Lorentz coframe $e^{A\dot{B}}$ corresponding to s instead of s itself.

The natural invariance properties of the Lagrangians with respect to the action of the local SL(2, C) group and the diffeomorphism group of spacetime imply that K and L depend on the components $\Gamma^A{}_B$ of a connection Γ only through the field strengths $\Theta^{A\dot{B}}$, $\Omega^A{}_B$, and F^Σ. In a general situation we may write[4,6,21,24–27]

$$K = K(e^{A\dot{B}}, \Theta^{A\dot{B}}, \Omega^A{}_B), \quad L = L(e^{A\dot{B}}, \phi^\Sigma, F^\Sigma) . \tag{3.1}$$

Variations of the gravitational and matter Lagrangians read

$$\begin{aligned} K &= \delta e^{A\dot{B}} \wedge V_{A\dot{B}} + \delta\Theta^{A\dot{B}} \wedge U_{A\dot{B}} + (\delta\Omega^A{}_B \wedge P_A{}^B + \text{c.c.}) , \\ L &= \delta e^{A\dot{B}} \wedge T_{A\dot{B}} + (\delta\phi^\Sigma \wedge J_\Sigma + \delta F^\Sigma \wedge W_\Sigma + \text{c.c.}) . \end{aligned} \tag{3.2}$$

Remark. The symbol c.c. denotes the complex conjugate expression.

Simultaneously, formulas (3.2) define the quantities $V_{A\dot{B}}$, $U_{A\dot{B}}$, $P_A{}^B$, J_Σ, W^Σ, and $T_{A\dot{B}}$: $V_{A\dot{B}}$ is an $H(2)$-valued three-form on M—the canonical energy-momentum three-form of the gravitational field, $U_{A\dot{B}}$ is an $H(2)$-valued two-form on M, and $P_A{}^B$ is a ${}^tC(2)$-valued two-form on M [${}^tC(2)$ is the space of complex traceless 2×2 matrices]. $U_{A\dot{B}}$ and $P_A{}^B$ represent the momenta canonically conjugate to the gravitational potentials $e^{A\dot{B}}$ and $\Gamma^A{}_B$, respectively.

$T_{A\dot{B}}$ is an $H(2)$-valued three-form on M—the canonical energy-momentum three-form of matter, J_Σ is a $(4-k)$-form of type $\hat{\rho}$ on M and W_Σ is a $(3-k)$-form of type $\hat{\rho}$ on M—they are called the matter current and the momentum of matter, respectively ($\hat{\rho}$ is the contragredient representation of ρ).

The variational principle $\delta\int(K+L)=0$ for the variational potentials $e^{A\dot{B}}$, $\Gamma^A{}_B$, and ϕ^Σ gives rise to the Euler-Lagrange equations

$$\delta(K+L)/\delta e^{A\dot{B}} = (E1)_{A\dot{B}} = (GE1)_{A\dot{B}} + T_{A\dot{B}} = 0 , \tag{3.3a}$$

$$\delta(K+L)/\delta\Gamma^A{}_B = (E2)_A{}^B = (GE2)_A{}^B + s_A{}^B = 0 , \tag{3.3b}$$

$$\delta(K+L)/\delta\phi^\Sigma = (EM)_\Sigma = J_\Sigma - (-1)^k DW_\Sigma = 0 . \tag{3.3c}$$

The ${}^tC(2)$-valued three-form $s_A{}^B$ on M, called the spin three-form, is defined by

$$s_A{}^B = W_\Sigma \wedge \phi^\Lambda \rho^\Sigma{}_{\Lambda A}{}^B + W_{\dot{\Sigma}} \wedge \phi^{\dot{\Lambda}} \rho^{\dot{\Sigma}}{}_{\dot{\Lambda} A}{}^B \tag{3.4}$$

and corresponds to the formal expression $s_A{}^B = \partial L/\partial\Gamma^A{}_B$. The gravitational parts of $(E1)_{A\dot{B}}$ and $(E2)_A{}^B$ read

$$(GE1)_{A\dot{B}} = V_{A\dot{B}} + DU_{A\dot{B}} , \tag{3.5a}$$

$$(GE2)_A{}^B = \tfrac{1}{2}(U_{A\dot{C}} \wedge e^{B\dot{C}} - U^{B\dot{C}} \wedge e_{A\dot{C}}) + DP_A{}^B . \tag{3.5b}$$

Remark. The reader should be aware that in the variational principle variations should be taken with respect to R-independent quantities $e^{A\dot{B}}, \Gamma^A{}_B, \Gamma^{\dot{A}}{}_{\dot{B}}, \phi^\Sigma, \phi^{\dot{\Sigma}} = (\phi^\Sigma)^*$. The Lagrangians, however, are real-valued forms; therefore the field equations (3.3) carry the full information.

The invariance of the gravitational and matter Lagrangians with respect to the action of the diffeomorphism group of spacetime and the local SL(2, C) group give rise to the conservation laws

$$D(GE1)_{A\dot{B}} = *(GE1)_{C\dot{D}} \wedge e_{A\dot{B}} \wedge *\Theta^{C\dot{D}} + *(GE2)_C{}^D \wedge e_{A\dot{B}} \wedge *\Omega^C{}_D + *(GE2)_{\dot{C}}{}^{\dot{D}} \wedge e_{A\dot{B}} \wedge *\Omega^{\dot{C}}{}_{\dot{D}} , \tag{3.6a}$$

$$D(GE2)_A{}^B = \tfrac{1}{2}[e_{A\dot{C}} \wedge (GE1)^{B\dot{C}} - e^{B\dot{C}} \wedge (GE1)_{A\dot{C}}] , \tag{3.6b}$$

and

$$DT_{A\dot{B}} = *T_{C\dot{D}} \wedge *\Theta^{C\dot{D}} \wedge e_{A\dot{B}} + [*s_C{}^D \wedge *\Omega^C{}_D + *\phi^\Sigma \wedge *D(EM)_\Sigma - (-1)^k *F^\Sigma \wedge *(EM)_\Sigma + \text{c.c.}] \wedge e_{A\dot{B}} , \tag{3.7a}$$

$$Ds_A{}^B = \tfrac{1}{2}(e_{A\dot{C}} \wedge T^{B\dot{C}} - e^{B\dot{C}} \wedge T_{A\dot{C}}) - \phi^\Sigma \wedge (EM)_\Lambda \rho^\Lambda{}_{\Sigma A}{}^B - \phi^{\dot{\Sigma}} \wedge (EM)_{\dot{\Lambda}} \rho^{\dot{\Lambda}}{}_{\dot{\Sigma} A}{}^B . \tag{3.7b}$$

Moreover, from the invariance of K and L with respect to the action of Diff$M$ we have

$$V_{A\dot{B}} = *K \wedge (*e_{A\dot{B}}) - U_{C\dot{D}} \wedge *(e_{A\dot{B}} \wedge *\Theta^{C\dot{D}}) - P_C{}^D \wedge *(e_{A\dot{B}} \wedge *\Omega^C{}_D) - P_{\dot{C}}{}^{\dot{D}} \wedge *(e_{A\dot{B}} \wedge *\Omega^{\dot{C}}{}_{\dot{D}}) , \tag{3.8a}$$

$$T_{A\dot{B}} = *L \wedge (*e_{A\dot{B}}) - W_\Sigma \wedge *(e_{A\dot{B}} \wedge *F^\Sigma) - W_{\dot{\Sigma}} \wedge *(e_{A\dot{B}} \wedge *F^{\dot{\Sigma}}) - J_\Sigma \wedge *(e_{A\dot{B}} \wedge *\phi^\Sigma) - J_{\dot{\Sigma}} \wedge *(e_{A\dot{B}} \wedge *\phi^{\dot{\Sigma}}) . \tag{3.8b}$$

In the above formulas the asterisk denotes the Hodge dual operation (cf. Appendix A).

If ϕ^Σ is a zero-form on spacetime then the term $e_{A\dot{B}} \wedge *\phi^\Sigma$ represents the trivial five-form on M. Also for the Maxwell theory (in curved spacetime) the terms with the matter current in (3.8) vanish for $J_\Sigma = 0$. In these situations formula (3.8b) coincides with the classical expression for the canonical energy-momentum tensor of matter. For more complicated theories, for instance, for the Proca or Rarita-Schwinger field the correct formula for the canonical energy-momentum three-form contains nontrivial current terms.

The conservation laws for matter fields (3.7) were derived in Riemann spacetime by Belinfante, Rosenfeld, and Pauli,[28] for the Riemann-Cartan spacetime the corresponding formulas were obtained by Sciama, Kibble, Trautman, and Hehl *et al.*[4,29,30] The gravitational conservation laws (3.6) have been given in Refs. 25, 26, and in Ref. 4 for the special case of the Einstein-Cartan theory.

The conservation laws (3.6) and (3.7) give rise to the contracted Bianchi identities:

$$D(E1)_{A\dot{B}}=*(E1)_{C\dot{D}}\wedge *\Theta^{C\dot{D}}\wedge e_{A\dot{B}}+[*(E2)_C{}^D\wedge *\Omega^C{}_D+*D(EM)_\Sigma\wedge *\phi^\Sigma-(-1)^k*(EM)_\Sigma\wedge *F^\Sigma+\text{c.c.}]\wedge e_{A\dot{B}}\ , \tag{3.9a}$$

$$D(E2)_A{}^B=\tfrac{1}{2}[e_{A\dot{C}}\wedge(E1)^{B\dot{C}}-e^{B\dot{C}}\wedge(E1)_{A\dot{C}}]-\phi^\Sigma\wedge(EM)_\Lambda\rho^\Lambda{}_{\Sigma A}{}^B-\phi^{\dot{\Sigma}}\wedge(EM)_{\dot{\Lambda}}\rho^{\dot{\Lambda}}{}_{\dot{\Sigma}A}{}^B\ . \tag{3.9b}$$

If $k=0$ then $D(EM)_\Sigma$ is the trivial five-form on spacetime and the contracted Bianchi identities (3.9) reduce to those obtained earlier in Refs. 4, 25, and 26. However, for $k>0$ the term $*D(EM)_\Sigma\wedge *\phi^\Sigma$ gives a nontrivial contribution to the right-hand side of (3.9a).

It is known in classical general relativity that the contracted Bianchi identities $\nabla_\nu G^\nu{}_\mu=0$ assure the maintenance of the Hamiltonian constraints in the process of evolution.[31] An analogous result for $SO_+(3,1)$ gauge theories of gravity with $SO_+(3,1)$-tensor-valued zero-forms as matter potentials has been presented in Ref. 22. It follows from (3.9) that for theories with matter potentials being k-forms ($k>0$) the time maintenance of constraints requires the additional condition $D(EM)_\Sigma=0$. For some theories, e.g., for the Einstein-Maxwell theory this condition is satisfied automatically, for others it leads to new constraints (e.g., the Rarita-Schwinger field[32]). This problem is discussed more profoundly in Sec. VII.

Let us make the following comment. The canonical energy-momentum three-form $T_{A\dot{B}}$ induces the following tensor density $\mathscr{T}_{\mu\nu}=-\mathscr{T}_{A\dot{B}\nu}e^{A\dot{B}}{}_\mu$ on spacetime (cf. Appendix D). Surprisingly, for the Proca-Maxwell fields this canonical energy-momentum tensor is symmetric (see Sec. VII). It could seem strange for, in the standard approach to field theories, the canonical energy-momentum tensor is nonsymmetric in general.[33] However, a detailed analysis of the classical and our formulas show that they coincide only if the matter potentials are zero-forms on spacetime.

We note that the symmetry condition for the energy-momentum tensor $\mathscr{T}_{\mu\nu}$ corresponding to three-form $T_{A\dot{B}}$ is given by

$$e^{A\dot{B}}\wedge *T_{A\dot{B}}=0 \tag{3.10a}$$

or

$$e_{A\dot{C}}\wedge T^{B\dot{D}}=e^{B\dot{D}}\wedge T_{A\dot{C}}\ . \tag{3.10b}$$

In fact, the latter equality can be reduced to

$$e_{A\dot{C}}\wedge T^{B\dot{C}}=e^{B\dot{C}}\wedge T_{A\dot{C}}\ . \tag{3.10c}$$

IV. THE ROTATIONAL AND TRANSLATIONAL GENERATORS IN THE SPACE OF FIELD POTENTIALS

The Lagrangian formulation of SL(2, C) gauge theories of gravity is invariant with respect to the action of the local SL(2, C) group and the diffeomorphism group of spacetime. For the SL(2, C) rotations their generators are the following operators from the space of field potentials to the space of infinitesimal variations of the field potentials (that is, to the tangent space of the space of potentials)

$$\delta_M e^{A\dot{B}}=-M^A{}_C e^{C\dot{B}}-M^{\dot{B}}{}_{\dot{D}}e^{A\dot{D}}\ , \tag{4.1a}$$

$$\delta_M\Gamma^A{}_B=DM^A{}_B\ , \tag{4.1b}$$

$$\delta_M\phi^\Sigma=-(\rho^\Sigma{}_{\Lambda A}{}^B M^A{}_B\phi^\Lambda+\rho^\Sigma{}_{\dot{\Lambda}\dot{A}}{}^{\dot{B}}M^{\dot{A}}{}_{\dot{B}}\phi^\Lambda)\ , \tag{4.1c}$$

where $M=M^A{}_B(\cdot)$ is a field of traceless matrices on spacetime, i.e., an element of the Lie algebra of the local SL(2, C) group.

In the traditional spacetime-covariant approach to theories of gravity, the generators of the action of Diff$M$ in the space of physical quantities are the corresponding Lie derivatives. The standard Lie derivatives, however, are not SL(2, C)-covariant operators and it is much more convenient to take the covariant Lie derivatives as generators of infinitesimal translations.

Let $\zeta^A{}_B$ be a fixed connection one-form on M. For differential forms of type ρ the covariant Lie derivative with respect to the connection $\zeta^A{}_B$ taken in the direction of a vector field Z on M is given by the formula

$${}_\zeta\mathscr{L}_Z\phi^\Sigma=Z\lrcorner\,{}_\zeta D\phi^\Sigma+{}_\zeta D(Z\lrcorner\,\phi^\Sigma)\ . \tag{4.2a}$$

Here, ${}_\zeta D$ denotes the covariant exterior derivative with respect to the auxiliary connection $\zeta^A{}_B$ and $\lrcorner$ is the contraction operation between vectors and forms on M. The definition (4.2a) generalizes the known formula for the standard Lie derivative—we replace the exterior derivative d by the covariant exterior derivative ${}_\zeta D$. A particular case of (4.2a) reads

$${}_\zeta\mathscr{L}_Z e^{A\dot{B}}=Z\lrcorner\,{}_\zeta\Theta^{A\dot{B}}+{}_\zeta D(Z\lrcorner\,e^{A\dot{B}})\ , \tag{4.2b}$$

where ${}_\zeta\Theta^{A\dot{B}}$ is the torsion of $\zeta^A{}_B$. For the connection one-form $\zeta^A{}_B$ we have

$${}_\zeta\mathscr{L}_Z\zeta^A{}_B=Z\lrcorner\,{}_\zeta\Omega^A{}_B\ , \tag{4.2c}$$

where ${}_\zeta\Omega^A{}_B$ is the curvature of $\zeta^A{}_B$.

For an arbitrary connection one-form $\Gamma^A{}_B$ the combination of (4.2a) and (4.2c) gives rise to the formula

$${}_\zeta\mathscr{L}_Z\Gamma^A{}_B=Z\lrcorner\,{}_\zeta\Omega^A{}_B+Z\lrcorner\,{}_\zeta D(\Gamma^A{}_B-\zeta^A{}_B)+{}_\zeta D[Z\lrcorner(\Gamma^A{}_B-\zeta^A{}_B)]\ . \tag{4.2d}$$

At fixed background connection $\zeta^A{}_B$, we define the operator of infinitesimal covariant translations $\delta_Z = {}_\zeta\mathscr{L}_Z$ acting from the space of field potentials to its tangent space. We have the following commutation relations for the operators δ_M and δ_Z:

$$[\delta_{Z_1}, \delta_{Z_2}] = \delta_{[Z_1, Z_2]} + \delta_{M_3} , \tag{4.3a}$$

$$[\delta_{M_1}, \delta_{M_2}] = \delta_{[M_1, M_2]} , \tag{4.3b}$$

$$[\delta_M, \delta_Z] = \delta_{M_4} , \tag{4.3c}$$

where

$$M_3{}^A{}_B = -(Z_1 \wedge Z_2) \lrcorner\, {}_\zeta\Omega^A{}_B ,$$

$$M_4^A{}_B = -Z \lrcorner\, {}_\zeta D M^A{}_B ,$$

$[Z_1, Z_2]$ is the commutator of the vector fields Z_1 and Z_2, and the commutator of two matrices is given by $[M_1, M_2]^A{}_B \doteq M_2{}^A{}_C M_1{}^C{}_B - M_1{}^A{}_C M_2{}^C{}_B$ (the right multiplication).

The commutation relations (4.3) satisfy the Jacobi identity and therefore they constitute an infinite-dimensional Lie algebra g. In g the operators of infinitesimal rotations form an ideal but infinitesimal covariant translations even do not form a subalgebra. The Lie algebra g generates a group G of transformations in the space of field potentials. It is clear that the transformations corresponding to the operators δ_M are simply the local SL(2, C) rotations. The geometric description of transformations induced by δ_Z is much more complicated and they are discused in Appendix C. G is a fibered space over the diffeomorphism group of spacetime and its fibers are isomorphic to the local SL(2, C) group.

In the SL(2, C)-covariant formulation, the Lagrangians of the theory under considerations must be invariant with respect to the action of G, that is, we have [cf. (3.2)]

$$\delta_X(K+L) = \delta_X e^{A\dot{B}} \wedge V_{A\dot{B}} + \delta_X \Theta^{A\dot{B}} \wedge U_{A\dot{B}} + (\delta_X \Omega^A{}_B \wedge P_A{}^B + \text{c.c.}) + \delta_X e^{A\dot{B}} \wedge T_{A\dot{B}} + (\delta_X \phi^\Sigma \wedge J_\Sigma + \delta_X F^\Sigma \wedge W_\Sigma + \text{c.c.}) , \tag{4.4}$$

where δ_X are operators from g. We rewrite infinitesimal variations of the total Lagrangian in the Noether form: for infinitesimal rotations we have

$$dI_M + (E1)_{A\dot{B}} \wedge \delta_M e^{A\dot{B}} + [(E2)_A{}^B \wedge \delta_M \Gamma^A{}_B + (EM)_\Sigma \wedge \delta_M \phi^\Sigma + \text{c.c}] = 0 \tag{4.5a}$$

and for infinitesimal covariant translations

$$dE_Z + (E1)_{A\dot{B}} \wedge \delta_Z e^{A\dot{B}} + [(E2)_A{}^B \wedge \delta_Z \Gamma^A{}_B + (EM)_\Sigma \wedge \delta_Z \phi^\Sigma + \text{c.c.}] = 0 . \tag{4.5b}$$

I_M and E_Z are scalar three-forms on spacetime, called the spin and energy-momentum three-form, respectively.

We have the relations

$$I_M = U_{A\dot{B}} \wedge \delta_M e^{A\dot{B}} + (P_A{}^B \wedge \delta_M \Gamma^A{}_B + W_\Sigma \wedge \delta_M \phi^\Sigma + \text{c.c.}) , \tag{4.6a}$$

$$E_Z = U_{A\dot{B}} \wedge \delta_Z e^{A\dot{B}} + (P_A{}^B \wedge \delta_Z \Gamma^A{}_B + W_\Sigma \wedge \delta_Z \phi^\Sigma + \text{c.c.}) - Z \lrcorner (K+L) . \tag{4.6b}$$

Making use of (3.3), (4.1), and (4.2) we obtain the Komar-type formulas[34]

$$I_M = (E2)_A{}^B M^A{}_B - d(P_A{}^B M^A{}_B) + \text{c.c.} , \tag{4.7a}$$

$$E_Z = -(Z \lrcorner e^{A\dot{B}})(E1)_{A\dot{B}} - [(Z \lrcorner Y^A{}_B)(E2)_A{}^B + (Z \lrcorner \phi^\Sigma) \wedge (EM)_\Sigma + \text{c.c.}] + d[(Z \lrcorner e^{A\dot{B}}) U_{A\dot{B}}]$$
$$+ d[(Z \lrcorner Y^A{}_B) P_A{}^B + (Z \lrcorner \phi^\Sigma) \wedge W_\Sigma + \text{c.c.}] . \tag{4.7b}$$

Here, $Y^A{}_B = \Gamma^A{}_B - \zeta^A{}_B$ is a spinor-valued one-form on spacetime determining the difference between the physical (geometric) connection $\Gamma^A{}_B$ and the auxiliary, background connection $\zeta^A{}_B$.

In the following section, using the three-forms I_M and E_Z we construct the Hamiltonian generators of rotations and covariant translations in the space of geometric (variational) potentials and cast the theory into an SL(2, C)-covariant Hamiltonian form.

Now we would like to point out the following problems.

(i) Formulas (4.6) and (4.7) generalize the results obtained in Refs. 22 and 25 for $SO_+(3,1)$ gauge theories of gravity. In the approach presented in those papers, however, the translational generators E_Z are not manifestly gauge invariant. The reason is that there we have taken the standard Lie derivatives as the generators of translations. The method of the present paper restores the gauge invariance of the translational generators but we have to fix an auxiliary connection $\zeta^A{}_B$ on spacetime. Of course, a particular choice of $\zeta^A{}_B$ does not affect the final form of the Hamilton equations as presented in Sec. V.

(ii) It follows from (4.7) that if we neglect the divergence terms, then the generators of the action of G are given by the left-hand sides of the field equations. It is remarkable that if matter is described by a k-form ϕ^Σ with $k > 0$, then also the left-hand sides of the matter field equations essentially contribute to E_Z [for $k = 0$ the term with $(EM)_\Sigma$ disappears].

