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Charged spinning fluids with magnetic dipole moment in the Einstein-Cartan theory

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A classical perfect charged spinning fluid with magnetic dipole moment in the Einstein-Cartan theory is described by using an Eulerian Lagrangian formalism. The field equations and equations of motion so obtained generalize those proposed by Ray and Smalley. We also clarify some open questions which appear in the works of Ray and Smalley and of de Ritis *et al.*

I. INTRODUCTION

At present, classical spinning fluids are often used as a simple model of media with internal degrees of freedom. The pioneer work of Halbwachs,[1] which describes perfect neutral spinning fluids in special relativity, has been extended to more complete theories like general relativity (GR)[2] and Einstein-Cartan theory (EC theory).[3–5] In the latter context, spin acquires a fundamental status, since it is dynamically coupled to geometry.[3]

In a previous work[6] we have presented Lagrangian treatment of charged spinning fluids with magnetic dipole moment in GR, which generalizes the work of Ray and Smalley[2] on neutral spinning fluids.

In this paper we consider the fluid treated in Ref. 6 in the framework of the EC theory. The spin-torsion interaction is introduced by using a minimal-coupling procedure. In the limit of vanishing electromagnetic (EM) fields we obtain the same energy-momentum tensor proposed by Ray and Smalley.[3] It is possible, however, to extend their results, by writing the complete differential system for the fluid. This complete system of equations allows us to perform a critical analysis on some topics treated by Ray and Smalley[3] and by de Ritis *et al.*[4] We clarify the dependence on spin of the proper internal energy and its relation to the Lagrangian treatment. The origin of the Mathisson force which appears in the generalized Euler equation is also clarified.

In Sec. II we present a brief review of the EC theory. In Sec. III we introduce the Lagrangian which describes the charged spinning fluid. Section IV analyzes with some detail the spin-torsion coupling induced by that Lagrangian. In Sec. V we obtain some of the alternative forms of the energy-momentum tensor and give the explicit form of the conservation laws. Section VI is devoted to point out some consequences of the Euler equation deduced in Sec. V.

II. THE EINSTEIN-CARTAN THEORY

The general theory of relativity is constructed in a Riemannian space-time. The arena of physics in the EC theory is the more general Riemann-Cartan manifold U_4 where the connection is asymmetric and its antisymmetric part

$$\Gamma^k_{[ij]}=S_{ij}{}^k \tag{2.1}$$

is the torsion tensor related to the spin by the U_4 field equations. The derivation of the equations which will be listed in this section can be found in the review of Hehl *et al.*[5]

The metric condition

$$\nabla_i g_{jk}=0$$

requires that the connection in U_4 be written as

$$\Gamma^i_{jk}=\left\{ {i \atop jk} \right\}-K_{jk}{}^i\ , \tag{2.2}$$

where

$$K_{ij}{}^k=-S_{ij}{}^k+S_j{}^k{}_i-S^k{}_{ji}$$

is the contortion tensor. Thus $K_{ij}{}^k=K_{ij}{}^k(S_{lm}{}^n,g_{lm})$, which is important in obtaining the field equations by a variational principle.

The Riemann curvature tensor is

$$R_{ijk}{}^l=2\partial_{[i}\Gamma_{j]k}{}^l+2\Gamma_{[i\,|\,m}{}^l\Gamma_{|\,j]k}{}^m$$

and the Einstein tensor is

$$G_{ij}=R_{kij}{}^k-\tfrac{1}{2}g_{ij}R\ ,$$

where the curvature scalar is defined as

$$R=R_{kl}{}^{lk}\ .$$

The field equations in the EC theory are obtained by varying the action

$$I=\int d^4x\left[\sqrt{-g}\ \frac{R}{2k}+L\right] \tag{2.3}$$

with respect to the metric, the torsion, and the other fields which can appear in the source Lagrangian L. We define $k=8\pi G$, where G is the gravitation constant (we use $c=1$). L is usually constructed by using a minimal-coupling procedure which consists in substituting the Riemannian covariant derivative of GR by the U_4 covariant derivative, constructed with the connection (2.2). This procedure can be broken in some cases, if we intend to maintain some fundamental principles, as gauge invariance of Maxwell or Yang-Mills fields[5] and mass conservation of classical fluids.[3] So we will define a Maxwell tensor in the usual way, i.e.,

$$F_{ij}=\partial_i A_j-\partial_j A_i \ ,$$

where A_i is the electromagnetic vector potential.

Variation of the total action with respect to g_{ij} gives

$$G^{ij}-\nabla_k^*(T^{ijk}-T^{jki}+T^{kij})\equiv k\sigma^{ij} \ , \quad (2.4)$$

where

$$\sigma^{ij}=\