(iii) Formulas (4.7b) have the simplest form when $\Gamma^A{}_B = \zeta^A{}_B$. Therefore one might be tempted to redefine the infinitesimal covariant translations taking always $\zeta^A{}_B = \Gamma^A{}_B$. If $'\delta_Z$ are operators given by (4.2) with $\zeta^A{}_B = \Gamma^A{}_B$ then we have instead of (4.3b) $[\delta_M, '\delta_Z] = 0$ and the operators $\delta_M, '\delta_Z$ do not form a Lie algebra. Even in such a case we can give a formal Hamiltonian formulation but the flows of Hamiltonian vector fields in the space of the symplectic variables will not form one-parameter groups of transformations.

(iv) A purely geometric approach to the Noether formulas of type (4.5) has been presented in Ref. 35. A natural modification of those results gives rise to the corresponding SL(2, C)- [or $SO_+(3,1)$-] covariant formulas.

V. AN SL(2, C)-COVARIANT HAMILTONIAN FORMULATION OF THEORIES OF GRAVITY

In relativistic field theories, the evolution picture can be defined only with respect to a fixed slicing of spacetime into a family of three-dimensional surfaces. Let $\{\sigma_t\}_{t\in R}$ be a one-parameter family of nonintersecting and diffeomorphic three-dimensional surfaces covering M and let $\{f_t\}_{t\in R}$ be a one-parameter subgroup of diffeomorphisms of spacetime preserving $\{\sigma_t\}_{t\in R}$, i.e., $f_t(\sigma_s)=\sigma_{t+s}$. We also assume that the orbits of $\{f_t\}_{t\in R}$ are σ transversal, that is, the vector field Z

$$Z(x)=\frac{d}{dt}f_t(x)\Big|_{t=0} , \quad x\in M , \tag{5.1}$$

is transversal to the surfaces $\sigma_t, t\in R$.

For any SL(2, C) gauge theory of gravity the space $\mathcal{G}$ of all conceivable geometric configurations of the field potentials e,Γ,ϕ on spacetime is an infinite dimensional manifold. Similarly, for an arbitrary surface σ belonging to the slicing we define the space $\mathcal{I}\mathcal{G}(\sigma)$ of initial values of the field potentials and their σ-transversal derivatives on σ. Each element $(e,\Gamma,\phi,\partial_Z e,\partial_Z\Gamma,\partial_Z\phi)$ of $\mathcal{I}\mathcal{G}(\sigma)$ uniquely determines the values of $e^{A\dot B},\Gamma^A{}_B,\phi^\Sigma$, $de^{A\dot B}$, $d\Gamma^A{}_B,d\phi^\Sigma$ on σ.

The space $\mathcal{I}\mathcal{G}(\sigma)$ carries a natural geometric structure, and according to standard rules, a vector X tangent to $\mathcal{I}\mathcal{G}(\sigma)$ at a fixed point is given by the values of variations of field potentials and the values of variations of their Z derivatives on σ

$$X=(\delta e^{A\dot B},\delta\Gamma^A{}_B,\delta\phi^\Sigma,\delta\partial_Z e^{A\dot B},\delta\partial_Z\Gamma^A{}_B,\delta\partial_Z\phi^\Sigma) . \tag{5.2}$$

Remarks. (i) From (5.2) we can compute $\delta de^{A\dot B},\delta d\Gamma^A{}_B,\delta d\phi^\Sigma$. (ii) The quantities $\delta e^{A\dot B},\delta\Gamma^A{}_B,\delta\phi^\Sigma$ are SL(2, C)-tensors but $\delta de^{A\dot B},\delta d\Gamma^A{}_B,\delta d\phi^\Sigma$ are not.

For Lagrangian SL(2, C) gauge theories of gravity, the space $\mathcal{I}\mathcal{G}(\sigma)$ carries a natural symplectic structure given by the symplectic two-form $\Omega(\sigma)$

$$\Omega(\sigma)(X_1,X_2) =\int_\sigma[\delta_1 U_{A\dot B}\wedge\delta_2 e^{A\dot B}+(\delta_1 P_A{}^B\wedge\delta_2\Gamma^A{}_B+\delta_1 W_\Sigma\wedge\delta_2\phi^\Sigma +\text{c.c.})] , \tag{5.3}$$

where X_1,X_2 are vectors tangent to $\mathcal{I}\mathcal{G}(\sigma)$; the variations $\delta U_{A\dot B},\delta e^{A\dot B}$, etc., are to be computed by means of (5.2) and the definition of the symplectic momenta (3.2). In (5.3) the symbol "$\wedge$" denotes not only the exterior product of differential forms but also the antisymmetrization with respect to the subscripts 1 and 2. It is clear that $\Omega(\sigma)$ is a closed two-form, i.e., $d\Omega(\sigma)=0$.

In order to assure the convergence of the integral in (5.3) we have to impose appropriate boundary conditions for components of the vector fields X_i, or to assume that σ is a compact manifold. In the present paper we consider mainly the case of compact σ's, nevertheless several of our results do not depend on this assumption.

We would like to emphasize that formula (5.3) yields a component representation of the general geometric definition of the symplectic two-form in Lagrangian field theories. Such a general construction was given in Refs. 36 and 37.

The second basic ingredient in the Hamiltonian formulation is the Hamilton function of translations, that is, the energy-momentum function $\mathcal{E}(\sigma)$. We define

$$\mathcal{E}_Z(\sigma)=\int_\sigma E_Z , \tag{5.4}$$

where E_Z is the energy-momentum three-form given by (4.6b). It follows from (4.7b) that $\mathcal{E}_Z(\sigma)$ is a well-defined function on $\mathcal{I}\mathcal{G}(\sigma)$ (of course we have to assume that σ is compact manifold, or to impose appropriate boundary conditions).

The energy-momentum three-form E_Z is the Noether current of covariant translations and we expect that the Hamiltonian vector field Y_E of the energy-momentum function generates just covariant translations in the space of symplectic variables. That is to say, the symplectic components of the vector Y_E coincide with the covariant Lie derivatives of the corresponding symplectic variables in the direction of Z. (We recall that the background connection $\zeta^A{}_B$ remains fixed.) This statement in the functional form reads

$$d\mathcal{E}_Z(\sigma)V=-2\Omega(\sigma)(Y_E,V) , \tag{5.5}$$

where

$$Y_E={}_\zeta\mathcal{L}_Z e^{A\dot B}\partial/\partial e^{A\dot B}+({}_\zeta\mathcal{L}_Z\Gamma^A{}_B\partial/\partial\Gamma^A{}_B+{}_\zeta\mathcal{L}_Z\phi^\Sigma\partial/\partial\phi^\Sigma+\text{c.c.})+{}_\zeta\mathcal{L}_Z U_{A\dot B}\partial/\partial U_{A\dot B} +({}_\zeta\mathcal{L}_Z P_A{}^B\partial/\partial P_A{}^B+{}_\zeta\mathcal{L}_Z W_\Sigma\partial/\partial W_\Sigma+\text{c.c.})+\cdots , \tag{5.6}$$

and

$$V=\delta e^{A\dot B}\partial/\partial e^{A\dot B}+\cdots \tag{5.7}$$

is an arbitrary vector tangent to the space $\mathcal{I}\mathcal{G}(\sigma)$. We have the following fundamental result.

Theorem 1. The functional Hamiltonian equation (5.5) and the evolution postulate (5.6) give rise to the system of equations equivalent to the variational EL equation (3.3).

Proof. At first, we compute the variation of the energy-momentum three-form E_Z. Making use of (3.3), (3.5), (3.8), and (4.7b) we obtain

$$\delta E_Z=[-{}_\xi\mathscr{L}_Z U_{A\dot B}\wedge\delta e^{A\dot B}-({}_\xi\mathscr{L}_Z P_A{}^B\wedge\delta\Gamma^A{}_B+{}_\xi\mathscr{L}_Z W_\Sigma\wedge\delta\phi^\Sigma+\text{c.c.})+\delta U_{A\dot B}\wedge{}_\xi\mathscr{L}_Z e^{A\dot B}$$
$$+(\delta P_A{}^B\wedge{}_\xi\mathscr{L}_Z\Gamma^A{}_B+\delta W_\Sigma\wedge{}_\xi\mathscr{L}_Z\phi^\Sigma+\text{c.c.})]$$
$$+(-Z\lrcorner[\delta e^{A\dot B}\wedge(E1)_{A\dot B}]-\{Z\lrcorner[\delta\Gamma^A{}_B\wedge(E2)_A{}^B]+Z\lrcorner[\delta\phi^\Sigma\wedge(EM)_\Sigma]+\text{c.c}\})$$
$$-d[(Z\lrcorner e^{A\dot B})\delta U_{A\dot B}+\delta e^{A\dot B}\wedge(Z\lrcorner U_{A\dot B})]-d[\delta\Gamma^A{}_B\wedge(Z\lrcorner P_A{}^B)+(Z\lrcorner Y^A{}_B)\delta P_A{}^B+\text{c.c.}]$$
$$-d[(-1)^{k+1}\delta\phi^\Sigma\wedge(Z\lrcorner W_\Sigma)+(Z\lrcorner\phi^\Sigma)\wedge\delta W_\Sigma+\text{c.c.}]$$
$$+d\delta[(Z\lrcorner e^{A\dot B})U_{A\dot B}]+d\{\delta[(Z\lrcorner Y^A{}_B)P_A{}^B]+\delta[(Z\lrcorner\phi^\Sigma)\wedge W_\Sigma]+\text{c.c.}\}\ . \tag{5.8}$$

We see that the first bracket of (5.8) exactly corresponds to the integrand in (5.3). Hence, neglecting the three-divergence terms we get from (5.5) and (5.8) that for arbitrary variations $\delta e^{A\dot B},\delta\Gamma^A{}_B,\delta\phi^\Sigma$ the formula holds

$$0=\int_\sigma(Z\lrcorner[\delta e^{A\dot B}\wedge(E1)_{A\dot B}]+\{Z\lrcorner[\delta\Gamma^A{}_B\wedge(E2)_A{}^B]+Z\lrcorner[\delta\phi^\Sigma\wedge(EM)_\Sigma]+\text{c.c.}\})\ . \tag{5.9}$$

If we assume that the vector field Z is transversal to σ then (5.9) gives rise to the field equations

$$(E1)_{A\dot B}=0\ ,\quad (E2)_A{}^B=0\ ,\quad (EM)_\Sigma=0\ .$$

In the above considerations we neglect three-divergences in the integral formulas. Of course, this procedure is justified if σ is a compact manifold without boundary. The field equations (3.3), however, can be derived from the Hamilton equation (5.5) also in the case of arbitrary σ's. We only have to restrict the space of the sample vectors V to those whose components have compact supports on σ. Such an assumption assures the convergence of the integrals in (5.5) and allows us to neglect three-divergences. There exists, however, another more subtle approach to the problem. Instead of the coarse condition of compact supports we may assume that the components of V asymptotically vanish at the infinity on σ. In such a case the boundary integrals contribute essentially to the generator of translations. We will discuss this problem briefly in Sec. VII.

For compact σ's the generator of the covariant translations is given by the left-hand sides of both the gravitational and matter equations. This result generalizes those presented earlier in the classical papers by Arnowitt, Deser, and Misner[31] (ADM) and several other authors [see Ref. 16 for general relativity and Ref. 22 for a general $SO_+(3,1)$ gauge theory of gravity]. In the latter paper matter was described by tensor-valued zero-forms on spacetime and therefore the left-hand sides of the matter field equations did not appear in the formulas for translational generators. The Noether current E_Z generates covariant translations in the space of symplectic variables; similarly, the Noether current I_M generates local rotations of the symplectic variables. We define the spin function on the space $\mathscr{S}\mathscr{G}(\sigma)$:

$$\mathscr{I}_M(\sigma)=\int_\sigma I_M\ . \tag{5.10}$$

We have the Hamilton equation for $\mathscr{I}_M$

$$d\mathscr{I}_M(\sigma)V=2\Omega(Y_I,V)\ , \tag{5.11}$$

where

$$Y_I=\delta_M e^{A\dot B}\partial/\partial e^{A\dot B}+(\delta_M\Gamma^A{}_B\partial/\partial\Gamma^A{}_B+\delta_M\phi^\Sigma\partial/\partial\phi^\Sigma+\text{c.c.})+\delta_M U_{A\dot B}\partial/\partial U_{A\dot B}$$
$$+(\delta_M P_A{}^B\partial/\partial P_A{}^B+\delta_M W_\Sigma\partial/\partial W_\Sigma+\text{c.c.})+\cdots \tag{5.12}$$

[δ_M is the operator of infinitesimal rotation given by (4.1)] and V is an arbitrary (sample) vector tangent to $\mathscr{S}\mathscr{G}(\sigma)$. The Hamilton equation (5.11) is the direct consequence of the formula

$$\delta I_M=\delta_M U_{A\dot B}\wedge\delta e^{A\dot B}-\delta U_{A\dot B}\wedge\delta_M e^{A\dot B}+(\delta_M P_A{}^B\wedge\delta\Gamma^A{}_B+\delta_M W_\Sigma\wedge\delta\phi^\Sigma-\delta P_A{}^B\wedge\delta_M\Gamma^A{}_B-\delta W_\Sigma\wedge\delta_M\phi^\Sigma+\text{c.c.})\ . \tag{5.13}$$

The generators $\mathscr{E}_Z$ and $\mathscr{I}_M$ fix the dynamics of the system and, as it has been shown in Ref. 22, their Hamiltonian vectors determine the gauge distribution of the symplectic form Ω. An interesting role of the spin function $\mathscr{I}_M$ in relations between the SO(3) covariant and the classical ADM Hamiltonian formulations of the Einstein-Cartan theory has been recently discussed in Ref. 38.

Our $SL(2,C)$-covariant Hamiltonian formulation treats all slicings of spacetime as equivalent (we do not assume that the slices are spacelike surfaces). Therefore, there are close relations between our approach and Kijowski-Tulczyjew's theory of symplectic spaces, their Lagrangian submanifolds and generating functions.[10] In our scheme, if σ is a compact surface then the dynamics preserves the symplectic form Ω and the motion is determined by the action of the full gauge group $G=\text{loc}\,SL(2,C)\times_b\text{Diff}M$ (see Appendix C). In the Kijowski-Tulczyjew theory, the integral (5.3) defining the symplectic two-form Ω_{KT} is taken over a closed compact surface $\sigma_{\rm cl}$. The dynamics in their approach is determined by maximal isotropic subspaces of $\Omega_{\rm KT}$ and their integral submanifolds (Lagrangian submanifolds of $\Omega_{\rm KT}$). It is clear that the Lagrangian subspaces of $\Omega_{\rm KT}$ consist of these vectors that, in our view, correspond to the dynamical Hamiltonian vectors of $\mathscr{E}_Z$ and $\mathscr{I}_M$. In the present section, we have proved directly that the functions $\mathscr{E}_Z$ and $\mathscr{I}_M$ constructed by means of the Noether currents E_Z and I_M are the dynamical generators of the theory. Another purely geometric proof of this statement was given in Ref. 35 but the complete analogy with the present results requires an SL(2, C)-covariant reformulation of the method used therein.

The pair of generators $(\mathscr{E},\mathscr{I})$ corresponding to the action of the gauge group G constitutes an object called the momentum mapping in the mathematical literature.[39] It follows from the above results that at fixed $(Z,M)\in g$ the dynamics preserves the zero-sets of the momentum mapping. Mathematical properties of the momentum mappings and their zero-sets for general relativity and the theory of Yang-Mills fields were profoundly investigated by Arms, Marsden, and Moncrief.[40]

The results of this section can be extended for Yang-Mills fields interacting with gravity and with matter fields. Let H be a semisimple Lie group, A be an h-valued connection one-form on spacetime. The action of H in the space of field potentials A and field strengths defines the H-Noether current B_h, $h\in h$ and $\mathscr{B}_h(\sigma)=\int_\sigma B_h$ is the corresponding dynamical generator. The Noether three-forms E_Z, I_M, and B_h are built by means of the left-hand sides of the gravitational, matter, and Yang-Mills equations. The Hamilton equations for $\mathscr{E}_Z,\mathscr{I}_M,\mathscr{B}_h$ determine the infinitesimal covariant translations, local SL(2, C) rotations and local H rotations of the symplectic positions e,Γ,ϕ,A and their conjugate momenta.

VI. THE ORTHOGONAL DECOMPOSITION OF DIFFERENTIAL FORMS AND THE SU(2)-COVARIANT CANONICAL VARIABLES

In the SL(2, C)-covariant Hamiltonian formulation a slicing of spacetime may be chosen arbitrarily. From the dynamical point of view, however, much more interesting are slicings determined by spacelike surfaces in M. The dynamical picture related to a spacelike slicing corresponds to the initial value problem for the system of partial differential equations of the theory. In subsequent sections we discuss initial value formulations induced by the symplectic structures of particular field theories.

Let $\{\sigma_t\}_{t\in R}$ be a fixed slicing of spacetime. From now on, we admit only such tetrad fields $e^{A\dot B}$ on spacetime that the submanifolds σ_t are spacelike surfaces with respect to the metric on M defined by $e^{A\dot B}$. Let n be a unit one-form on M orthogonal to the slicing whose orientation is consistent with the σ-transversal vector field Z (5.1), i.e.,

$$Z \lrcorner\, n>0 \tag{6.1}$$

(this condition requires the external orientability of the slicing). In subsequent considerations, a very important role is played by the decomposition of Hermitian matrices into their diagonal and traceless parts. In particular, for the tetrad field we have

$$e^{A\dot B}={}^de\sigma_{(0)}^{A\dot B}+{}^te^{A\dot B}\ . \tag{6.2}$$

The diagonal part ${}^de=\sigma_{(0)}^{A\dot B}e_{A\dot B}$ of $e^{A\dot B}$ presents a unit SU(2)-scalar one-form, i.e., $\langle {}^de,{}^de\rangle=-1$ and ${}^te^{A\dot B}$ is the traceless part of $e^{A\dot B}$, i.e., ${}^te_{A\dot B}\sigma_{(0)}^{A\dot B}=0$. We have also

$$\langle {}^de,{}^te^{A\dot B}\rangle=0\ . \tag{6.3}$$

The decomposition (6.2) is invariant with respect to SU(2) rotations of tetrads but is not SL(2, C) invariant (see Appendix B). Therefore we expect that after an appropriate local SL(2, C) rotation of the tetrad field, its diagonal part coincides with the one-form n. Let $H^A{}_B(\cdot)$ be a field of SL(2, C) matrices on spacetime such that if

$$\tilde e^{A\dot B}=H^{-1A}{}_CH^{-1\dot B}{}_{\dot D}e^{C\dot D} \tag{6.4}$$

then

$${}^d\tilde e=\tilde e_{A\dot B}\sigma_{(0)}^{A\dot B}=n\ . \tag{6.5}$$

Of course, the field $H^A{}_B(\cdot)$ is determined only up to an x dependent SU(2) factor. We recall that each element $S\in$SL(2, C) can be uniquely decomposed into the product $S=H\cdot U$ (the Cartan decomposition[41]) where H is an unimodular, positively definite, Hermitian matrix, i.e., $H\in S_+H(2)$ and $U\in$SU(2). Hence we restrict fields of SL(2, C) matrices in (6.4) to fields of $S_+H(2)$ matrices. The unique field of $S_+H(2)$ matrices satisfying (6.4) and (6.5) can be constructed in the following way. Let $n=n_{A\dot B}e^{A\dot B}$ be the decomposition of the one-form n with respect to the basis $(e^{A\dot B})$, and $n^{A\dot B}={}^dn\sigma_{(0)}^{A\dot B}+{}^tn^{A\dot B}$ be the decomposition of the Hermitian matrix $[n^{A\dot B}]$ into its diagonal and traceless parts.

We assume that the one-forms ${}^de$ and n determine the same time-orientation of spacetime, that is,

$$\langle {}^de,n\rangle<0 \quad \text{or equivalently } {}^dn>0\ . \tag{6.6}$$

Now, we observe that the matrices $[2n^{A\dot C}\sigma_{(0)B\dot C}]$ and $[2n_{B\dot C}\sigma_{(0)}^{A\dot C}]$ are Hermitian, unimodular, and positively definite. As a matter of fact, if

$$[2n^{A\dot C}\sigma_{(0)B\dot C}]=\begin{bmatrix} a & z\\ z^* & b\end{bmatrix}\quad a,b\in R,\ \ z\in C$$

then

$$\text{(i) } 1=-\langle n,n\rangle=n_{A\dot B}n^{A\dot B}=\det[2n^{A\dot C}\sigma_{(0)B\dot C}] =ab-zz^*\ ,$$

$$\text{(ii) } {}^dn=\tfrac{1}{2}(a+b)>0\ .$$

It follows from (i) and (ii) that both the eigenvalues of the matrix $[2n^{A\dot C}\sigma_{(0)B\dot C}]$ are positive, that is, it is positively definite. We may define the unimodular Hermitian and positively definite matrices

$$[H^A{}_B]=[2n^{A\dot{C}}\sigma_{(0)B\dot{C}}]^{1/2}$$

and (6.7)

$$[H^{-1A}{}_B]=[2n_{B\dot{C}}\sigma^{A\dot{C}}_{(0)}]^{1/2} .$$

It is easy to check that

$$H^A{}_B=\sqrt{2}(1+{}^dn)^{-1/2}\,{}^tn^{A\dot{C}}\sigma_{(0)B\dot{C}}+\frac{1}{\sqrt{2}}(1+{}^dn)^{1/2}\delta^A{}_B , \tag{6.8a}$$

$$H^{-1A}{}_B=\sqrt{2}(1+{}^dn)^{-1/2}\,{}^tn_{B\dot{C}}\sigma^{A\dot{C}}_{(0)}+\frac{1}{\sqrt{2}}(1+{}^dn)^{1/2}\delta^A{}_B , \tag{6.8b}$$

and that the relations (6.4) and (6.5) hold.

Remark. The set $S_+H(2)$ does not form a group. If, however, $H\in S_+H(2)$ then the inverse matrix H^{-1} also belongs to $S_+H(2)$.

The H rotation of tetrads defined by (6.4) gives rise to the following transformations of the connection coefficients and the matter field potentials:

$$\tilde{\Gamma}^A{}_B=H^{-1A}{}_CH^D{}_B\Gamma^C{}_D+H^{-1A}{}_CdH^C{}_B , \tag{6.9}$$

$$\tilde{\phi}^\Sigma=\rho^\Sigma{}_\Lambda(H^{-1})\phi^\Lambda . \tag{6.10}$$

In the language of $SO_+(3,1)$ group the H rotations correspond to the Lorentz boosts. As a matter of fact, the Cartan decomposition of the Lorentz group reads[41] $L=B\cdot R$, where $R\in SO(3)$ and B is a boost transformation. In the $SO_+(3,1)$ gauge theories, the tilde operations are realized by means of the corresponding boost transformations.[22] The field of $S_+H(2)$ matrices (6.8) is the lift of a corresponding field of the boost matrices given in Ref. 22. We would like to emphasize that such a lift always exists although, in general, we are not able to lift an arbitrary field of Lorentz matrices to a field of $SL(2,C)$ matrices. The reason is that the double covering map Λ (2.8) is induced by the double covering map $SU(2)\to SO(3)$ and the correspondence $S_+H(2)\leftrightarrow$boosts is one-to-one.

In the $SL(2,C)$ covariant formulation of theories of gravity, the field potentials are $SL(2,C)$-object-valued differential forms on spacetime. In the SU(2)-covariant picture, however, basic geometric quantities are one-parameter families of SU(2)-object-valued differential forms on three-dimensional surfaces in M. In order to decompose differential forms on spacetime into families of differential forms on slices, we need appropriate bases in the tangent and cotangent spaces of M. First of all, we assume that local coordinates in M are consistent with the slicing, that is to say, the surfaces of the slicing are labeled by a real parameter x^0 and local coordinates on particular slices are (x^s). Such a structure admits the following transformations of local coordinates (x^0,x^s):

$$x^{0\prime}=x^{0\prime}(x^0) , \quad x^{s\prime}=x^{s\prime}(x^0,x^s) . \tag{6.11}$$

In such coordinates the components of the σ-normal vector field $\mathcal{N}$ are $(1/N,-N^s/N)$, where N and N^s are ADM's lapse and shift of the metric $\underline{g}=g_{\mu\nu}dx^\mu\otimes dx^\nu$ on M (cf. Appendix A). The σ-normal one-form $n=n_\mu dx^\mu$ is the anti-Riemannian dual of $\mathcal{N}$, that is, $n_\mu=-g_{\mu\nu}N^\nu$ or equivalently $n=N\,dx^0$. We take the following basis in the tangent space of M:

$$\bar{\partial}_0=\mathcal{N}=\frac{1}{N}\partial_0-\frac{N^s}{N}\partial_x , \quad \bar{\partial}_s=\partial_s . \tag{6.12a}$$

The dual basis of one-forms on M is

$$\overline{dx}^{\,0}=n=N\,dx^0 , \quad \overline{dx}^{\,s}=dx^s+N^s\cdot dx^0 . \tag{6.12b}$$

In this basis the metric on spacetime can be written as

$$\underline{g}=-\overline{dx}^{\,0}\otimes\overline{dx}^{\,0}+g_{\bar{s}\bar{r}}\overline{dx}^{\,s}\otimes\overline{dx}^{\,r} , \quad \text{where } g_{\bar{s}\bar{r}}=g_{sr} . \tag{6.13}$$

Having an $SL(2,C)$-spinor-valued differential k-form on spacetime

$$\phi^\Sigma=\sum_{\mu_1<\cdots<\mu_k}\phi^\Sigma{}_{\mu_1\cdots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k} \tag{6.14}$$

we perform the H rotation (6.10), take the basis (6.12b), and obtain the following SU(2)-spinor-valued k form on M:

$$\tilde{\phi}^\Sigma=\sum_{\mu_1<\cdots<\mu_k}\tilde{\phi}^\Sigma{}_{\bar{\mu}_1\cdots\bar{\mu}_k}\overline{dx}^{\,\mu_1}\wedge\cdots\wedge\overline{dx}^{\,\mu_k} . \tag{6.15}$$

We decompose the k-form $\tilde{\phi}^\Sigma$ into its σ-tangential and σ-transversal parts

$$\tilde{\phi}^\Sigma={}_{\|}\tilde{\phi}^\Sigma+{}_{\perp}\tilde{\phi}^\Sigma . \tag{6.16}$$

The σ-tangential part of $\tilde{\phi}^\Sigma$

$${}_{\|}\tilde{\phi}^\Sigma=\sum_{1\le s_1<\cdots<s_k}\tilde{\phi}^\Sigma{}_{\bar{s}_1\cdots\bar{s}_k}\overline{dx}^{\,s_1}\wedge\cdots\wedge\overline{dx}^{\,s_k} \tag{6.17}$$

can be projected onto the surfaces $\{\sigma_{x^0}\}_{x^0\in R}$ and we obtain the following one-parameter family of SU(2)-spinor-valued k forms on the slices:

$${}_{\|}\hat{\phi}^\Sigma=\sum_{1\le s_1<\cdots<s_k}\hat{\phi}^\Sigma{}_{s_1\cdots s_k}dx^{s_1}\wedge\cdots\wedge dx^{s_k} , \tag{6.18}$$

where $\hat{\phi}^\Sigma{}_{s_1\cdots s_k}=\tilde{\phi}^\Sigma{}_{\bar{s}_1\cdots\bar{s}_k}$.

In order to project the σ-transversal part of $\tilde{\phi}^\Sigma$

$${}_{\perp}\tilde{\phi}^\Sigma=\sum_{1\le s_2<\cdots<s_k}\tilde{\phi}^\Sigma{}_{\bar{0}\bar{s}_2\cdots\bar{s}_k}\overline{dx}^{\,0}\wedge\overline{dx}^{\,s_2}\wedge\cdots\wedge\overline{dx}^{\,s_k} \tag{6.19}$$

onto the slices we transvect it with the normal field $\mathcal{N}$ and, then, pull back the $(k-1)$-form $\mathcal{N}\lrcorner{}_{\perp}\tilde{\phi}^\Sigma$ onto slices. We get the following one-parameter family of SU(2)-spinor-valued $(k-1)$-forms on slices:

$${}_{\perp}\hat{\phi}^\Sigma=\sum_{1\le s_2<\cdots<s_k}\hat{\phi}^\Sigma{}_{\perp s_2\cdots s_k}dx^{s_2}\wedge\cdots\wedge dx^{s_k} , \tag{6.20}$$

where $\hat{\phi}^\Sigma{}_{\perp s_2\cdots s_k}=\tilde{\phi}^\Sigma{}_{\bar{0}\bar{s}_2\cdots\bar{s}_k}$.

This procedure applied to the Hodge dual representation of a $(4-k)$-form $\tilde{A}$ (cf. Appendix A)

$$\tilde{A}=\frac{1}{k!}\frac{1}{\sqrt{g}}\tilde{\mathcal{A}}_{\bar{\mu}_1\cdots\bar{\mu}_k}*(\overline{dx}^{\,\mu_1}\wedge\cdots\wedge\overline{dx}^{\,\mu_k}) \tag{6.21}$$

gives rise to

$$ {}_{\|}\hat{A}=\frac{1}{(k-1)!\sqrt{g}}\hat{\mathscr{A}}^{\perp}{}_{s_2\cdots s_k}{}^3*(dx^{s_2}\wedge\cdots\wedge dx^{s_k}) , \quad (6.22)$$

$$ {}_{\perp}\hat{A}=\frac{(-1)^k}{k!\sqrt{g}}\hat{\mathscr{A}}_{s_1\cdots s_k}{}^3*(dx^{s_1}\wedge\cdots\wedge dx^{s_k}) , \quad (6.23)$$

where $\hat{\mathscr{A}}^{\perp}{}_{s_2\cdots s_k}=\tilde{\mathscr{A}}^{0}{}_{\bar{s}_2\cdots\bar{s}_k}$, $\hat{\mathscr{A}}_{s_1\cdots s_k}=\tilde{\mathscr{A}}_{\bar{s}_1\cdots\bar{s}_k}$, and the ^{3*} is the Hodge dual on σ (cf. Appendix A).

If we apply formulas (6.2) and (6.3) to the tetrad one-form $\tilde{e}^{A\dot{B}}$ and perform the (3 + 1) decomposition then we get

$$ {}^{d}_{\|}\hat{e}=0={}^{t}_{\perp}\hat{e}^{A\dot{B}} ,\quad {}^{d}_{\perp}\hat{e}=1={}^{d}\hat{e}_{\perp} ,$$
$$ {}^{t}\hat{e}^{A\dot{B}}={}_{\|}\hat{e}^{A\dot{B}}={}^{t}\hat{e}^{A\dot{B}}{}_{s}dx^{s} . \quad (6.24)$$

It means that instead of 16 tetrad components $e^{A\dot{B}}{}_{\mu}$ we have only 9 independent triad components ${}^{t}\hat{e}^{A\dot{B}}{}_{s}$ (the components of the triad one-form ${}_{\|}\hat{e}^{A\dot{B}}$ on slices). The remaining independent variables are the lapse N and the shift N^s as well as the three components ${}^{t}n_{A\dot{B}}$.

Remark. By virtue of the relation ${}^{d}n=(1-{}^{t}n_{A\dot{B}}\cdot{}^{t}n^{A\dot{B}})^{1/2}$, the quantity ${}^{d}n$ is not independent.

A connection one-form $\Gamma^{A}{}_{B}$ generates the family ${}_{\perp}\hat{\Gamma}^{A}{}_{B}=\hat{\Gamma}^{A}{}_{B\perp}$ of zero-forms on slices, and the family ${}_{\|}\hat{\Gamma}^{A}{}_{B}=\hat{\Gamma}^{A}{}_{Bs}dx^{s}$ of one-forms on slices. These quantities can be decomposed into their Hermitian and anti-Hermitian parts

$$ {}_{\perp}\hat{\Gamma}^{A}{}_{B}={}^{h}_{\perp}\hat{\Gamma}^{A}{}_{B}+{}^{a}_{\perp}\hat{\Gamma}^{A}{}_{B} ,\quad {}_{\|}\hat{\Gamma}^{A}{}_{B}={}^{h}_{\|}\hat{\Gamma}^{A}{}_{B}+{}^{a}_{\|}\hat{\Gamma}^{A}{}_{B} . \quad (6.25)$$

The su(2)-valued one-form ${}^{a}_{\|}\hat{\Gamma}^{A}{}_{B}={}^{a}\hat{\Gamma}^{A}{}_{Bs}dx^{s}$ is the connection one-form on slices generated by the connection one-form $\Gamma^{A}{}_{B}$ on M (Refs. 8 and 42). In the subsequent section we show that the quantities N, N^s, ${}^{a}\hat{\Gamma}^{A}{}_{B\perp}$, and ${}^{t}n_{A\dot{B}}$ have a clear geometric interpretation—they are the gauge variables corresponding to the action of the Diff$M$, the local SU(2) and the local $S_{+}H(2)$ rotations, respectively.

We would like to recall that in the standard ADM formulation, where the x^0 coordinate is fixed, the lapse N represents a family of functions on slices and the components of shift N^s represent a family of vector fields on slices. In our approach, where we admit more general transformations (6.11) of local coordinates in M, these quantities have much more complicated transformation rules (cf. Appendix A). However, the quantity ${}^{3}d\ln N=\partial_s\ln N\cdot dx^s$ defines a family of one-forms on slices even if we admit transformations (6.11). This fact will be often used in subsequent sections.

It is interesting to note that the transition from the SL(2, C) picture to the SU(2) one can be described as a symmetry-breaking process.[8]

VII. THE SU(2)-COVARIANT HAMILTONIAN FORMULATION OF GAUGE THEORIES OF GRAVITY

We have shown in Sec. V that Lagrangian SL(2, C) gauge theories of gravity carry a natural Hamiltonian structure. Now, we discuss a special case of the general Hamiltonian theory presented there. We assume that the slicing of spacetime $\{\sigma_t\}$ consists of spacelike, compact surfaces without boundaries, and deal with SU(2)-object-valued differential forms on slices. The technique of the (3 + 1) decomposition developed in the previous section, enables us to reformulate the results of Sec. V and to obtain the (3 + 1) form of field equations. In terms of (3 + 1) variables we get the following formula for the symplectic two-form (5.3):

$$\Omega(\sigma)(X_1,X_2)=\int_{\sigma}[\delta_1{}^{t}_{\|}\hat{U}_{A\dot{B}}\wedge\delta_2{}^{t}_{\|}\hat{e}^{A\dot{B}}+(\delta_1{}_{\|}\hat{P}_{A}{}^{B}\wedge\delta_2{}_{\|}\hat{\Gamma}^{A}{}_{B}+\delta_1{}_{\|}\hat{W}_{\Sigma}\wedge\delta_2{}_{\|}\hat{\phi}^{\Sigma}+\text{c.c.})+\delta_1 m^{A\dot{B}}\wedge\delta_2{}^{t}_{\|}n_{A\dot{B}}] , \quad (7.1)$$

where $m^{A\dot{B}}$ is an ${}^{t}H(2)$-valued three-form on σ given by

$$m^{A\dot{B}}=-2{}^{h}_{\|}(\hat{E}2)_{C}{}^{A}\cdot\sigma^{C\dot{B}}_{(0)}-2\cdot{}^{t}n^{A\dot{B}}\cdot{}^{t}n_{C\dot{D}}\cdot{}^{d}n^{-1}(1+{}^{d}n)^{-1}{}^{h}_{\|}(\hat{E}2)_{E}{}^{C}\cdot\sigma^{E\dot{D}}_{(0)}+(1+{}^{d}n)^{-1}[{}^{t}n^{C\dot{B}}{}^{a}_{\|}(\hat{E}2)_{C}{}^{A}+{}^{t}n^{A\dot{C}}{}^{a}_{\|}(\hat{E}2)_{\dot{C}}{}^{\dot{B}}] . \quad (7.2)$$

In local coordinates $m^{A\dot{B}}=\mathscr{m}^{A\dot{B}}dx^1\wedge dx^2\wedge dx^3$. The differential forms ${}^{t}_{\|}\hat{e}^{A\dot{B}}$, ${}_{\|}\hat{\Gamma}^{A}{}_{B}$, ${}_{\|}\hat{\phi}^{\Sigma}$, ${}^{t}n_{A\dot{B}}$, ${}^{t}_{\|}\hat{U}_{A\dot{B}}$, ${}_{\|}\hat{P}_{A}{}^{B}$, ${}_{\|}\hat{W}_{\Sigma}$, and $m^{A\dot{B}}$ on σ, involved in (7.1), are called (3 + 1)-symplectic variables of the theory under consideration. In local coordinates they are represented by the following quantities (see Appendix D): ${}^{t}\hat{e}^{A\dot{B}}{}_{s}$, $\hat{\Gamma}^{A}{}_{Bs}$, $\hat{\phi}^{\Sigma}{}_{s_1\cdots s_k}$, ${}^{t}n_{A\dot{B}}$, ${}^{t}\hat{\mathscr{U}}_{A\dot{B}}{}^{\perp s}$, $\hat{\mathscr{P}}_{A}{}^{B\perp s}$, $\hat{\mathscr{W}}_{\Sigma}{}^{\perp s_1\cdots s_k}$, $\mathscr{m}^{A\dot{B}}$, and formula (7.1) can be rewritten as

$$\Omega(\sigma)(X_1,X_2)=\int_{\sigma}[\delta_1{}^{t}\hat{\mathscr{U}}_{A\dot{B}}{}^{\perp s}\wedge\delta_2{}^{t}\hat{e}^{A\dot{B}}{}_{s}+(\delta_1\hat{\mathscr{P}}_{A}{}^{B\perp s}\wedge\delta_2\hat{\Gamma}^{A}{}_{Bs}+\delta_1\hat{\mathscr{W}}_{\Sigma}{}^{\perp s_1\cdots s_k}\wedge\delta_2\hat{\phi}^{\Sigma}{}_{s_1\cdots s_k}+\text{c.c.})$$
$$+\delta_1\mathscr{m}^{A\dot{B}}\wedge\delta_2{}^{t}n_{A\dot{B}}]dx^1\wedge dx^2\wedge dx^3 . \quad (7.1')$$

Remarks. (i) In (7.1′) the symbol "∧" denotes only the antisymmetrization with respect to subscripts 1 and 2. (ii) Passing from (5.3) to (7.1) we have omitted some three-divergence terms (cf. Ref. 22). The formula for the energy-momentum function $\mathscr{E}(\sigma)$ reads

$$\mathscr{E}_{Z}(\sigma)=-\int_{\sigma}\{{}^{\perp}Z\cdot{}^{d}_{\|}(\hat{E}1)+({}^{\|}Z\;\lrcorner\;{}^{t}_{\|}\hat{e}^{A\dot{B}})\cdot{}^{t}_{\|}(\hat{E}1)_{A\dot{B}}$$
$$+[({}^{\perp}Z\,{}_{\perp}\hat{Y}^{A}{}_{B})\cdot{}_{\|}(\hat{E}2)_{A}{}^{B}+({}^{\|}Z\;\lrcorner\;{}_{\|}\hat{Y}^{A}{}_{B})\cdot{}_{\|}(\hat{E}2)_{A}{}^{B}+({}^{\perp}Z{}_{\perp}\hat{\phi}^{\Sigma})\wedge{}_{\|}(\hat{E}M)_{\Sigma}+({}^{\|}Z\;\lrcorner\;{}_{\|}\hat{\phi}^{\Sigma})\wedge{}_{\|}(\hat{E}M)_{\Sigma}+\text{c.c.}]\} . \quad (7.3)$$

Remark. The σ-normal and σ-tangential components of the vector field Z are given by ${}^{\perp}Z = Z \lrcorner n = NZ^0$, ${}^{\parallel}Z = Z - {}^{\perp}Z\mathcal{N} = (Z^s + N^s \cdot Z^0)\partial_s$.

In the $SL(2,C)$-covariant Hamiltonian approach, the components of the evolution vector Y_E are given by means of the covariant Lie derivatives of corresponding geometric quantities (5.6). Therefore we need $(3+1)$-decompositions of $SL(2,C)$-covariant Lie derivatives. In a general case, when Z is an arbitrary vector field on spacetime and $\zeta^A{}_B$ is a metric compatible connection, the corresponding formulas were presented in Ref. 38. Those formulas have a relatively simple form if (i) transformations of M generated by the vector field Z preserve the slicing. That is, if

$$Z^0 = Z^0(x^0)\ , \quad Z^s = Z^s(x^0, x^s)\ . \tag{7.4}$$

(ii) The auxiliary connection $\zeta^A{}_B$ is consistent with the slicing, i.e.,

$${}_{\zeta}Dn_{A\dot{B}} = 0 \quad \text{or} \quad {}^{h}_{\perp}\hat{\zeta}^A{}_B = 0 = {}^{h}_{\parallel}\hat{\zeta}^A{}_B\ . \tag{7.5}$$

The conditions (7.4) and (7.5) are invariant with respect to the coordinate transformations (6.11) and with respect to local SU(2) rotations. If the consistency conditions (7.4) and (7.5) hold then the σ-parallel part of the $SL(2,C)$-covariant Lie derivative of a spinor-valued k-form ϕ^Σ on M consists of two terms. One of them is the SU(2)-covariant "time" derivative ${}^3_{\zeta}\mathcal{D}_0$ of ${}_{\parallel}\hat{\phi}^\Sigma$ computed with respect to the ∂_0 connection ${}^{a}_{\perp}\hat{\zeta}^A{}_B$ and the latter is the SU(2)-covariant Lie derivative of the k-form ${}_{\parallel}\hat{\phi}^\Sigma$ on σ, computed with respect to the ∂_s connection ${}^{a}_{\parallel}\hat{\zeta}^A{}_B$ and taken in the direction of the σ-parallel part of Z (see Appendix E). Similarly, for a spinor-valued field of k-vector densities on M $\mathcal{F}^{\Sigma\mu_1\cdots\mu_k}$ the σ-transversal part of its $SL(2,C)$-covariant Lie derivative splits into the SU(2)-covariant time derivative ${}^3_{\zeta}\mathcal{D}_0$ of ${}^{\perp}\hat{\mathcal{F}}^\Sigma$ and the SU(2)-covariant Lie derivative of ${}^{\perp}\hat{\mathcal{F}}^\Sigma$, computed with respect to the connection ${}^3\zeta = {}^{a}_{\parallel}\hat{\zeta}^A{}_B$, in the direction of ${}^{\parallel}Z$ (see Appendix E).

Therefore, we may expect that in the $(3+1)$ picture the sympletic components of the evolution vector Y_E are given by means of the relations

$$\delta_Y \hat{e}^{A\dot{B}}{}_s = {}^{\perp}Z\, {}^3_{\zeta}\mathcal{D}_0 \hat{e}^{A\dot{B}}{}_s + {}^3_{\zeta}\mathcal{L}_Z \hat{e}^{A\dot{B}}{}_s \tag{7.6}$$

and analogous formulas for other $(3+1)$ symplectic variables. The conditions (i) and (ii) give rise to the relation

$$\delta_Y {}^t n_{A\dot{B}} = 0\ . \tag{7.7}$$

Remark. For a general Z and ζ we have

$$\delta_Y n^{A\dot{B}} = Z \lrcorner {}_{\zeta}Dn^{A\dot{B}} + e^{A\dot{B}}{}_s \cdot N \cdot \partial^s Z^0\ .$$

We have an SU(2)-covariant version of Theorem 1.

Theorem 1. In the $(3+1)$ picture, the Hamilton equation (5.5) and the evolution postulates (7.6) and (7.7) give rise to the system of equations

$${}_{\parallel}(\hat{E}1)_{A\dot{B}} = 0\ , \quad {}^t_{\perp}(\hat{E}1)_{A\dot{B}} = 0\ , \quad {}_{\parallel}(\hat{E}2)_A{}^B = 0\ , {}_{\perp}(\hat{E}2)_A{}^B = 0\ ,$$
$${}_{\parallel}(\hat{E}M)_\Sigma = 0\ , \quad {}_{\perp}(\hat{E}M)_\Sigma = 0\ , \quad {}^3_{\zeta}\mathcal{D}_0 m^{A\dot{B}} = 0\ . \tag{7.8}$$

It follows from (7.2) and the above formulas that $m^{A\dot{B}} = 0$ on σ.

Proof. The variations of the energy-momentum function (7.3) can be expressed in the form

$$\begin{aligned}
\delta\mathcal{E}_Z = \int_\sigma \Bigg\{ & (-\delta_Y {}^t\hat{\mathcal{U}}_{A\dot{B}}{}^{\perp s}\delta^t e^{A\dot{B}}{}_s + \delta_Y {}^t\hat{e}^{A\dot{B}}{}_s \delta^t\hat{\mathcal{U}}_{A\dot{B}}{}^{\perp s}) + (-\delta_Y \hat{\mathcal{P}}_A{}^{B\perp s}\delta\hat{\Gamma}^A{}_{Bs} + \delta_Y \hat{\Gamma}^A{}_{Bs}\delta\hat{\mathcal{P}}_A{}^{B\perp s} + \text{c.c.}) \\
& + \frac{1}{k!}(-\delta_Y \hat{\mathcal{W}}_\Sigma{}^{\perp s_1\cdots s_k}\delta\hat{\phi}^\Sigma{}_{s_1\cdots s_k} + \delta_Y \hat{\phi}^\Sigma{}_{s_1\cdots s_k}\delta\hat{\mathcal{W}}_\Sigma{}^{\perp s_1\cdots s_k} + \text{c.c.}) \\
& - {}^{\perp}Z \Bigg[{}^t(\hat{\mathcal{E}}1)_{A\dot{B}}{}^s \delta^t\hat{e}^{A\dot{B}}{}_s + ((\hat{\mathcal{E}}2)_A{}^{Bs}\delta\hat{\Gamma}^A{}_{Bs} + \text{c.c.}) \\
& \qquad + \frac{\delta N}{N}[{}^d(\hat{\mathcal{E}}1)^{\perp} + ((\hat{\mathcal{E}}2)_A{}^{B\perp}\cdot\hat{Y}^A{}_{B\perp} + \text{c.c.})] + \delta\hat{\Gamma}^A{}_{B\perp}((\hat{\mathcal{E}}2)_A{}^{B\perp} + \text{c.c.}) \\
& \qquad + \frac{\delta N^s}{N}[{}^t\hat{e}^{A\dot{B}}{}_s \cdot {}^t(\hat{\mathcal{E}}1)_{A\dot{B}}{}^{\perp} + ((\hat{\mathcal{E}}2)_A{}^{B\perp}\cdot\hat{Y}^A{}_{Bs} + \text{c.c.})] \\
& \qquad + \left[\frac{1}{k!}(\hat{\mathcal{E}}\mathcal{M})_\Sigma{}^{s_1\cdots s_k}\delta\hat{\phi}^\Sigma{}_{s_1\cdots s_k} + \frac{1}{(k-1)!}(\hat{\mathcal{E}}\mathcal{M})_\Sigma{}^{\perp s_2\cdots s_k}\delta\hat{\phi}^\Sigma{}_{\perp s_2\cdots s_k} + \text{c.c.})\right]\Bigg] \\
& + \text{three-divergences} \Bigg\} dx^1 \wedge dx^2 \wedge dx^2\ .
\end{aligned} \tag{7.9}$$

In the above formula $(\hat{\mathcal{E}}1)_{A\dot{B}}{}^s$, etc., denote the coordinate representations of the left-hand sides of the field equations (cf. Appendix D). Taking into account that variations $\delta^t\hat{e}^{A\dot{B}}{}_s, \delta\hat{\Gamma}^A{}_{Bs}, \delta\hat{\Gamma}^{\dot{A}}{}_{\dot{B}s}, \delta N, \delta N^s, \delta^t n_{A\dot{B}}, \delta\hat{\phi}^\Sigma{}_{s_1\cdots s_k}$, $\delta\hat{\phi}^{\dot{\Sigma}}{}_{s_1\cdots s_k}, \delta\hat{\phi}^\Sigma{}_{\perp s_1\cdots s_k}, \delta\hat{\phi}^{\dot{\Sigma}}{}_{\perp s_1\cdots s_k}, \delta\hat{\Gamma}^A{}_{B\perp}, \delta\hat{\Gamma}^{\dot{A}}{}_{\dot{B}\perp}$ are arbitrary and comparing (7.1), (7.6), and (7.7) we get the result.

The analysis of Eqs. (7.8) shows that we lack the equation

$${}^d_{\perp}(\hat{E}1) = 0 \ [\text{or } {}^d(\hat{\mathcal{E}}1)^s = 0]\ . \tag{7.10}$$

If, however, we apply the ${}^3\mathcal{D}_0$ operator to formula (7.2) and make use of Eqs. (7.8) and the contracted Bianchi identities (3.9b) then we get the missing equation (7.10).

Remark. The time-maintenance condition ${}^{3}_{\xi}\mathscr{D}_0 m^{A\dot{B}}=0$ gives rise, in fact, to the equations equivalent to a part of the contracted Bianchi identities (3.9). It follows from the $(3+1)$ form of (3.9) that (7.10) is the consequence of the remaining field equations. The $(3+1)$-Hamiltonian formulation naturally divides the field equations into two groups

$$_{\|}(\hat{E}1)_{A\dot{B}}=0\ ,\quad {}^{a}_{\|}(\hat{E}2)_A{}^B=0\ ,\quad {}_{\|}(\hat{E}M)_\Sigma=0\ , \tag{7.11a}$$

$$\begin{gathered} {}^{t}_{\perp}(\hat{E}1)_{A\dot{B}}=0\ ,\quad {}_{\perp}(\hat{E}2)_A{}^B=0\ ,\\ {}_{\perp}(\hat{E}M)_\Sigma=0\ ,\quad m^{A\dot{B}}=0\ . \end{gathered} \tag{7.11b}$$

Remark. In the set (7.11b) we may replace the equation $m^{A\dot{B}}=0$ by ${}^{h}_{\|}(\hat{E}2)_A{}^B=0$ or by ${}^{d}_{\perp}(\hat{E}1)=0$.

The following result shows that such a decomposition of the system of field equations is very important for the initial value formulation of the theory.

Theorem 2. If Eqs. (7.11a) are satisfied on an initial surface σ, Eqs. (7.11b) and the equations

$$_{\perp}(D(EM)_{\hat{\Sigma}}=0 \tag{7.11c}$$

are satisfied on spacetime then Eqs. (7.11a) are also satisfied on the whole M.

This result follows from the contracted Bianchi identities (3.9) and it generalizes a similar result in the Einstein theory of gravity.[31] The essential difference that appears in our general scheme, in comparison with the previous results, is condition (7.11c). In theories with matter potentials described by zero-forms, Eq. (7.11c) is trivial for $D(EM)_\Sigma$ is a five-form on spacetime. In such cases we get a simpler version of Theorem 2 formulated in Ref. 22. Also in some important cases, e.g., in electrodynamics, Eq. (7.11c) holds automatically. In a general case, however, this equation is not trivial.

Bearing in mind the statement of Theorem 2, we might expect that Eqs. (7.11a) are the initial value constraints of the theory and Eqs. (7.11b) and (7.11c) define the evolution of the system. Such a classification, however, is not true in general.

In further considerations we are going to describe the constraints for initial values of symplectic variables and the dynamical evolution of initial data. First of all, we eliminate the variables $m^{A\dot{B}}$ from the theory setting them, in virtue of (7.11b), equal to zero. After this partial reduction of the system we have the following $(3+1)$-symplectic variables:

$${}^{t}_{\|}\hat{e}^{A\dot{B}}\ ,\ {}^{t}_{\|}\hat{U}_{A\dot{B}}\ ,\ {}_{\|}\hat{\Gamma}^{A}{}_{B}\ ,\ {}_{\|}\hat{P}_A{}^B\ ,\ {}_{\|}\hat{\phi}^{\Sigma}\ ,\ {}_{\|}\hat{W}_\Sigma\ . \tag{7.12}$$

Now, the main problem is whether we are able to express the left-hand sides of Eqs. (7.11a) by symplectic variables (7.12). The answer to this question depends on the properties of the σ-tangential part of the matter current ${}_{\|}\hat{J}_\Sigma$. If ${}_{\|}\hat{J}_\Sigma$ can be expressed by symplectic variables (7.12) then the same is true for the left-hand sides of Eqs. (7.11a). For some theories it is the case. In electrodynamics, the situation is trivial for $J=0$. A nontrivial example is provided by the Rarita-Schwinger Lagrangian

$$\begin{aligned} L_{\rm RS}=\frac{1}{\sqrt{2}}(\phi^A\wedge e_{A\dot{B}}\wedge D\psi^{\dot{B}}+D\psi_{\dot{A}}\wedge e^{\dot{A}B}\wedge\phi_B)\\ +m[\phi^A\wedge *\psi_A+2*e_{A\dot{B}}\wedge\phi^A*(\psi_C\wedge *e^{C\dot{B}})]+\text{c.c.}\ , \end{aligned} \tag{7.13}$$

where ϕ^A and $\psi_{\dot{A}}$ are Weyl-spinor-valued one-forms on spacetime. Of course, there are also theories for which the σ-tangential component of the matter current cannot be expressed by the symplectic variables. At the end of the present section we discuss such an example—the Proca field.

If ${}_{\|}\hat{J}_\Sigma$ cannot be directly expressed in terms of symplectic variables then we compute it from the field equation ${}_{\|}(\hat{E}M)_\Sigma=0$. We obtain

$${}_{\|}\hat{J}_\Sigma=(-1)^k{}_{\|}(DW_\Sigma)\hat{}\ . \tag{7.14}$$

It follows from the formulas of Appendix F that the right-hand side of (7.14) is a function of symplectic variables and their σ-tangential derivatives. The relations (7.14) enable us to eliminate ${}_{\|}\hat{J}_\Sigma$ from the formula for the σ-tangential part of the matter energy-momentum three-form ${}_{\|}\hat{T}_{A\dot{B}}$ [cf. (3.8b)], moreover, the σ-tangential component of the spin three-form ${}_{\|}\hat{s}_A{}^B$ can be expressed by the symplectic variables ${}_{\|}\hat{\phi}^\Sigma$ and ${}_{\|}\hat{W}_\Sigma$, cf. (3.4). Finally, we are able to express the left-hand sides of the gravitational constraints

$${}_{\|}(\hat{E}1)_{A\dot{B}}=0\ ,\quad {}^{a}_{\|}(\hat{E}2)_A{}^B=0 \tag{7.15}$$

by means of symplectic variables (7.12). In a general case, that is, when the equation ${}_{\|}(\hat{E}M)_\Sigma=0$ is not a symplectic constraint, field equations (7.11) can be divided into three groups:

(A) The dynamical equations for symplectic variables;

(B) the Hamiltonian symplectic constraints (7.15) preserved in the process of evolution;

(C) other nondynamical equations.

Also the field variables form three distinct sets:

(i) The symplectic variables (7.12);

(ii) the gauge variables $N, N^s, {}^{a}_{\perp}\Gamma^{A}{}_{B}, {}^{t}n_{A\dot{B}}$;

(iii) other nondynamical variables.

The nondynamical equations (C) involve values of the fields on the surface σ and do not involve the time derivatives of field variables. If we pose the initial value problem on a surface σ then we have to: first, specify values of the symplectic variables on σ in such a way that they satisfy the Hamiltonian constraints (7.15) and second, fix spacetime values of the gauge variables.

Serious troubles cause the nondynamical field variables (iii) and the nondynamical equations (C). In some cases we are able to eliminate the nondynamical variables by means of the equations (C). A very important example is provided by the Einstein-Cartan-Dirac theory (see Secs. VIII and IX). There are, however, examples where a direct elimination of the nondynamical variables and nondynamical equations is not possible. In such cases we time-differentiate the nondynamical equations (C) and the obtained time-maintenance conditions help us to determine the evolution of nondynamical variables. Nontrivial examples of such a procedure have been recently considered in Ref. 43 for the ECSK theory with vector matter fields.

In a general case, we have seven gravitational symplectic constraints (7.15) and ten gauge variables. We should, however, remember that for the complete set of symplectic variables (7.3) we have three additional Hamiltonian constraints

$$m^{A\dot{B}}=0 \; . \tag{7.16}$$

That is, the total number of symplectic constraints is at least 10. Therefore, we may expect a close relation between 10 symplectic constraints (7.15), (7.16), and 10 gauge variables

$$N, N^s, {}^a_{\perp}\hat{\Gamma}^A{}_B \; , \; {}^t n_{A\dot{B}} \; . \tag{7.17}$$

First of all, we explain why quantities (7.17) are called gauge variables. We recall that in its SL(2, C)-covariant formulation the theory is invariant with respect to the full gauge group $G = \mathrm{loc}\, \mathrm{SL}(2,C) \times_b \mathrm{Diff}M$. This implies that the set of solutions of field equations is determined up to a transformation parametrized by elements of G. In the infinitesimal picture, we pass from the action of G in the set of solutions of field equations to the action of the Lie algebra g of G in the set of solutions of the linearized field equations. It can be proven directly that the action of g in the set of linearized solutions generates an isomorphism between g and the tangent space of the space of gauge variables (7.17). That is, the mapping

$$g \ni (Z,M) \to ((\delta_Z+\delta_M)N, (\delta_Z+\delta_M)N^s, (\delta_Z+\delta_M){}^a\hat{\Gamma}^A{}_{B\perp}, (\delta_Z+\delta_M){}^t n_{A\dot{B}}) \tag{7.18}$$

defined by means of (4.1) and (4.2), is one-to-one.

The detailed analysis of the map (7.18) shows that infinitesimal covariant translations generated by the vector field Z on spacetime lead to arbitrary values of $\delta_Z N, \delta_Z N^s$. That is why, we call N and N^s the translational gauge variables. Infinitesimal rotations generated by x-dependent fields of matrices M give rise to arbitrary values of $\delta_M {}^a\hat{\Gamma}^A{}_{B\perp}$ and $\delta_M {}^t n_{A\dot{B}}$. The quantities ${}^a\hat{\Gamma}^A{}_{B\perp}$ and ${}^t n_{A\dot{B}}$ are called the rotational gauge variables corresponding to local SU(2) rotations and to local $S_+H(2)$ rotations (boost transformations), respectively.

Remark. The role of gauge variables N, N^s (or equivalently $g_{0\mu}$) in the classical Einstein-Hilbert theory was investigated already at the very beginning of its development. Later several authors contributed to the problem, see Refs. 31, 44, and 45. The diffeomorphism group approach to gauge variables in classical gravity was discussed in Refs. 12, 46, 35, and 47. For a general $\mathrm{SO}_+(3,1)$ gauge theory the rotational gauge variables were found in Ref. 22. In that paper, however, instead of quantities ${}^a\hat{\Gamma}^A{}_{B\perp}$ and ${}^t n_{A\dot{B}}$ the equivalent quantities $\Gamma^A{}_{B0}$ were considered.

We would like to point out that the number of symplectic constraints can be essentially larger than the number of Hamiltonian constraints (7.15) and (7.16). The reason is that a degeneracy of the gravitational or matter Lagrangian can lead to kinematical (primary) constraints in the set of symplectic variables (7.12).

If all the kinematical constraints are known, then we search for constraints in the complete set of symplectic variables. It is clear that the gauge variables ${}^t n_{A\dot{B}}$ are completely independent quantities; they do not enter field equations (7.8) and have no relations to other elements of complete set of (3 + 1)-symplectic variables. A much more subtle question is whether we could expect any relation between the variables $m^{A\dot{B}}$ and the elements of (7.12). The analysis of formula (7.2) shows that for some classes of gravitational Lagrangians the variables $m^{A\dot{B}}$ are functions of the reduced symplectic variables (7.12). An example is provided by the Hehl-von der Heyde Lagrangian[21]

$$K_{HH} = -\tfrac{1}{2}[\Theta^{A\dot{B}} \wedge *\Theta_{A\dot{B}} + (e_{C\dot{D}} \wedge *\Theta^{C\dot{D}}) \wedge *(e_{A\dot{B}} \wedge *\Theta^{A\dot{B}})] - \left[\frac{1}{4a}\Omega^A{}_B \wedge *\Omega^B{}_A + \text{c.c.} \right] \tag{7.19a}$$

and by the Yang Lagrangian

$$K_Y = \tfrac{1}{4}(\Omega^A{}_B \wedge *\Omega^B{}_A + \text{c.c.}) \tag{7.19b}$$

as well.

Our analysis shows that if the kinematics of the theory, that is, the definition of symplectic variables does not yield symplectic constraints then the only symplectic constraints are those given by dynamics. In the vacuum case, when matter is absent, we have ten Hamiltonian constraints (7.15), (7.16), and three of these can always be reduced. The situation changes for some special classes of matter Lagrangians. As we have pointed earlier, in some cases the left-hand side of the equation

$$_{\parallel}(\hat{E}M)_{\Sigma} = 0 \tag{7.20}$$

is a function of symplectic variables (7.12) and their σ-tangential derivatives. In such a case the set of Hamiltonian constraints (B) consists of Eqs. (7.15), (7.16), and (7.20). If, simultaneously, the time-maintenance condition (7.11c) is a direct consequence of the (remaining) field equations then the Hamiltonian constraints (B) are preserved in time automatically. Then we expect an additional gauge invariance of the theory and an additional set of gauge variables. A standard example is given by Maxwell electrodynamics (see below).

Example. The Proca field and Maxwell field. The electromagnetic potential is an R-valued one-form $A = A_\mu dx^\mu$. The Lagrangian reads

$$L = \tfrac{1}{2}F \wedge *F + \frac{m^2}{2} A \wedge *A = \sqrt{-g}\left[\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A_\mu A^\mu \right] \times dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \; ,$$

where $F=dA=\frac{1}{2}F_{\mu\nu}dx^{\mu}\wedge dx^{\nu}, F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$. The momenta are equal to $\mathscr{W}^{\mu\nu}=\sqrt{-g}\,F^{\mu\nu}$ (or $W=*F$) and the matter current $\mathscr{J}^{\mu}=m^2\sqrt{-g}\,A^{\mu}$ (or $J=m^2*A$).

Remark. For real field we use a modified version of (3.2) that does not contain (c.c.) terms. The symplectic variables are $\hat{A}_s$ and $\hat{\mathscr{E}}^{s}=\sqrt{g}\hat{F}^{\perp s}$. If $m\neq 0$ then the σ-tangential part of the matter current $_{\|}\hat{J}=m^2\sqrt{g}\hat{A}^{\perp}dx^1\wedge dx^2\wedge dx^3$, cannot be expressed by symplectic variables. Equation (7.20) reads

$$\sqrt{g}\;m^2\hat{A}^{\perp}+\partial_s\hat{\mathscr{E}}^{s}=0\;. \tag{7.21}$$

In the case of the Proca field, that is, if $m\neq 0$, the constraint (7.21) is not symplectic. This equation belongs to the system (C) of nondynamical equations. Fortunately, this equation can be solved for the nonsymplectic variable $\hat{A}^{\perp}$ and thereby eliminated. Simultaneously, the nondynamical variable $\hat{A}^{\perp}$ is reduced.

If $m=0$, Maxwell electrodynamics, the situation is completely different. Equation (7.21) gives us a symplectic constraint. The theory is invariant with respect to the gradient gauge transformation $A\rightarrow A'=A+d\chi$ and $\hat{A}^{\perp}$ is the gauge variable corresponding to this gauge transformation.

The three-form of energy-momentum for the Proca-Maxwell field is given by the formula

$$T_{A\dot{B}}=*L\wedge *e_{A\dot{B}}-*F_{\wedge}*(e_{A\dot{B}}\wedge *F)$$
$$-m^2*A\wedge *(e_{A\dot{B}}\wedge *A)\;. \tag{7.22}$$

It is an easy exercise to check that $T_{A\dot{B}}$ fulfills the symmetry condition (3.10). In fact we have

$$\mathscr{T}_{\mu\nu}=g_{\mu\nu}L-\sqrt{-g}\;F_{\mu\lambda}F_{\nu}{}^{\lambda}+m^2\sqrt{-g}\;A_{\mu}A_{\nu}\;. \tag{7.22$'$}$$

For spatially closed spacetimes, the energy-momentum function $\mathscr{E}$ is the generator of spacetime translations. The function $\mathscr{E}$, however, vanishes when computed on solutions of field equations. That is, the energy-momentum of spatially closed universes is zero.[48] In spatially open cases, the field equations can be obtained from the Hamiltonian formulation only if appropriate boundary conditions are imposed. If we assume that solutions of field equations are asymptotically flat at the spatial infinity then the dynamics can be determined by the Hamilton equation (5.5) with a modified energy-momentum function $\mathscr{E}_{\rm tot}(\sigma)$. Now, in the Hamilton analysis only some boundary integrals vanish and the formula for $\mathscr{E}_{\rm tot}(\sigma)$ contains volume integrals over σ as well as boundary integrals over the asymptotic boundary of σ. For the Einstein theory this procedure gives rise to the ADM energy-momentum formula.[31,49] Because of the recent results concerning the positivity of the ADM energy[50] it would be interesting to obtain asymptotic energy-momentum formulas for other SL(2,C) gauge theories of gravity. This problem can be solved by analyzing boundary terms in (5.8). The question whether the *ad hoc* imposed boundary conditions are reasonable requires the knowledge of several examples of exact solutions for particular theories. Fortunately, some attempts in this direction have been already done.[51]

While the above-mentioned papers present the standard Hamiltonian formulation, a new, interesting approach to the canonical formulation of general relativity in asymptotically flat spacetimes has been recently given by Ashtekar and Horowitz.[52] In this paper, the authors by means of Witten's spinor technique define natural asymptotically constant time-translations and find the corresponding dynamical generator.

VIII. SPINOR MATTER FIELDS AND THEIR SYMPLECTIC VARIABLES

As we already know, the invariance of the theory in question with respect to the action of the gauge group $G=\mathrm{loc\,SL}(2,C)\times_b \mathrm{Diff}M$ gives rise to ten kinematical (primary) gravitational constraints

$$\mathscr{U}_{A\dot{B}}{}^{00}=0\;,\quad \mathscr{P}_{A}{}^{B00}=0\;. \tag{8.1a}$$

The time-maintenance condition applied to the system (8.1a) leads to ten dynamical (secondary) gravitational constraints

$$(\hat{\mathscr{E}}1)_{A\dot{B}}{}^{\perp}=0\;,\quad (\hat{\mathscr{E}}2)_{A}{}^{B\perp}=0\;. \tag{8.2a}$$

If matter is present, we have further kinematical (primary) constraints

$$\mathscr{W}_{\Sigma}{}^{0\nu_1\cdots\nu_k}=0\;,\text{if at least one of indices } \nu_i \text{ is equal zero}\;, \tag{8.1b}$$

as well as secondary matter constraints

$$(\hat{\mathscr{E}}\mathscr{M})_{\Sigma}{}^{\perp s_2\cdots s_k}=0\;. \tag{8.2b}$$

The structure of gravitational constraints (8.2a) has been thoroughly investigated in Sec. VII, these equations are symplectic constraints. The structure of matter constraints (8.2b) essentially depends on the particular choice of matter Lagrangian. In some cases these equations are symplectic constraints in others they are not.

In the present section, however, we investigate another feature of first order spinor matter Lagrangians. For such Lagrangians, the field momenta are proportional to field potentials and these relations lead to additional symplectic constraints. For some classical fields, e.g., Dirac, Weyl, Fierz-Pauli, those symplectic constraints can be completely reduced. The method we use is based on an appropriate redefinition of symplectic variables such that the symplectic two-form Ω preserves its diagonal form. Until now, it is not clear for which classes of first order matter Lagrangians such a reduction can be performed.

For second-order matter Lagrangians. examples are known, where such a reduction is not possible; in such cases we have additional (differential) secondary constraints.[43]

For the SL(2,C)-covariant formulation the most natural is the spinor representation of Dirac matrices.[53] The matter potential of the Dirac field is given by a pair $(\phi^A,\psi_{\dot{A}})$ of zero-forms with values in Weyl spinors. The transformation rules hold

$$'\phi^A=S^{-1A}{}_C\phi^C\;,\quad '\psi_{\dot{A}}=S^{\dot{C}}{}_{\dot{A}}\psi_{\dot{C}}\;. \tag{8.3}$$

The Dirac Lagrangian reads

$$L_D=\frac{i}{\sqrt{2}}(\phi^{\dot A}\wedge *e_{\dot A B}\wedge D\phi^B+\psi_A\wedge *e^{A\dot B}\wedge D\psi_{\dot B}) -m\phi^A*\psi_A+\text{c.c.} \quad (8.4)$$

The momenta W_Σ are given by a pair $(\Phi_A,\Psi^{\dot A})$ of three-forms on spacetime

$$\Phi_A=-\frac{i}{\sqrt{2}}*e_{A\dot B}\phi^{\dot B}\,,\quad \Psi^{\dot A}=-\frac{i}{\sqrt{2}}*e^{\dot AB}\psi_B\,. \quad (8.5)$$

In the (3 + 1) language we have

$${}_{\|}\hat\Phi_A=\frac{i}{\sqrt{2}}\sigma_{(0)A\dot B}\hat\phi^{\dot B}\eta(\sigma)\,,\quad {}_{\|}\hat\Psi^{\dot A}=\frac{i}{\sqrt{2}}\sigma_{(0)}^{\dot AB}\hat\psi_B\eta(\sigma)\,, \quad (8.6)$$

where $\eta(\sigma)$ is the volume element on σ (cf. Appendix A).

The matter part of the symplectic two-form Ω can be written

$$\Omega_m(\sigma)(X_1,X_2)=\int_\sigma\left[\frac{i}{\sqrt{2}}\sigma_{(0)A\dot B}[\delta_1(\hat\phi^{\dot B}\eta(\sigma))\wedge\delta_2\hat\phi^A+\delta_1\hat\phi^{\dot B}\wedge\delta_2(\hat\phi^A\eta(\sigma))] + \frac{i}{\sqrt{2}}\sigma_{(0)}^{\dot AB}[\delta_1(\hat\psi_B\eta(\sigma))\wedge\delta_2\hat\psi_{\dot A}+\delta_1\hat\psi_B\wedge\delta_2(\hat\psi_{\dot A}\eta(\sigma))]\right]\,. \quad (8.7)$$

It is clear that for the Dirac field interacting with gravity the symplectic variables in (8.7) are not independent quantities. We can, however, improve the situation by introducing new symplectic variables. Let q be a half-form on σ such that $qq=\eta(\sigma)$. Then formula (8.7) takes the form

$$\Omega_m(\sigma)(X_1,X_2)=\int_\sigma[\delta_1(i\sqrt{2}\,\sigma_{(0)A\dot B}\hat\phi^{\dot B}q)\wedge\delta_2(\hat\phi^A q)+\delta_1(i\sqrt{2}\,\sigma_{(0)}^{\dot AB}\hat\psi_B q)\wedge\delta_2(\hat\psi_{\dot A}q)]\,. \quad (8.8)$$

We see that the proper symplectic variables are spinor-valued half-forms on σ

$$\hat\phi^A q,\hat\psi_{\dot A}q,i\sqrt{2}\,\sigma_{(0)A\dot B}\hat\phi^{\dot B}q,i\sqrt{2}\,\sigma_{(0)}^{\dot AB}\hat\psi_B q\,. \quad (8.9)$$

These quantities are linearly independent over the field of real numbers.

Remark. The mathematical notion of a half-form on the manifold enables us to extract the square root q of the volume three-form $\eta(\sigma)$. In general, half-forms exist only on manifolds equipped with metalinear structures.[37] In our case, such a metalinear structure exists by virtue of the orientability of σ.

In local coordinates we define the R-independent quantities

$$\xi^A=\sqrt[4]{g}\,\hat\phi^A\,,\quad \xi^{\dot A}=\sqrt[4]{g}\,\hat\phi^{\dot A}\,,\quad \chi_A=\sqrt[4]{g}\,\hat\psi_A\,,\quad \chi_{\dot A}=\sqrt[4]{g}\,\hat\psi_{\dot A}\,, \quad (8.9')$$

and formula (8.8) reads

$$\Omega_m(\sigma)(X_1,X_2)=\int_\sigma[\delta_1(i\sqrt{2}\,\sigma_{(0)A\dot B}\xi^{\dot B})\wedge\delta_2\xi^A+\delta_1(i\sqrt{2}\,\sigma_{(0)}^{\dot AB}\chi_B)\wedge\delta_2\chi_{\dot A}]dx^1\wedge dx^2\wedge dx^3\,. \quad (8.8')$$

The diagonal expression (8.8′) indicates that the Dirac equations written in terms of variables (8.9′) should have a simple and elegant dynamical form. Such a formulation of the Dirac equations in the Einstein-Cartan theory is presented in Sec. IX.

The method of diagonalizing the matter part of the symplectic two-form by means of half-densities on σ can be also applied to the Fierz-Pauli Lagrangian[54]

$$L_{\rm FP}=\frac{i}{\sqrt{2}}(\phi^{\dot A\dot A_1\cdots\dot A_k}_{B_1\cdots B_k}\wedge *e_{\dot AB}\wedge D\phi^{BB_1\cdots B_k}_{\dot A_1\cdots\dot A_k}+\psi^{\dot B_1\cdots\dot B_k}_{AA_1\cdots A_k}\wedge *e^{A\dot B}\wedge D\psi^{A_1\cdots A_k}_{\dot B\dot B_1\cdots\dot B_k})+m\phi^{\dot A_0\cdots\dot A_k}_{B_1\cdots B_k}\psi^{B_1\cdots B_k}_{\dot A_0\cdots\dot A_k}+\text{c.c.}\,, \quad (8.10)$$

where ϕ and ψ are zero-forms with values in SL$(2,C)$ symmetric spinors. We get

$$\Omega_m(\sigma)(X_1,X_2)=\int_\sigma[\delta_1(i\sqrt{2}\,\sigma_{(0)\dot A(B}\hat\phi^{\dot A\dot A_1\cdots\dot A_k}_{B_1\cdots B_k)}q)\wedge\delta_2(\hat\phi^{BB_1\cdots B_k}_{\dot A_1\cdots\dot A_k}q)+\delta_1(i\sqrt{2}\,\sigma_{(0)}^{A(\dot B}\hat\psi^{\dot B_1\cdots\dot B_k)}_{AA_1\cdots A_k}q)\wedge\delta_2(\hat\psi^{A_1\cdots A_k}_{\dot B\dot B_1\cdots\dot B_k}q)]\,. \quad (8.11)$$

In our symplectic approach, the technique of spinor-valued half-forms eliminates the primary constraints, which are generic for first-order Lagrangians, and leads to an elegant dynamical description of matter fields.

We would like to point out also that some other authors used spinor-valued half-densities (they called them weighted spinors) as canonical variables in the standard Bergmann-Dirac analysis of the Einstein-Dirac field.[55]

IX. THE DIRAC FIELD INTERACTING WITH EINSTEIN-CARTAN GRAVITY

The coupled Einstein-Dirac system was investigated by several authors in the framework of Riemannian geometry[55–57] as well as in the framework of Riemann-Cartan geometry.[58] Our analysis of the problem, however, is different from those presented in the above-mentioned papers. We reduce the Einstein-Cartan-Dirac field equations to a system consisting of dynamical equations and seven first-class initial-value constraints. No second-class constraints appear in our formulation. The total ECD Lagrangian reads

$$L_{\mathrm{tot}}=*(e^{B\dot{C}}\wedge e_{A\dot{C}})\wedge\Omega^{A}{}_{B}+*(e^{\dot{B}C}\wedge e_{\dot{A}C})\wedge\Omega^{\dot{A}}{}_{\dot{B}}+L_{D}\ . \tag{9.1}$$

The Dirac Lagrangian L_D has been analyzed in the previous section. Now we discuss the gravitational part of the total Lagrangian. We have the following formulas for the gravitational canonical momenta:

$$U_{A\dot{B}}=0\ ,\quad P_{A}{}^{B}=*(e^{B\dot{C}}\wedge e_{A\dot{C}})\ . \tag{9.2}$$

In the (3 + 1) decomposition

$${}^{a}_{\|}\hat{P}_{A}{}^{B}=0\ ,\quad {}^{h}_{\|}\hat{P}_{A}{}^{B}=\sigma_{(0)A\dot{C}}\,{}^{3}*{}^{t}_{\|}\hat{e}^{B\dot{C}}-\sigma^{B\dot{C}}_{(0)}\,{}^{3}*{}^{t}_{\|}\hat{e}_{A\dot{C}}\ , \tag{9.3}$$

(we recall that ${}^{3}*$ denotes the Hodge operator on σ, see Appendix A). The gravitational part of the symplectic two-form is

$$\Omega_{g}=\int_{\sigma}[\delta_{1}(2\sqrt{g}\cdot{}^{t}\hat{e}_{A\dot{B}}{}^{s})\wedge\delta_{2}(-2\sigma^{C\dot{B}}_{(0)}\cdot{}^{h}\hat{\Gamma}^{A}{}_{Cs})+\delta_{1}m^{A\dot{B}}\wedge\delta_{2}{}^{t}n_{A\dot{B}}]dx^{1}\wedge dx^{2}\wedge dx^{3}\ . \tag{9.4}$$

The dynamical gravitational symplectic variables

$$e^{s}{}_{A\dot{B}}=2\sqrt{g}\cdot{}^{t}\hat{e}_{A\dot{B}}{}^{s}\ ,\quad K_{s}{}^{A\dot{B}}=-2\sigma^{C\dot{B}}_{(0)}\cdot{}^{h}\hat{\Gamma}^{A}{}_{Cs} \tag{9.5}$$

have a simple geometric interpretation: $e^{s}{}_{A\dot{B}}$ are the components of an ${}^{t}H(2)$-valued vector density on σ, and $K_{s}{}^{A\dot{B}}$ represent the second fundamental form of the embedding $\sigma\to M$ (cf. Ref. 22). According to the general scheme proposed in Sec. VII we divide the ECD equations into three groups:

(A) 18 dynamical gravitational equations

$${}^{t}_{\perp}(\hat{E}1)_{A\dot{B}}=0\ ,\quad {}^{h}_{\perp}(\hat{E}2)_{A}{}^{B}=0\ , \tag{9.6a}$$

and 8 (dynamical) matter equations

$$(\hat{E}\phi)_{A}=0\ ,\quad (\hat{E}\psi)^{\dot{A}}=0\ , \tag{9.6b}$$

(B) 7 gravitational symplectic constraints

$${}_{\|}(\hat{E}1)_{A\dot{B}}=0\ ,\quad {}^{a}_{\|}(\hat{E}2)_{A}{}^{B}=0\ , \tag{9.7}$$

(C) 12 nondynamical equations

$${}^{h}_{\|}(\hat{E}2)_{A}{}^{B}=0\ ,\quad {}^{a}_{\perp}(\hat{E}2)_{A}{}^{B}=0\ . \tag{9.8}$$

Also the field variables split into three groups:

(i) 18 gravitational and 8 matter symplectic variables

$$e^{s}{}_{A\dot{B}},K_{s}{}^{A\dot{B}}\ (\text{or }{}^{h}\hat{\Gamma}^{A}{}_{Bs})\ , \tag{9.9a}$$

$$\xi^{A},\xi^{\dot{A}},\chi_{A},\chi_{\dot{A}}\ , \tag{9.9b}$$

(ii) 10 gravitational gauge variables

$$N,N^{s},{}^{a}\hat{\Gamma}^{A}{}_{B\perp},{}^{t}n_{A\dot{B}}\ , \tag{9.10}$$

(iii) 12 nondynamical variables

$${}^{h}\hat{\Gamma}^{A}{}_{B\perp},{}^{a}\hat{\Gamma}^{A}{}_{Bs}\ . \tag{9.11}$$

In the Einstein-Cartan theory we are able to solve nondynamical equations (9.8) algebraically and to eliminate nondynamical variables (9.11). This fact follows from the explicit form of Eqs. (9.8)

$${}^{h}\hat{\Gamma}^{A}{}_{C\perp}\sigma^{C\dot{B}}_{(0)}=\frac{1}{2}{}^{t}\hat{e}^{A\dot{B}s}\partial_{s}\ln N-\frac{1}{4}\frac{1}{\sqrt{g}}{}^{h}\hat{\mathfrak{s}}_{C}{}^{A\perp}\sigma^{C\dot{B}}_{(0)}+\frac{1}{8\sqrt{g}}{}^{t}\hat{e}^{C\dot{B}}{}_{s}\,{}^{a}\hat{\mathfrak{s}}_{C}{}^{As}+\frac{1}{8\sqrt{g}}{}^{t}\hat{e}^{A\dot{C}}{}_{s}\cdot{}^{a}\hat{\mathfrak{s}}_{\dot{C}}{}^{\dot{B}s}\ , \tag{9.8a'}$$

$${}^{t}\hat{Q}^{A\dot{B}}{}_{rs}=\frac{1}{2\sqrt{g}}({}^{t}\hat{e}^{A\dot{B}}{}_{s}\cdot{}^{t}\hat{e}_{C\dot{D}r}-{}^{t}\hat{e}^{A\dot{B}}{}_{r}\cdot{}^{t}\hat{e}_{C\dot{D}s})({}^{h}\hat{\mathfrak{s}}_{E}{}^{C\perp}\sigma^{E\dot{D}}_{(0)}+\tfrac{1}{2}{}^{t}\hat{e}^{E\dot{D}}{}_{p}\cdot{}^{a}\hat{\mathfrak{s}}_{E}{}^{Cp}+\tfrac{1}{2}{}^{t}\hat{e}^{C\dot{E}}{}_{p}\cdot{}^{a}\hat{\mathfrak{s}}_{\dot{E}}{}^{\dot{D}p})+\frac{1}{2\sqrt{g}}{}^{t}\hat{e}^{A\dot{B}}{}_{p}\cdot{}^{t}\hat{e}^{C\dot{D}}{}_{r}({}^{t}\hat{e}_{E\dot{D}s}\cdot{}^{a}\hat{\mathfrak{s}}_{C}{}^{Ep}+{}^{t}\hat{e}_{C\dot{E}s}\cdot{}^{a}\hat{\mathfrak{s}}_{\dot{D}}{}^{\dot{E}p})\ . \tag{9.8b'}$$

For the Dirac field the spin three-form is given by the formula

$$s_{A}{}^{B}=\frac{i}{2\sqrt{2}}(\phi_{A}\phi_{\dot{C}}-\psi_{A}\psi_{\dot{C}})*e^{B\dot{C}}-\frac{i}{2\sqrt{2}}(\phi^{B}\phi^{\dot{C}}-\psi^{B}\psi^{\dot{C}})*e_{A\dot{C}}\ . \tag{9.12}$$

Hence, we have

$$\hat{\mathfrak{s}}_{A}{}^{B\perp}=\frac{i}{2\sqrt{2}}[(\xi_{A}\xi_{\dot{C}}-\chi_{A}\chi_{\dot{C}})\sigma^{(0)B\dot{C}}-(\xi^{B}\xi^{\dot{C}}-\chi^{B}\chi^{\dot{C}})\sigma^{(0)}_{A\dot{C}}]\ ,$$

$$\hat{\mathfrak{s}}_{A}{}^{Br}=\frac{i}{2\sqrt{2}}[(\xi_{A}\xi_{\dot{C}}-\chi_{A}\chi_{\dot{C}})\,{}^{t}\hat{e}^{B\dot{C}r}-(\xi^{B}\xi^{\dot{C}}-\chi^{B}\chi^{\dot{C}})\,{}^{t}\hat{e}_{A\dot{C}}{}^{r}]\ . \tag{9.12'}$$

Moreover, it is easy to show that

$${}^{t}_{\|}\hat{\Theta}^{A\dot{B}}={}^{3}d({}^{t}_{\|}\hat{e}^{A\dot{B}})+{}^{a}_{\|}\hat{\Gamma}^{A}{}_{C}\wedge{}^{t}_{\|}\hat{e}^{C\dot{B}}+{}^{a}_{\|}\hat{\Gamma}^{\dot{B}}{}_{\dot{D}}\wedge{}^{t}_{\|}\hat{e}^{A\dot{D}}\ , \tag{9.13}$$

that is to say, ${}^{t}\hat{Q}^{A\dot{B}}{}_{rs}$ are the components of torsion of the connection ${}^{a}_{\|}\hat{\Gamma}^{A}{}_{B}$ on σ (cf. Appendix E). The torsion of a metric compatible connection determines this connection uniquely (the explicit formula can be found in Ref. 38). Making use of relations (9.12′) and (9.13) we solve Eqs. (9.8′) for ${}^{h}\hat{\Gamma}^{A}{}_{B\perp}$ and ${}^{a}\hat{\Gamma}^{A}{}_{Bs}$.

Remark. For the Einstein-Cartan theory coupled to tensor matter fields with quadratic Lagrangians, the components of spin (9.12′) depend also on the three-connection coefficients ${}^{a}\hat{\Gamma}^{A}{}_{Bs}$. In such a case Eq. (9.8b′) constitutes a complicated linear system for ${}^{a}\hat{\Gamma}^{A}{}_{Bs}$. It has been recently shown that such a system can be solved effectively but in some cases its solutions have singular points.[38,43]

The explicit SO(3)-covariant formulation of the dynamical gravitational equations and the constraints has been presented in Ref. 38. Those formulas rewritten in the SU(2)-spinor language read

$$\begin{aligned}{}^{3}\mathscr{D}_{0}{}^{h}\hat{\Gamma}^{A}{}_{Bs}={}&{}^{3}\mathscr{D}_{s}{}^{h}\hat{\Gamma}^{A}{}_{B\perp}+{}^{h}\hat{\Gamma}^{A}{}_{B\perp}\partial_{s}\ln N+({}^{t}\hat{e}_{C\dot{D}}{}^{r}\cdot{}^{3}R^{C}{}_{Brs}+{}^{t}\hat{e}_{B\dot{C}}{}^{r}\cdot{}^{3}R^{\dot{C}}{}_{\dot{D}rs})\sigma^{A\dot{D}}_{(0)}+{}^{t}\hat{e}^{C\dot{D}r}\sigma_{(0)E\dot{D}}({}^{h}\hat{\Gamma}^{A}{}_{Br}{}^{h}\hat{\Gamma}^{E}{}_{Cs}-{}^{h}\hat{\Gamma}^{A}{}_{Bs}{}^{h}\hat{\Gamma}^{E}{}_{Cr})\\&+\frac{1}{2\sqrt{g}}\sigma^{(0)A\dot{C}}({}^{t}\hat{\mathscr{T}}_{E\dot{F}}{}^{r}\cdot{}^{t}\hat{e}^{E\dot{F}}{}_{s}\cdot{}^{t}\hat{e}_{BCr}-\tfrac{1}{2}{}^{t}\hat{e}_{B\dot{C}s}\,tr\hat{\mathscr{T}})\,,\end{aligned}\tag{9.6a′}$$

$${}^{3}\mathscr{D}_{0}e^{s}{}_{A\dot{B}}=2(e^{s}{}_{A\dot{B}}\cdot{}^{t}\hat{e}^{C\dot{D}r}-e^{sC\dot{D}}\cdot{}^{t}\hat{e}_{A\dot{B}}{}^{r}){}^{h}\hat{\Gamma}^{E}{}_{Cr}\cdot\sigma_{(0)E\dot{D}}-2{}^{h}\hat{\mathfrak{Z}}_{A}{}^{Cs}\cdot\sigma_{(0)C\dot{B}}\,,$$

where ${}^{3}R^{A}{}_{Brs}$ is the Riemann tensor of the connection ${}^{a}_{\parallel}\hat{\Gamma}^{A}{}_{B}$ on σ, and $tr\hat{\mathscr{T}}$ is given by $tr\hat{\mathscr{T}}={}^{d}\hat{\mathscr{T}}^{\perp}+{}^{t}\hat{\mathscr{T}}_{A\dot{B}}{}^{s}\cdot{}^{t}\hat{e}^{A\dot{B}}{}_{s}$;

$$\begin{aligned}&{}^{d}(\hat{\mathscr{E}}1)^{\perp}=\sqrt{g}\cdot{}^{3}R+4\sqrt{g}\,{}^{t}\hat{e}^{A\dot{B}r}\cdot{}^{t}\hat{e}^{C\dot{D}s}\cdot\sigma_{(0)E\dot{B}}\cdot\sigma_{(0)F\dot{D}}({}^{h}\hat{\Gamma}^{E}{}_{Ar}{}^{h}\hat{\Gamma}^{F}{}_{Cs}-{}^{h}\hat{\Gamma}^{E}{}_{As}{}^{h}\hat{\Gamma}^{F}{}_{Cr})+{}^{d}\hat{\mathscr{T}}^{\perp}=0\,,\\&{}^{t}(\hat{\mathscr{E}}1)_{A\dot{B}}{}^{\perp}=2\cdot e^{sC\dot{D}}\,{}^{t}\hat{e}_{A\dot{B}}{}^{r}\cdot\sigma^{(0)}_{E\dot{D}}({}^{3}\mathscr{D}_{s}{}^{h}\hat{\Gamma}^{E}{}_{Cr}-{}^{3}\mathscr{D}_{r}{}^{h}\hat{\Gamma}^{E}{}_{Cs})+{}^{t}\hat{\mathscr{T}}_{A\dot{B}}{}^{\perp}=0\,,\\&{}^{a}(\hat{\mathscr{E}}2)_{A}{}^{B\perp}=2\sqrt{g}\,({}^{h}\hat{\Gamma}^{B}{}_{Es}\cdot{}^{t}\hat{e}_{A\dot{C}}{}^{s}\cdot\sigma^{(0)E\dot{C}}+{}^{h}\hat{\Gamma}^{E}{}_{As}\cdot{}^{t}\hat{e}^{B\dot{C}s}\cdot\sigma^{(0)}_{E\dot{C}})+{}^{a}\hat{\mathfrak{Z}}_{A}{}^{B\perp}=0\,,\end{aligned}\tag{9.7′}$$

where ${}^{3}R$ is the Ricci scalar of curvature ${}^{3}R^{A}{}_{Brs}$ on σ,

$${}^{3}R=-{}^{3}R^{A}{}_{Brs}\cdot{}^{t}\hat{e}_{A\dot{C}}{}^{r}\cdot{}^{t}\hat{e}^{B\dot{C}s}+\text{c.c.}$$

The SU(2)-covariant formulation of the Dirac equations reads

$$\begin{aligned}&\sigma_{(0)A\dot{B}}\cdot{}^{3}\mathscr{D}_{0}\xi^{A}=\tfrac{1}{2}\xi^{A}\cdot{}^{3}\mathscr{D}_{s}{}^{t}\hat{e}_{A\dot{B}}{}^{s}+{}^{t}\hat{e}_{A\dot{B}}{}^{s}\cdot{}^{3}\mathscr{D}_{s}\xi^{A}-\frac{i}{\sqrt{2}}m\chi_{\dot{B}}+\tfrac{1}{2}\xi^{A}\cdot{}^{t}\hat{e}_{C\dot{B}}{}^{s}\cdot{}^{h}\hat{\Gamma}^{C}{}_{As}-\tfrac{1}{2}\xi^{A}\cdot{}^{t}\hat{e}_{A\dot{C}}{}^{s}\cdot{}^{h}\hat{\Gamma}^{\dot{C}}{}_{\dot{B}s}+\tfrac{1}{2}\xi^{A}\cdot{}^{t}\hat{e}_{A\dot{B}}{}^{s}\partial_{s}\ln N\,,\\&\sigma^{A\dot{B}}_{(0)}\cdot{}^{3}\mathscr{D}_{0}\chi_{\dot{B}}=\tfrac{1}{2}\chi_{\dot{B}}{}^{3}\cdot\mathscr{D}_{s}{}^{t}\hat{e}^{A\dot{B}s}+{}^{t}\hat{e}^{A\dot{B}s}\cdot{}^{3}\mathscr{D}_{s}\chi_{\dot{B}}-\frac{i}{\sqrt{2}}m\xi^{A}+\tfrac{1}{2}\chi_{\dot{B}}{}^{t}\hat{e}^{C\dot{B}s}\cdot{}^{h}\hat{\Gamma}^{A}{}_{Cs}-\tfrac{1}{2}\chi_{\dot{C}}{}^{t}\hat{e}^{A\dot{B}s}\cdot{}^{h}\hat{\Gamma}^{\dot{C}}{}_{\dot{B}s}+\tfrac{1}{2}\chi_{\dot{B}}{}^{t}\hat{e}^{A\dot{B}s}\partial_{s}\ln N\,.\end{aligned}\tag{9.6b′}$$

Remark. SU(2)-covariant operators ${}^{3}\mathscr{D}_{0}$ and ${}^{3}\mathscr{D}_{s}$ are defined in Appendix E.

As given above, the manifestly covariant dynamical formulation of the ECD system enables us to pose the Cauchy-Kowalewska initial-value problem on the surface σ. We fix the following quantities: (a) initial values of symplectic variables (9.9) on the initial surface σ in such a way that constraints (9.7) hold and (b) arbitrary values of gauge variables (9.10) on spacetime.

If these data are analytic functions then in a neighborhood of each point on σ we have a unique local (analytic) solution. We know, however, that much more interesting is the hyperbolic initial-value problem, which in classical general relativity enables us to find solutions in a neighborhood of the initial surface. We may expect the ECD system to be hyperbolic only if we impose a certain gauge condition. In Einstein's theory the harmonic gauge[12,59] is natural, although it is not convenient for the (3 + 1) picture. Recently, however, Choquet-Bruhat and Ruggieri[60] have shown that in general relativity there exists another gauge condition appropriate to the hyperbolic analysis of the (3 + 1) decomposition. In this gauge the shift N^s vanishes and the lapse N satisfies a certain differential equation. It would be interesting to investigate whether Choquet-Ruggieri-type conditions could be found for more general theories of gravity, particularly, for the Einstein-Cartan-Dirac system.

Having the correctly posed initial-value problem, we investigate the question of independent degrees of freedom. In the ECD theory we have $18+8=26$ real symplectic variables whose initial values are subject to 7 constraints. These constraints are first class in the Dirac terminology[15] and therefore they reduce 14 degrees of freedom. Thus, we expect to have only 12 degrees of freedom in the phase space.[61] Four of them describe the gravitational field and the remaining eight the Dirac field. For spatially closed spacetimes, a rigorous proof of this statement can be obtained by means of symplectic methods elaborated in Refs. 12, 47, and 62. The analysis of independent degrees of freedom based on the symplectic technique gives rise to the corresponding splitting of the (formal) tangent space of the set of solutions. The problem of how the decomposition of the tangent space reflects the genuine structure of the set of solutions requires much more subtle methods. For the Einstein theory, very profound results in that direction were obtained by Arms, Fischer, Isenberg, Marsden, and Moncrief.[13,63] These authors showed that the set of Einstein metrics on a spatially closed spacetimes is a manifold with singularities and that singular points are symmetric solutions of Einstein equations. Moreover, the action of Diff$M$ admits a slice, that is, we are able to decompose (locally) the set of Einstein metrics into orbits of the diffeomorphism group of spacetime.

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APPENDIX A: THE HODGE DUALITY

Let M be an oriented, four-dimensional, Lorentzian manifold with the metric $g=g_{\mu\nu}dx^{\mu}\otimes dx^{\nu}$. We denote by $\epsilon_{\alpha_1\cdots\alpha_4}$ and $\epsilon^{\alpha_1\cdots\alpha_4}$ the Levi-Civita symbols in four dimensions ($\epsilon_{0123}=\epsilon^{0123}=1$). The volume element on M is given by

$$\eta(M)=\frac{\sqrt{-g}}{4!}\epsilon_{\alpha_1\cdots\alpha_4}dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_4}\,, \tag{A1}$$

where $g=\det[g_{\mu\nu}]$.

The Hodge dual operator $*$ transforms the space Λ^kT^*M of k-covectors on M onto the space $\Lambda^{4-k}T^*M$ of $(4-k)$-covectors. The basic formula defining this operator reads

$$\begin{aligned}*(dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_k})&=\frac{\sqrt{-g}}{(4-k)!}g^{\alpha_1\beta_1}\cdots g^{\alpha_k\beta_k}\\&\quad\times\epsilon_{\beta_1\cdots\beta_4}dx^{\beta_{k+1}}\wedge\cdots\wedge dx^{\beta_4}\,.\end{aligned} \tag{A2}$$

We have the relations

$$**=(-1)^{k+1}id\;; \tag{A3}$$

if ϕ,ψ are k-covectors then

$$*\phi\wedge\psi=*\psi\wedge\phi\;; \tag{A4}$$

if $Z=Z^{\mu}\partial_{\mu}$ is a vector on M and $\check{Z}=\check{Z}_{\nu}dx^{\nu}$ is the corresponding covector on M, i.e., $\check{Z}_{\nu}=g_{\nu\mu}Z^{\mu}$, then for an arbitrary k covector ϕ on M

$$*(\check{Z}\wedge *\phi)=-Z\,\lrcorner\,\phi\,. \tag{A5}$$

For a spinor-valued differential k-form

$$A^{\Sigma}=\frac{1}{k!}A^{\Sigma}{}_{\nu_1\cdots\nu_k}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_k} \tag{A6}$$

its Hodge-dual representation reads

$$A^{\Sigma}=\frac{(\sqrt{-g}\,)^{-1}}{(4-k)!}\mathscr{A}^{\Sigma}{}_{\mu_1\cdots\mu_{4-k}}*(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{4-k}})\,, \tag{A7}$$

where $\mathscr{A}^{\Sigma}{}_{\mu_1\cdots\mu_{4-k}}=g/k!\epsilon_{\mu_1\cdots\mu_{4-k}\nu_1\cdots\nu_k}A^{\Sigma\nu_1\cdots\nu_k}$ are the components of a tensor density of weight $+1$ on M.

Some important formulas in the (3+1) decomposition. We assume that local coordinates (x^0,x^s) on M are consistent with the slicing. The lapse and shift are defined by

$$N=(-g^{00})^{-1/2},\;\; N^s=N^2g^{0s}\,. \tag{A8}$$

We have the relations

$$\begin{aligned}&g_{0s}=N_s=g_{sr}N^r\,,\\&g_{00}=-N^2+N_s\cdot N^s\,,\\&g^{rs}={}^3g^{rs}-\frac{N^r\cdot N^s}{N^2}\,,\end{aligned} \tag{A9}$$

where $[{}^3g^{rs}]$ is the inverse matrix of $[g_{rs}]$.

The relations (6.12) between holonomic and anholonomic bases may be rewritten as[35]

$$\overline{dx}^{\mu}=A^{\bar{\mu}}{}_{\nu}dx^{\nu}\,,\;\;\bar{\partial}_{\mu}=A^{-1\nu}{}_{\bar{\mu}}\partial_{\nu}\,, \tag{A10}$$

where

$$\begin{aligned}&A^{\bar{0}}{}_0=N\,,\;\;A^{\bar{s}}{}_0=N^s\,,\;\;A^{\bar{0}}{}_s=0\,,\\&A^{\bar{s}}{}_r=\delta^s{}_r\,,A^{-1\,0}{}_{\bar{0}}=\frac{1}{N}\,,\;\;A^{-1\,s}{}_{\bar{0}}=-\frac{N^s}{N}\,,\\&A^{-1\,0}{}_{\bar{s}}=0\,,\;\;A^{-1s}{}_{\bar{r}}=\delta^s{}_r\,.\end{aligned} \tag{A11}$$

For any spacetime tensor $V^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_l}$ we define

$$\begin{aligned}\bar{V}^{\alpha_1\cdots\alpha_k}_{\beta_1\cdots\beta_l}&=V^{\bar{\alpha}_1\cdots\bar{\alpha}_k}_{\bar{\beta}_1\cdots\bar{\beta}_l}\\&=A^{\bar{\alpha}_1}{}_{\mu_1}\cdots A^{\bar{\alpha}_k}{}_{\mu_k}A^{-1\nu_1}{}_{\bar{\beta}_1}\cdots A^{-1\nu_l}{}_{\bar{\beta}_l}V^{\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l}\,.\end{aligned} \tag{A12}$$

The quantity $\bar{V}$ represents a family of tensors on surfaces of the slicing and the valence of these tensors is determined by nonzero indices. In order to emphasize that in the (3 + 1) decomposition the index "0" is not a tensor index, we often replace it by the symbol $\perp$. For instance, if V^{α} is a spacetime vector then

$$\begin{aligned}&\bar{V}^{\perp}=V^{\bar{0}}\;\text{ is a scalar on }\sigma\,,\\&\bar{V}^{s}=V^{\bar{s}}\;\text{ is a vector on }\sigma\,.\end{aligned} \tag{A13}$$

In general, three-geometric objects defined by the quantity $\bar{V}$ are three-tensors with respect to transformations (6.11).

For the metric tensor we get

$$\begin{aligned}&\bar{g}_{\perp\perp}=-1\,,\;\;\bar{g}_{\perp s}=0\,,\;\;\bar{g}_{sr}=g_{sr}\,,\\&\bar{g}^{\perp\perp}=-1\,,\;\;\bar{g}^{\perp s}=0\,,\;\;\bar{g}^{sr}={}^3g^{sr}\,.\end{aligned} \tag{A14}$$

For a spacetime scalar density F of weight r we define

$$\bar{\mathscr{F}}=N^{-r}\mathscr{F}\,. \tag{A15}$$

In particular

$$\overline{\sqrt{-g}}=N^{-1}\sqrt{-g}=\sqrt{\bar{g}}=(\det[\bar{g}_{rs}])^{1/2}\,. \tag{A16}$$

The bar operation can also be defined for the holonomic components $\Gamma^{\mu}{}_{\nu\lambda}$ of a connection on spacetime. The corresponding formulas are given in Ref. 35.

Applying the bar operation to components of spacetime tensors and connections, we obtain families of three-tensors and three-connections on slices. In the complete theory, however, we have two quantities having specific properties with respect to transformations (6.11). They are the lapse and the shift. We have

$$N'=\left[\frac{\partial x^{0'}}{\partial x^0}\right]^{-1}\cdot N\ ,\quad N^{s'}=\left[\frac{\partial x^0}{\partial x^{0'}}\right]\left[\frac{\partial x^{s'}}{\partial x^r}N^r-\frac{\partial x^{s'}}{\partial x^0}\right]. \tag{A17}$$

Let us observe that $\partial_s \ln N$ are the components of a three-tensor on σ.

The metric $g_{\mu\nu}$, the orientation of spacetime, and an external orientation of the surface σ induce the metric $\bar{g}_{rs}$ on σ and its internal orientation. The volume element on σ is given by

$$\eta(\sigma)=\frac{\sqrt{\bar{g}}}{3!}\epsilon_{s_1s_2s_3}dx^{s_1}\wedge dx^{s_2}\wedge dx^{s_3}\ . \tag{A18}$$

The formula for the Hodge operator ^{3*} on σ reads

$$ {}^3*(dx^{s_1}\wedge\cdots\wedge dx^{s_k}) =\frac{\sqrt{\bar{g}}}{(3-k)!}\bar{g}^{s_1r_1}\cdots\bar{g}^{s_kr_k}\epsilon_{r_1r_2r_3}dx^{r_{k+1}}\wedge\cdots\wedge dx^{r_3}\ . \tag{A19}$$

Instead of (A3) we have

$$ {}^3*{}^3*=id\ . \tag{A20}$$

In Sec. VI we use the formula

$$*(n\wedge\phi)\,|_\sigma=-{}^3*(\phi\,|_\sigma) \tag{A21}$$

valid for an arbitrary differential form ϕ on M.

APPENDIX B: SOME FORMULAS OF THE SPINOR CALCULUS

If the Infeld-van der Waerden matrices $[\sigma^{A\dot{B}}_{(\alpha)}]$ are defined by (2.5) then we have the numerical relations

$$\sigma_{(\alpha)A\dot{B}}=(-1)^\alpha\sigma^{A\dot{B}}_{(\alpha)}\ . \tag{B1}$$

We have

$$\sigma_{(\alpha)A\dot{B}}\sigma^{(\alpha)C\dot{D}}=-\delta^C{}_A\delta^{\dot{D}}{}_{\dot{B}}\ ,\quad \sigma_{(\alpha)A\dot{B}}\sigma^{A\dot{B}}_{(\beta)}=-\eta_{(\alpha)(\beta)}\ . \tag{B2}$$

In the theory of SU(2) spinors, $\delta^{A\dot{B}}=\sqrt{2}\sigma^{A\dot{B}}_{(0)}$ and $\delta_{A\dot{B}}=\sqrt{2}\sigma_{(0)A\dot{B}}$ are unit matrices invariant with respect to the action of SU(2). The Kronecker symbols $\delta^{A\dot{B}}$ and $\delta_{A\dot{B}}$ yield the canonical isomorphism of spaces of dotted and undotted SU(2) spinors. Also we have

$$\sigma_{(s)A\dot{B}}\sigma^{(s)C\dot{D}}=\tfrac{1}{2}\delta_{A\dot{B}}\delta^{C\dot{D}}-\delta^C{}_A\delta^{\dot{D}}{}_{\dot{B}}\ ,$$
$$\sigma_{(r)A\dot{B}}\sigma^{A\dot{B}}_{(s)}=-\delta_{(r)(s)}\ . \tag{B2'}$$

SU(2)-invariant decompositions of complex 2×2 matrices. Any complex 2×2 matrix $[M^{A\dot{B}}]$ can be decomposed into its diagonal and traceless parts

$$M^{A\dot{B}}={}^dM\sigma^{A\dot{B}}_{(0)}+{}^tM^{A\dot{B}}\ , \tag{B3}$$

where ${}^dM=M^{A\dot{B}}\sigma_{(0)A\dot{B}}$ and ${}^tM^{A\dot{B}}$ is traceless, i.e., ${}^tM^{A\dot{B}}\delta_{A\dot{B}}=0$. Any complex 2×2 matrix $[M^A{}_B]$ can be decomposed into its Hermitian and anti-Hermitian parts

$$M^A{}_B={}^hM^A{}_B+{}^aM^A{}_B\ , \tag{B4}$$

where

$${}^hM^A{}_B=\tfrac{1}{2}(M^A{}_B+M^{\dot{C}}{}_{\dot{D}}\delta^{A\dot{D}}\delta_{B\dot{C}})\ ,$$
$${}^aM^A{}_B=\tfrac{1}{2}(M^A{}_B-M^{\dot{C}}{}_{\dot{D}}\delta^{A\dot{D}}\delta_{B\dot{C}})\ .$$

The decompositions (B3) and (B4) are invariant with respect to the appropriate actions of the group SU(2).

APPENDIX C: THE FULL GRAVITATIONAL GAUGE GROUP

The diffeomorphism group of spacetime acts naturally in the space of tensor fields on M. In particular, for a differential form ϕ we have

$$A(\Phi)\phi=(\Phi^{-1})^*\phi\ , \tag{C1}$$

where $\Phi\in\mathrm{Diff}M$, and the asterisk denotes the pullback operation for differential forms. The Lie algebra of Diff$M$ consists of (smooth) vector fields on spacetime being generators of one-parameter families (subgroups) of diffeomorphisms. The action of the Lie algebra of Diff$M$ in the space of tensor fields is given by the standard Lie derivative $\mathscr{L}_Z$. We have the consistency relations

$$[\mathscr{L}_{Z_1},\mathscr{L}_{Z_2}]=\mathscr{L}_{[Z_1,Z_2]}\ . \tag{C2}$$

If we apply formula (C1) to SL$(2,C)$-spinor-valued differential forms then we indeed get an action of Diff$M$ but this action is not SL$(2,C)$ covariant. Therefore we need the following construction:

(i) We assume that there exists a neighborhood $\mathscr{O}_0$ of the zero-vector field on M (in a suitable topology in the space of vector fields), a neighborhood $\mathscr{U}_{id}$ of the identity in Diff$M$ (in a suitable topology in Diff$M$), and a diffeomorphic exponential mapping exp: $\mathscr{O}_0\to\mathscr{U}_{id}$.

Remark. The problem of how to choose a topology (a differentiable structure) in Diff$M$ and in its Lie algebra in order to get an exponential mapping is not trivial. It is known that even for compact M, C^k topologies are not appropriate. However, for compact M the groups Diff$M$ carry the structure of the inverse Hilbert limit Lie groups[64] modeled on H^s spaces and the corresponding exponential mappings are local diffeomorphisms. We may assume that under some, even restrictive, conditions these results can be generalized for noncompact manifolds.

(ii) Let $Z\in\mathscr{O}_0$ and $\Phi=\exp(Z)$. The curve $[0,1]\ni t\to\Phi_t=\exp(tZ)\in\mathrm{Diff}M$ joins the identity and Φ. For each point $x\in M$ we have the curve $t\to\Phi_t(x)$ joining the points x and $\Phi(x)$.

(iii) Let $S(\rho)$ be the spinor space corresponding to a representation ρ of SL$(2,C)$. Each element $s\in S(\rho)$ is represented by its components s^Σ with respect to a chosen basis in $S(\rho)$. $S(\rho)$-valued differential k-forms on M are elements of the tensor product $S(\rho)\otimes\Lambda^kT^*M$ and simple elements $s\otimes\phi$ have the coordinate representation

$$s^{\Sigma}\otimes\phi_{\mu_1\cdots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}=\phi^{\Sigma}{}_{\mu_1\cdots\mu_k}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_k}\ . \qquad (C3)$$

Covariant translations generated by Diff$M$ are defined by the rule

$$_{\zeta}A(\Phi)(s\otimes\phi)=s'\otimes\phi'\ , \qquad (C4)$$

where $\phi'=A(\Phi)\phi$ and s' is the result of the parallel transport of s along the curve $t\to\Phi_t(x)$ from the point x to the point $\Phi(x)$ (a background connection ζ determining the parallel transport is fixed).

Infinitesimal generators of covariant translations are the covariant Lie derivatives (4.2). It follows from commutation relations (4.3a) that the composition of two covariant translations is not a translation. We explain this fact, observing that, in order to perform two consecutive transformations (C4) for diffeomorphisms Φ_1 and Φ_2, we transport differential forms by means of the diffeomorphism $\Phi_3=\Phi_2\circ\Phi_1$, as well as simultaneously parallel transport spinors along the curve $x_1(t)=\Phi_{1t}(x)$ and then along the curve $x_2(t)=\Phi_{2t}(\Phi_1(x))$. For Φ_3 we have the curve $x_3(t)=\Phi_{2t}\circ\Phi_{1t}(x)$. The final point for the composition of $x_1(t)$ and $x_2(t)$ coincides with the final point of $x_3(t)$ but these curves do not coincide. That is why, the composition of transformations (C4) for Φ_1 and Φ_2 is equal to the translation corresponding to Φ_3 composed with a local SL$(2,C)$ rotation.

Our considerations show that the set of pairs (S,Φ), where S is a field of SL$(2,C)$ matrices on M and $\Phi\in\mathscr{U}_{id}\subset$Diff$M$, has a bundle structure over $\mathscr{U}_{id}$ (over Diff$M$). The fiber over a fixed point of the base is isomorphic to the local SL$(2,C)$ group. We write $G_{\mathscr{U}_{id}}=\mathrm{loc\,SL}(2,C)\times_b\mathscr{U}_{id}$ [$G=G_{\mathrm{Diff}M}=\mathrm{loc\,SL}(2,C)\times_b\mathrm{Diff}M$]. The group G acts in the space of SL$(2,C)$-spinor-valued differential forms. Of course, making use of the results of Sec. IV we can extend this action onto sl$(2,C)$-valued connection forms on M.

The essential formulations and the statements of the theory are invariant with respect to the action of G. Therefore this group is the full gravitational gauge group for SL$(2,C)$ theories. The local SL$(2,C)$ group is a normal subgroup of G and the quotient group $G_{\mathscr{G}}=G/\mathrm{loc\,SL}(2,C)$ is isomorphic to the group of transformations generated by the standard action of Diff$M$. In special cases, if the connection $\zeta^A{}_B$ is flat, i.e., $_{\zeta}\Omega^A{}_B=0$, the covariant translations form a subgroup and G is the semidirect product of the loc SL$(2,C)$ and Diff$M$.[22] The bundle structure of G depends essentially on the choice of the connection ζ. It follows from the commutation relations (4.3) that the group corresponding to different connections are locally isomorphic if curvatures of these connections coincide.

APPENDIX D: FIELD EQUATIONS IN LOCAL COORDINATES

According to the general rule [(A6) and (A7)] we have

$$2\sqrt{-g}\ U_{A\dot B}=\mathscr{U}_{A\dot B\mu\nu}*(dx^\mu\wedge dx^\nu)\ ,$$

$$\mathscr{U}_{AB}{}^{\nu}=\mathscr{U}_{\dot C(A}{}^{\mu\nu}e_{B)}{}^{\dot C}{}_{\mu}\ ,$$

$$2\sqrt{-g}\ P_A{}^B=\mathscr{P}_A{}^B{}_{\mu\nu}*(dx^\mu\wedge dx^\nu)\ , \qquad (D1)$$

$$(k+1)!\sqrt{-g}\,W_\Sigma=\mathscr{W}_{\Sigma\nu_1\cdots\nu_{k+1}}*(dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{k+1}})$$

[the symbol (AB) denotes symmetrization in the indices A,B]

$$\sqrt{-g}\ V_{A\dot B}=\mathscr{V}_{A\dot B\nu}*dx^\nu\ ,\quad V_{AB}=V_{\dot C(A}{}^{\nu}e_{B)}{}^{\dot C}{}_{\nu}\ ,$$

$$\sqrt{-g}\ T_{A\dot B}=\mathscr{T}_{A\dot B\nu}*dx^\nu\ ,\quad \mathscr{T}_{AB}=\mathscr{T}_{\dot C(A}{}^{\nu}e_{B)}{}^{\dot C}{}_{\nu}\ ,$$

$$\sqrt{-g}\ s_A{}^B=\mathscr{s}_A{}^B{}_\nu*dx^\nu\ , \qquad (D2)$$

$$k!\sqrt{-g}\ J_\Sigma=\mathscr{J}_{\Sigma\nu_1\cdots\nu_k}*(dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_k})\ .$$

$$\sqrt{-g}\ K=\mathscr{K}*1\ ,\quad \sqrt{-g}\ L=\mathscr{L}*1\ ,$$

$$\sqrt{-g}\,(E1)_{A\dot B}=(\mathscr{E}1)_{A\dot B\nu}*dx^\nu\ ,$$

$$(\mathscr{E}1)_{AB}=(\mathscr{E}1)_{\dot C(A}{}^{\nu}e_{B)}{}^{\dot C}{}_{\nu}\ , \qquad (D3)$$

$$\sqrt{-g}\,(E2)_A{}^B=(\mathscr{E}2)_A{}^B{}_\nu*dx^\nu\ ,$$

$$k!\sqrt{-g}\,(EM)_\Sigma=(\mathscr{E}\mathscr{M})_{\Sigma\nu_1\cdots\nu_k}*(dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_k})\ .$$

It follows from (E3) that for a $(3-k)$-form W_Σ given by (D1)

$$k!\sqrt{-g}\ DW_\Sigma=\mathscr{D}_\lambda\mathscr{W}_{\Sigma\nu_1\cdots\nu_k}{}^{\lambda}*(dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_k})\ . \qquad (D4)$$

For the field strengths we have

$$\Theta^{A\dot B}=\tfrac{1}{2}Q^{A\dot B}{}_{\mu\nu}dx^\mu\wedge dx^\nu\ ,\quad \Omega^A{}_B=\tfrac{1}{2}R^A{}_{B\mu\nu}dx^\mu\wedge dx^\nu\ , \qquad (D5)$$

$$(k+1)!F^\Sigma=F^\Sigma{}_{\nu_1\cdots\nu_{k+1}}dx^{\nu_1}\wedge\cdots\wedge dx^{\nu_{k+1}}\ .$$

The field equations (3.3) in the component form read

$$(\mathscr{E}1)_{A\dot B}{}^{\mu}=\mathscr{T}_{A\dot B}{}^{\mu}+\mathscr{V}_{A\dot B}{}^{\mu}-\mathscr{D}_\lambda\mathscr{U}_{A\dot B}{}^{\lambda\mu}=0\ ,$$

$$(\mathscr{E}2)_A{}^{B\mu}=\mathscr{s}_A{}^{B\mu}+\mathscr{U}_A{}^{B\mu}-\mathscr{D}_\lambda\mathscr{P}_A{}^{B\lambda\mu}+0\ , \qquad (D6)$$

$$(\mathscr{E}\mathscr{M})_\Sigma{}^{\mu_1\cdots\mu_k}=\mathscr{J}_\Sigma{}^{\mu_1\cdots\mu_k}-\mathscr{D}_\lambda\mathscr{W}_\Sigma{}^{\lambda\mu_1\cdots\mu_k}=0\ .$$

The conversion laws (3.7) read

$$\mathscr{D}_\lambda \mathscr{T}_{A\dot B}{}^{\lambda} = -\mathscr{T}_{C\dot D}{}^{\lambda} Q^{C\dot D}{}_{\mu\lambda} e_{A\dot B}{}^{\mu} - e_{A\dot B}{}^{\mu}\Bigg[\mathfrak{s}_C{}^{D\lambda} R^C{}_{D\mu\lambda} + \frac{1}{k!}(\mathscr{E}\mathscr{M})_\Sigma{}^{\lambda_1\cdots\lambda_k} F^\Sigma_{\mu\lambda_1\cdots\lambda_k} + \frac{1}{(k-1)!}\mathscr{D}_\lambda(\mathscr{E}\mathscr{M})_\Sigma{}^{\lambda\lambda_2\cdots\lambda_k}\phi^\Sigma_{\mu\lambda_2\cdots\lambda_k} + \text{c.c.}\Bigg], \tag{D7}$$

$$\mathscr{D}_\lambda \mathfrak{s}_A{}^{B\lambda} = -\mathscr{T}_A{}^B - \frac{1}{k!}(\mathscr{E}\mathscr{M})_\Sigma{}^{\lambda_1\cdots\lambda_k}\phi^\Lambda{}_{\lambda_1\cdots\lambda_k}\rho^\Sigma{}_{\Lambda A}{}^B - \frac{1}{k!}(\mathscr{E}\mathscr{M})_{\dot\Sigma}{}^{\lambda_1\cdots\lambda_k}\phi^{\dot\Lambda}{}_{\lambda_1\cdots\lambda_k}\rho^{\dot\Sigma}{}_{\dot\Lambda A}{}^B .$$

The contracted Bianchi identities (3.9) read

$$\mathscr{D}_\lambda(\mathscr{E}1)_{A\dot B}{}^{\lambda} = -(\mathscr{E}1)_{C\dot D}{}^{\lambda} Q^{C\dot D}{}_{\mu\lambda} e_{A\dot B}{}^{\mu} - e_{A\dot B}{}^{\mu}\Bigg[(\mathscr{E}2)_C{}^{D\lambda} R^C{}_{D\mu\lambda} - \frac{1}{k!}(\mathscr{E}\mathscr{M})_\Sigma{}^{\lambda_1\cdots\lambda_k} F^\Sigma_{\mu\lambda_1\cdots\lambda_k} + \frac{1}{(k-1)!}\mathscr{D}_\lambda(\mathscr{E}\mathscr{M})_\Sigma{}^{\lambda\lambda_2\cdots\lambda_k}\phi^\Sigma_{\mu\lambda_2\cdots\lambda_k} + \text{c.c.}\Bigg], \tag{D8}$$

$$D_\lambda(\mathscr{E}2)_A{}^{B\lambda} = -(\mathscr{E}1)_A{}^B - \frac{1}{k!}(\mathscr{E}\mathscr{M})_\Sigma{}^{\lambda_1\cdots\lambda_k}\phi^\Lambda{}_{\lambda_1\cdots\lambda_k}\rho^\Sigma{}_{\Lambda A}{}^B - \frac{1}{k!}(\mathscr{E}\mathscr{M})_{\dot\Sigma}{}^{\lambda_1\cdots\lambda_k}\phi^{\dot\Lambda}{}_{\lambda_1\cdots\lambda_k}\rho^{\dot\Sigma}{}_{\dot\Lambda A}{}^B .$$

APPENDIX E: SU(2)-COVARIANT DIFFERENTIAL OPERATORS AND DECOMPOSITION OF THE COVARIANT EXTERIOR DERIVATIVE

In the present paper we deal with SL(2, C)-spinor-valued differential forms on spacetime. Generally, we may consider SL(2, C)-spinor-valued spacetime tensor fields and tensor densities. Let $\mathscr{F}^\Sigma = (\mathscr{F}^{\Sigma\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l})$ be a spacetime tensor density of weight r with values in a spinor space. We define the following SL(2, C)-covariant derivative

$$\begin{aligned}\mathscr{D}_\lambda \mathscr{F}^{\Sigma\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} ={}& \partial_\lambda \mathscr{F}^{\Sigma\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} - r\gamma^\tau{}_{\tau\lambda}\mathscr{F}^{\Sigma\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} + \rho^\Sigma{}_{\Lambda A}{}^B\Gamma^A{}_{B\lambda}\mathscr{F}^{\Lambda\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} + \rho^\Sigma{}_{\Lambda\dot A}{}^{\dot B}\Gamma^{\dot A}{}_{\dot B\lambda}\mathscr{F}^{\Lambda\mu_1\cdots\mu_k}_{\nu_1\cdots\nu_l} + \gamma^{\mu_1}{}_{\alpha\lambda}\mathscr{F}^{\Sigma\alpha\cdots\mu_k}_{\nu_1\cdots\nu_l} + \cdots \\ &+ \gamma^{\mu_k}{}_{\alpha\lambda}\mathscr{F}^{\Sigma\mu_1\cdots\alpha}_{\nu_1\cdots\nu_l} - \gamma^\beta{}_{\nu_1\lambda}\mathscr{F}^{\Sigma\mu_1\cdots\mu_k}_{\beta\cdots\nu_l} - \cdots - \gamma^\beta{}_{\nu_l\lambda}\mathscr{F}^{\Sigma\mu_1\cdots\mu_k}_{\nu_1\cdots\beta} .\end{aligned} \tag{E1}$$

where $\gamma^\epsilon{}_{\tau\lambda}$ are components of the Riemannian connection on M. In particular, if A^Σ is a spinor-valued k-form on M given by (A6) then

$$DA^\Sigma = \frac{1}{k!}\mathscr{D}_{\nu_0}A^\Sigma{}_{\nu_1\cdots\nu_k}dx^{\nu_0}\wedge\cdots\wedge dx^{\nu_k} . \tag{E2}$$

For the dual representation (A7) we obtain

$$DA^\Sigma = \frac{1}{(3-k)!\sqrt{-g}}\mathscr{D}_\lambda A^\Sigma{}_{\mu_1\cdots\mu_{3-k}}{}^{\lambda} * (dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{3-k}}) . \tag{E3}$$

In the (3 + 1) picture we work with the tilde-overbar components of geometric objects. The quantities $\tilde{\mathscr{F}}^{\Sigma\bar\mu_1\cdots\bar\mu_k}_{\bar\nu_1\cdots\bar\nu_l}$ and $\tilde{\mathscr{D}}_{\bar\lambda}\tilde{\mathscr{F}}^{\Sigma\bar\mu_1\cdots\bar\mu_k}_{\bar\nu_1\cdots\bar\nu_l}$ can be computed by means of formulas (6.10) and (A12). The tilde-overbar components of the covariant derivative can be also computed from (E1) directly if we take the formal tilde-overbar expression of the right-hand side of this relation. The corresponding formulas for $\bar\partial_\lambda$ and $\bar\gamma^\epsilon{}_{\tau\lambda}$ have been presented in Ref. 35 [$\bar\partial_\lambda$ are defined by (6.12a)].

Now we discuss SU(2)-covariant geometric objects on the initial (spacelike) surface σ. Let $\hat{\mathscr{F}}^\Sigma = (\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}_{r_1\cdots r_l})$ be a spatial tensor density of weight r with values in an SU(2)-spinor space. We define the σ-tangential covariant derivaitve of $\hat{\mathscr{F}}^\Sigma$

$$\begin{aligned}{}^3\mathscr{D}_p\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}_{r_1\cdots r_l} ={}& \partial_p\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}_{r_1\cdots r_l} - r\bar\gamma^q{}_{qp}\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}_{r_1\cdots r_l} + \rho^\Sigma{}_{\Lambda A}{}^{B\,a}\hat\Gamma^A{}_{Bp}\hat{\mathscr{F}}^{\Lambda s_1\cdots s_k}_{r_1\cdots r_l} + \rho^\Sigma{}_{\Lambda\dot A}{}^{\dot B\,a}\hat\Gamma^{\dot A}{}_{\dot Bp}\hat{\mathscr{F}}^{\Lambda s_1\cdots s_k}_{r_1\cdots r_l} \\ &+ \bar\gamma^{s_1}{}_{qp}\hat{\mathscr{F}}^{\Sigma q\cdots s_k}_{r_1\cdots r_l} + \cdots + \bar\gamma^{s_k}{}_{qp}\hat{\mathscr{F}}^{\Sigma s_1\cdots q}_{r_1\cdots r_l} - \bar\gamma^q{}_{r_1p}\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}_{q\cdots r_l} - \cdots - \bar\gamma^q{}_{r_lp}\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}_{r_1\cdots q} .\end{aligned} \tag{E4}$$

Here $\bar\gamma^r{}_{sp} = \gamma^{\bar r}{}_{\bar s\bar p}$ are components of the Riemannian connection on σ induced by the connection $\gamma^\epsilon{}_{\tau\lambda}$ on M, ${}^a\hat\Gamma^A{}_{Bp}$ are the components of an su(2)-valued connection one-form on σ.

Remark. As it was mentioned in Appendix B, for SU(2)-spinors the matrices $\delta_{A\dot B}$ and $\delta^{A\dot B}$ realize an isomorphism between dotted and undotted spinor spaces. We maintain, however, the formal distinction between these spaces in order to have formulas manifestly corresponding to the four-dimensional picture.

In order to define a σ-normal SU(2)-covariant derivative, we need connection coefficients for the $\bar\partial_0$ operator. It has been shown recently in Ref. 38 that for SU(2)-spinor indices we may take anti-Hermitian matrices ${}^a\hat\Gamma^A{}_{B\perp}$ and for spatial indices appropriate connection coefficients are the following objects $\sigma^s{}_r = N^{-1}\cdot\partial_r N^s$. We define

$$ {}^3\mathscr{D}_0\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}{}_{r_1\cdots r_t}=\overline{\partial}_0\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}{}_{r_1\cdots r_t}-r\sigma^p{}_p\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}{}_{r_1\cdots r_t}+\rho^\Sigma{}_{\Lambda A}{}^{Ba}\hat{\Gamma}{}^A{}_{B\perp}\hat{\mathscr{F}}^{\Lambda s_1\cdots s_k}{}_{r_1\cdots r_t}+\rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}a}\hat{\Gamma}{}^{\dot{A}}{}_{\dot{B}\perp}\hat{\mathscr{F}}^{\Lambda s_1\cdots s_k}{}_{r_1\cdots r_t} $$
$$ +\sigma^{s_1}{}_p\hat{\mathscr{F}}^{\Sigma p\cdots s_k}{}_{r_1\cdots r_t}+\cdots+\sigma^{s_k}{}_p\hat{\mathscr{F}}^{\Sigma s_1\cdots p}{}_{r_1\cdots r_t}-\sigma^p{}_{r_1}\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}{}_{p\cdots r_t}-\cdots-\sigma^p{}_{r_t}\hat{\mathscr{F}}^{\Sigma s_1\cdots s_k}{}_{r_1\cdots p}\ . \tag{E5} $$

The differential operators (E4) and (E5) are covariant with respect to local (x^0,x^s)-dependent SU(2) rotations of triads ${}^t_{||}\hat{e}^{A\dot{B}}$ on the initial surface σ and are consistent with the transformations (6.11).

Remark. The connection coefficients $\sigma^r{}_s$ can be obtained by the following construction. For the nonholonomic basis (6.12b), we define the object of anholonomity

$$ d(\overline{dx}^\lambda)=\tfrac{1}{2}\sigma^\lambda{}_{\overline{\mu\nu}}\overline{dx}^\mu\wedge\overline{dx}^\nu\ . \tag{E6} $$

We get $\sigma^{\bar{0}}{}_{\bar{r}\bar{0}}=\partial_{\bar{r}}\ln N$, $\sigma^{\bar{r}}{}_{\bar{s}\bar{0}}=N^{-1}\cdot\partial_{\bar{s}}N^r$, $\sigma^\lambda{}_{\overline{rs}}=0$. We see that $\sigma^r{}_s=\sigma^{\bar{r}}{}_{\bar{s}\bar{0}}$.

For an SU(2)-spinor-valued k form $\hat{\phi}^\Sigma$ on σ

$$ \hat{\phi}^\Sigma=\frac{1}{k!}\hat{\phi}^\Sigma{}_{s_1\cdots s_k}dx^{s_1}\wedge\cdots\wedge dx^{s_k} \tag{E7} $$

the SU(2)-covariant exterior derivative is given by

$$ {}^3D\hat{\phi}^\Sigma=\frac{1}{k!}{}^3\mathscr{D}_{s_0}\hat{\phi}^\Sigma{}_{s_1\cdots s_k}dx^{s_0}\wedge\cdots\wedge dx^{s_k}\ . \tag{E8} $$

Formulas (6.18) and (6.20) enable us to perform the (3 + 1) decomposition of the covariant exterior derivative. We get

$$ {}_{||}(DA^\Sigma)\hat{}={}^3D_{||}\hat{A}^\Sigma+\rho^\Sigma{}_{\Lambda A}{}^{B}{}^h_{||}\hat{\Gamma}{}^A{}_B\wedge{}_{||}\hat{A}^\Lambda+\rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}{}^h_{||}\hat{\Gamma}{}^{\dot{A}}{}_{\dot{B}}\wedge{}_{||}\hat{A}^\Lambda\ , \tag{E9} $$

$$ {}_\perp(DA^\Sigma)\hat{}=\frac{1}{k!}{}^3\mathscr{D}_0\hat{A}^\Sigma{}_{s_1\cdots s_k}dx^{s_1}\wedge\cdots\wedge dx^{s_k}+\rho^\Sigma{}_{\Lambda A}{}^{B}{}^h_\perp\hat{\Gamma}{}^A{}_B\,{}_{||}\hat{A}^\Lambda+\rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}{}^h_\perp\hat{\Gamma}{}^{\dot{A}}{}_{\dot{B}}\,{}_{||}\hat{A}^\Lambda-{}^3D_\perp\hat{A}^\Sigma-\rho^\Sigma{}_{\Lambda A}{}^{B}{}^h_{||}\hat{\Gamma}{}^A{}_B\wedge{}_\perp\hat{A}^\Lambda $$
$$ -\rho^\Sigma{}_{\Lambda\dot{A}}{}^{\dot{B}}{}^h_{||}\hat{\Gamma}{}^{\dot{A}}{}_{\dot{B}}\wedge{}_\perp\hat{A}^\Lambda-{}^3d\ln N\wedge{}_\perp\hat{A}^\Sigma\ . \tag{E10} $$

For the Hodge-dual representation of a form A^Σ (A7) the first term in formula (E10) is replaced by

$$ \frac{(\sqrt{g}\,)^{-1}}{(3-k)!}({}^3\mathscr{D}_0\hat{\mathscr{A}}^{\Sigma\perp s_1\cdots s_{3-k}})\overline{g}_{s_1r_1}\cdots\overline{g}_{s_{3-k}r_{3-k}}{}^3{*}(dx^{r_1}\wedge\cdots\wedge dx^{r_{3-k}})\ . \tag{E11} $$

Making use of the above formulas and taking into account that the anticommutator of two ${}^tH(2)$ matrices is an ${}^dH(2)$ matrix, and the anticommutator of an ${}^tH(2)$ matrix and an ${}^dH(2)$ matrix is an ${}^tH(2)$ matrix, we get the (3 + 1) decomposition of the torsion two-form $\Theta^{A\dot{B}}=De^{A\dot{B}}$:

$$ {}^d_{||}\hat{\Theta}^{A\dot{B}}={}^h_{||}\hat{\Gamma}{}^A{}_C\wedge{}^t_{||}\hat{e}^{C\dot{B}}+{}^h_{||}\hat{\Gamma}{}^{\dot{B}}{}_{\dot{D}}\wedge{}^t_{||}\hat{e}^{A\dot{D}}\ ,\quad {}^t_{||}\hat{\Theta}^{A\dot{B}}={}^3D{}^t_{||}\hat{e}^{A\dot{B}}\ , $$
$$ {}^d_\perp\hat{\Theta}^{A\dot{B}}={}^h_\perp\hat{\Gamma}{}^A{}_C\cdot{}^t_{||}\hat{e}^{C\dot{B}}+{}^h_\perp\hat{\Gamma}{}^{\dot{B}}{}_{\dot{D}}\cdot{}^t_{||}\hat{e}^{A\dot{D}}-{}^3d\ln N\ ,\quad {}^t_\perp\hat{\Theta}^{A\dot{B}}={}^3\mathscr{D}_0{}^t\hat{e}^{A\dot{B}}{}_sdx^s-2{}^h_{||}\hat{\Gamma}{}^A{}_C\sigma^{C\dot{B}}_{(0)}\ . \tag{E12} $$

It follows from (E12) that the two-form ${}^t_{||}\hat{\theta}^{A\dot{B}}$ is torsion of the connection ${}^a_{||}\hat{\Gamma}{}^A{}_B$ on σ. Analogously we have

$$ {}^a_\perp\hat{\Omega}{}^A{}_B={}^3\mathscr{D}_0{}^a\hat{\Gamma}{}^A{}_{Bs}dx^s+{}^h_\perp\hat{\Gamma}{}^A{}_C\cdot{}^h_{||}\hat{\Gamma}{}^C{}_B-{}^h_\perp\hat{\Gamma}{}^C{}_B\cdot{}^h_{||}\hat{\Gamma}{}^A{}_C\ ,\quad {}^h_\perp\hat{\Omega}{}^A{}_B={}^3\mathscr{D}_0{}^h\hat{\Gamma}{}^A{}_{Bs}dx^s-{}^3D{}^h_\perp\hat{\Gamma}{}^A{}_B-{}^h_\perp\hat{\Gamma}{}^A{}_B\cdot{}^3d\ln N\ , $$
$$ {}^a_{||}\hat{\Omega}{}^A{}_B={}^3\Omega^A{}_B+{}^h_{||}\hat{\Gamma}{}^A{}_C\wedge{}^h_{||}\hat{\Gamma}{}^C{}_B\ ,\quad {}^h_{||}\hat{\Omega}{}^A{}_B={}^3D{}^h_{||}\hat{\Gamma}{}^A{}_B\ . \tag{E13} $$

Here

$$ {}^3\Omega^A{}_B={}^3d{}^a_{||}\hat{\Gamma}{}^A{}_B+{}^a_{||}\hat{\Gamma}{}^A{}_C\wedge{}^a_{||}\hat{\Gamma}{}^C{}_B \tag{E14} $$

is the curvature two-form of the connection ${}^a_{||}\hat{\Gamma}{}^A{}_B$ on σ and

$$ {}^3\mathscr{D}_0{}^a\hat{\Gamma}{}^A{}_{Bs}=\overline{\partial}_0{}^a\hat{\Gamma}{}^A{}_{Bs}-\partial_s{}^a\hat{\Gamma}{}^A{}_{B\perp}+{}^a\hat{\Gamma}{}^A{}_{C\perp}\cdot{}^a\hat{\Gamma}{}^C{}_{Bs}-{}^a\hat{\Gamma}{}^A{}_{Cs}\cdot{}^a\hat{\Gamma}{}^C{}_{B\perp}-{}^a\hat{\Gamma}{}^A{}_{B\perp}\partial_s\ln N-{}^a\hat{\Gamma}{}^A{}_{Br}N^{-1}\cdot\partial_sN^r\ . \tag{E15} $$

Remark. We note that ${}^h_{||}\hat{\Gamma}$ and ${}^h_\perp\hat{\Gamma}$ are SU(2)-spinor-valued differential forms on σ. They obey tensor transformation rules with respect to the action of the group SU(2) (Ref. 8).

The (3 + 1) decomposition of SL(2, C)-covariant Lie derivative

If a connection ζ on spacetime satisfies the condition ${}^h\tilde{\zeta}^A{}_B=0$ and if $Z^0=Z^0(x^0)$ [cf. (7.4) and (7.5)], then for a spinor-valued k-form A^Σ (A6) we have

$$ {}_{\|}({}_{\zeta}\mathscr{L}_Z A^{\Sigma})\hat{} = {}^{\perp}Z \cdot {}^{3}_{\zeta}\mathscr{D}_{0\|}\hat{A}^{\Sigma} + {}^{3}_{\zeta}\mathscr{L}_{Z\|}\hat{A}^{\Sigma} , \tag{E16} $$

$$ {}_{\perp}({}_{\zeta}\mathscr{L}_Z A^{\Sigma})\hat{} = {}^{\perp}Z \cdot {}^{3}_{\zeta}\mathscr{D}_{0\perp}\hat{A}^{\Sigma} + {}^{3}_{\zeta}\mathscr{L}_{Z\perp}\hat{A}^{\Sigma} + ({}^{3}_{\zeta}\mathscr{D}_0{}^{\perp}Z + {}^{\|}Z \lrcorner {}^{3}d\ln N)_{\perp}\hat{A}^{\Sigma} + ({}^{3}_{\zeta}\mathscr{D}_0{}^{\|}Z)\lrcorner_{\|}\hat{A}^{\Sigma} . \tag{E17} $$

For the component description of A^{Σ} (E16) reads

$$ \begin{aligned} {}_{\zeta}\mathscr{L}_Z \hat{A}^{\Sigma}{}_{s_1\cdots s_k} &= {}^{\perp}Z \cdot {}^{3}_{\zeta}\mathscr{D}_0 \hat{A}^{\Sigma}{}_{s_1\cdots s_k} + {}^{3}_{\zeta}\mathscr{L}_Z \hat{A}^{\Sigma}{}_{s_1\cdots s_k} , \\ {}_{\zeta}\mathscr{L}_Z \hat{\mathscr{A}}^{\Sigma\perp s_2\cdots s_{4-k}} &= {}^{\perp}Z \cdot {}^{3}_{\zeta}\mathscr{D}_0 \hat{\mathscr{A}}^{\Sigma\perp s_2\cdots s_{4-k}} + {}^{3}_{\zeta}\mathscr{L}_Z \hat{\mathscr{A}}^{\Sigma\perp s_2\cdots s_{4-k}} . \end{aligned} \tag{E16′} $$

In these formulas ${}^{3}_{\zeta}\mathscr{D}_0$ is the covariant "time" derivative (E5) computed with respect to the background connection ${}^{a}\hat{\zeta}^{A}{}_{B\perp}$ instead of the dynamical one ${}^{a}\hat{\Gamma}^{A}{}_{B\perp}$. ${}^{3}_{\zeta}\mathscr{L}_Z$ denotes the covariant Lie derivative on particular slices (with respect to the background connection ${}^{3}\zeta$) taken in the direction of σ-tangential part of the vector field Z.

APPENDIX F: THE GRAVITATIONAL CONSTRAINTS IN THE SU(2)-COVARIANT FORM

$$ \begin{aligned} {}^{d}(\mathscr{C}1)\hat{}^{\perp} = \overline{\mathscr{K}} + \overline{\mathscr{L}} - \hat{\mathscr{U}}_{C\dot{D}}{}^{\perp s}\cdot\hat{Q}^{C\dot{D}}{}_{\perp s} + {}^{3}\mathscr{D}_s{}^{d}\hat{\mathscr{U}}^{\perp s} - 2{}^{t}\hat{\mathscr{U}}_{A\dot{C}}{}^{\perp s}\cdot{}^{h}\hat{\Gamma}^{A}{}_{Bs}\sigma^{B\dot{C}}_{(0)} \\ - \left[\hat{\mathscr{P}}_{C}{}^{D\perp s}\cdot\hat{R}^{C}{}_{D\perp s} + \frac{1}{k!}\hat{\mathscr{W}}_{\Sigma}{}^{\perp s_1\cdots s_k}\hat{F}^{\Sigma}{}_{\perp s_1\cdots s_k} + \frac{1}{(k-1)!}\hat{\mathscr{J}}_{\Sigma}{}^{\perp s_2\cdots s_k}\hat{\phi}^{\Sigma}{}_{\perp s_2\cdots s_k} + \text{c.c.} \right] = 0 , \end{aligned} \tag{F1} $$

$$ \begin{aligned} {}^{t}(\mathscr{C}1)\hat{}_{A\dot{B}}{}^{\perp} = {}^{3}\mathscr{D}_s{}^{t}\hat{\mathscr{U}}_{A\dot{B}}{}^{\perp s} - 2{}^{d}\hat{\mathscr{U}}^{\perp s}\cdot{}^{h}\hat{\Gamma}^{C}{}_{As}\sigma_{(0)C\dot{B}} \\ + {}^{t}\hat{e}_{A\dot{B}}{}^{r}\left[\tfrac{1}{2}\hat{\mathscr{U}}_{C\dot{D}}{}^{\perp s}\cdot\hat{Q}^{C\dot{D}}{}_{rs} + \hat{\mathscr{P}}_{C}{}^{D\perp s}\cdot\hat{R}^{C}{}_{Drs} + \frac{1}{k!}\hat{\mathscr{W}}_{\Sigma}{}^{\perp s_1\cdots s_k}\hat{F}^{\Sigma}{}_{rs_1\cdots s_k} + \frac{1}{(k-1)!}\hat{\mathscr{J}}_{\Sigma}{}^{\perp s_2\cdots s_k}\hat{\phi}^{\Sigma}{}_{rs_2\cdots s_k} \right. \\ \left. + \text{c.c} \right] = 0 , \end{aligned} \tag{F2} $$

$$ (\mathscr{C}2)\hat{}_{A}{}^{B\perp} = \hat{\mathfrak{I}}_{A}{}^{B\perp} + \hat{\mathscr{U}}_{A}{}^{B\perp} + {}^{3}\mathscr{D}_s\hat{\mathscr{P}}_{A}{}^{B\perp s} - {}^{h}\hat{\Gamma}^{C}{}_{As}\hat{\mathscr{P}}_{C}{}^{B\perp s} + {}^{h}\hat{\Gamma}^{B}{}_{Cs}\hat{\mathscr{P}}_{A}{}^{C\perp s} = 0 , \tag{F3} $$

$$ (\mathscr{C}\mathscr{M})\hat{}_{\Sigma}{}^{\perp s_2\cdots s_k} = \hat{\mathscr{J}}_{\Sigma}{}^{\perp s_2\cdots s_k} + {}^{3}\mathscr{D}_r\hat{\mathscr{W}}_{\Sigma}{}^{\perp rs_2\cdots s_k} - \hat{\mathscr{W}}_{\Lambda}{}^{\perp rs_2\cdots s_k}\cdot{}^{h}\hat{\Gamma}^{A}{}_{Br}\rho^{\Lambda}{}_{\Sigma A}{}^{B} - \hat{\mathscr{W}}_{\Lambda}{}^{\perp rs_2\cdots s_k}\cdot{}^{h}\hat{\Gamma}^{\dot{A}}{}_{\dot{B}r}\rho^{\Lambda}{}_{\Sigma\dot{A}}{}^{\dot{B}} = 0 . \tag{F4} $$

Addendum. In their recent paper,[65] Blagojević and Nikolić discussed the Hamiltonian dynamics of the theories of gravity with Lagrangians quadratic in curvature and torsion. They have found several canonical constraints for particular Lagrangians and investigated the time maintenance of these constraints. We would like to note, however, that the complete treatment of particular theories requires a much more complicated analysis than that presented in Ref. 65. Recently, one of us has solved the problem for the Yang theory[66] giving its complete set of canonical constraints, dynamical equations, and gauge transformations.[67]

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[1]A. Salam, Rev. Mod. Phys. **52**, 525 (1980).

[2]A. Salam and J. Strathdee, Ann. Phys. (N.Y.) **141**, 316 (1982).

[3]P. van Nieuwenhuizen, Phys. Rep. **68**, 189 (1981); L. Castellani, P. Fré, and P. van Nieuwenhuizen, Ann. Phys. (N.Y.) **136**, 398 (1981); Y. Ne'eman, in *Cosmology and Gravitation*, edited by P. Bergmann and V. de Sabbata (Plenum, New York, 1980).

[4]A. Trautman, Symp. Math. **12**, 139 (1973).

[5]A. Trautman, in *General Relativity and Gravitation*, edited by A. Held (Plenum, New York, 1980); Acta Phys. Austriaca Suppl. **23**, 401 (1981); in *Geometric Techniques in Gauge Theories*, Proceedings of the Fifth Scheveningen Conference on Differential Equations, The Netherlands, 1981, edited by R. Martini, and E. M. De Jager (Lecture Notes in Mathematics **926**) (Springer, Berlin, 1982).

[6]F. W. Hehl, in *Cosmology and Gravitation* (Ref. 3).

[7]A. Tseytlin, Phys. Rev. D **26**, 3327 (1982).

[8]M. Antonowicz and W. Szczyrba, Lett. Math. Phys. (to be published).

[9]E. Sezgin and P. Nieuwenhuizen, Phys. Rev. D **21**, 3269 (1980).

[10]J. Kijowski and W. M. Tulczyjew, *A Symplectic Framework for Field Theories* (Lecture Notes in Physics **107**) (Springer, Berlin, 1979).

[11]J. Kijowski, Gen. Relativ. Gravit. **9**, 857 (1978); M. Ferraris and J. Kijowski, Lett. Math. Phys. **5**, 127 (1981); Gen. Relativ. Gravit. **14**, 37 (1982); **14**, 165 (1982).

[12]A. E. Fischer and J. E. Marsden, in *Proceedings of the International School of Physics, Enrico Fermi Course LXVII*, edited by J. Ehlers (North-Holland, Amsterdam, 1979); in *General Relativity: An Einstein Centenary Survey*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).

[13]A. Fischer, J. Marsden, and V. Moncrief, Ann. Inst. Henri Poincaré **A33**, 147 (1980); J. Arms, J. Marsden, and V. Moncrief, Ann. Phys. (N.Y.) **144**, 81 (1982).

[14]J. M. Arms, Math. Proc. Cambridge Philos. Soc. **90**, 361 (1981).

[15]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964); A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Roma, 1976); K. Sundermeyer, *Constrained Dynamics* (Lecture Notes in Physics **169**) (Springer, Berlin, 1982).

[16]J. A. Isenberg and J. M. Nester, in *General Relativity and*

Gravitation (Ref. 5).

[17]R. Penrose, *Battelle Rencontres in Mathematics and Physics* (Benjamin, New York, 1968); R. Geroch and G. T. Horowitz, *General Relativity: An Einstein Centenary Survey* (Ref. 12).

[18]J. W. Milnor, Enseign. Math. **9**, 198 (1963); C.J. Isham, Proc. R. Soc. London **A364**, 591 (1978); J. Avis and C. J. Isham, in *Recent Developments in Gravitation, Cargèse 1978,* edited by M. Lévy and S. Deser (Plenum, New York, 1979); Commun. Math. Phys. **72**, 103 (1980).

[19]R. Geroch, J. Math. Phys. **9**, 1739 (1968).

[20]N. Steenrod, *The Topology of Fibre Bundles* (Princeton, New Jersey, 1951); J. Milnor and J. Stasheff, *Characteristic Classes* (Princeton University Press, Princeton, New Jersey, 1974).

[21]F. W. Hehl, J. Nitsch, and P. von der Heyde, in *General Relativity and Gravitation* (Ref. 5).

[22]W. Szczyrba, Phys. Rev. D **25**, 2548 (1982).

[23]S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (Interscience, New York, 1963).

[24]P. B. Yasskin, Ph.D. dissertation, University of Maryland, 1979 (unpublished).

[25]W. Szczyrba, in *Cosmology and Gravitation* (Ref. 3).

[26]M. Schweitzer, in *Cosmology and Gravitation* (Ref. 3).

[27]W. Kopczyński, J. Phys. A **15**, 493 (1982).

[28]L. Rosenfeld, Acad. R. Belg. **18**, No. 6 (1940); F. J. Belinfante, Physica **6**, 887 (1939); W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).

[29]D. W. Sciama, Proc. Cambridge Philos. Soc. **54**, 72 (1958); in Recent Developments in General Relativity (PWN-Polish Scientific, Warsaw, 1964); T. W. B. Kibble, J. Math. Phys. **2**, 212 (1961).

[30]F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. **48**, 393 (1976).

[31]R. Arnowitt, S. Deser, and C. Misner, in *Gravitation: An Introduction to Current Research,* edited by L. Witten (Wiley, New York, 1962).

[32]J. Isenberg, J. Nester, and R. Skinner (unpublished).

[33]L. Landau and E. Lifschitz, *Field Theory* (Nauka, Moscow, 1973).

[34]A. Komar, Phys. Rev. **113**, 934 (1959).

[35]W. Szczyrba, J. Math. Phys. **22**, 1926 (1981).

[36]J. Kijowski and W. Szczyrba, Commun. Math. Phys. **46**, 183 (1976); P. L. Garciá, in *Differential Geometrical Methods in Mathematical Physics,* edited by K. Bleuler and A. Reetz (Lecture Notes in Mathematics **570**) (Springer, Berlin, 1977).

[37]V. Guillemin and S. Sternberg, *Geometric Asymptotics,* Math. Survey 14 (American Mathematical Society, Providence, 1977).

[38]W. Szczyrba, Ann. Phys. (N.Y.) **158**, 320 (1984).

[39]R. Abraham and J. Marsden, *Foundations of Mechanics* (Benjamin, New York, 1978).

[40]J. M. Arms, J. E. Marsden, and V. Moncrief, Commun. Math. Phys. **78**, 455 (1981).

[41]A. Barut and R. Raczka, *Theory of Group Representations and Applications* (PWN-Polish Scientific, Warsaw, 1977).

[42]W. Szczyrba, Commun. Math. Phys. **60**, 215 (1978); J. A. Isenberg and J. M. Nester, Ann. Phys. (N.Y.) **107**, 56 (1977).

[43]W. Szczyrba (unpublished).

[44]A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnétisme* (Masson, Paris, 1955).

[45]J. A. Wheeler, in *Relativity, Groups and Topology,* edited by B. DeWitt and C. DeWitt (Gordon and Breach, New York, 1964).

[46]A. Fischer and J. Marsden, J. Math. Phys. **13**, 546 (1972).

[47]W. Szczyrba, Commun. Math. Phys. **51**, 163 (1976); Diss. Math. **150**, 1 (1977).

[48]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[49]B. DeWitt, Phys. Rev. **160**, 1113 (1967); T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974); Y. Choquet-Bruhat, A. Fischer, and J. M. Marsden, in *Proceedings of the International School of Physics Enrico Fermi, Course LXVII* (Ref. 12); J. W. York, Jr., in *Essays in General Relativity,* edited by F. Tipler (Academic, New York, 1980); L. Faddeev, Usp. Fiz. Nauk. **136**, 435 (1982) [Sov. Phys. Usp. **25**, 130 (1982).

[50]R. Schoen and S. T. Yau, Commun. Math. Phys. **65**, 45 (1979); **79**, 231 (1981); E. Witten, *ibid.* **80**, 381 (1981).

[51]P. Baekler, F. W. Hehl, and E. W. Mielke, in *Proceedings of the Second Marcel Grossman Meeting on General Relativity,* edited by R. Ruffini (North-Holland, Amsterdam, 1982); P. Baekler, F. W. Hehl, and H. J. Lenzen (unpublished).

[52]A. Ashtekar and G. Horowitz, report, 1983 (unpublished).

[53]W. Beresteckij, E. Lifschitz, and L. Pitaevskij, *Relativistic Quantum Theory* (Nauka, Moscow, 1968).

[54]M. Fierz, Helv. Phys. Acta **12**, 3 (1938); M. Fierz and W. Pauli, Proc. R. Soc. London **A173**, 211 (1939).

[55]J. E. Nelson and C. Teitelboim, Ann. Phys. (N.Y.) **116**, 86 (1978); M. Henneaux, Phys. Rev. D **21**, 857 (1980).

[56]P. A. M. Dirac, in *Recent Developments in General Relativity* (PWN-Polish Scientific, Warsaw, 1964).

[57]M. Henneaux, Gen. Relativ. Gravit. **9**, 1031 (1978).

[58]T. W. B. Kibble, J. Math. Phys. **4**, 1433 (1963); B. K. Datta, Nuovo Cimento **6B**, 1 (1971).

[59]Y. Choquet-Bruhat, in *Gravitation: An Introduction to Current Research* (Ref. 31); S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Spacetime* (Cambridge University Press, Cambridge, 1973).

[60]Y. Choquet-Bruhat and T. Ruggieri, Commun. Math. Phys. **89**, 269 (1983).

[61]L. Castellani, Ann. Phys. (N.Y.) **143**, 357 (1982).

[62]V. Moncrief, J. Math. Phys. **16**, 1556 (1975).

[63]J. A. Isenberg and J. E. Marsden, Phys. Rep. **89**, 179 (1982).

[64]D. G. Ebin and J. E. Marsden, Ann. Math. **92**, 102 (1970); H. Omori, in *Proceedings in Pure and Applied Mathematics,* Vol. 15 (American Mathematical Society, Providence, 1970); *Infinite Dimensional Lie Transformation Groups* (Lecture Notes in Mathematics **427**) (Springer, Berlin, 1974).

[65]M. Blagojević and I. Niklolić, Phys. Rev. D **28**, 2455 (1983).

[66]C. N. Yang, Phys. Rev. Lett. **33**, 445 (1974).

[67]W. Szczyrba, report, Warsaw, 1984 (unpublished).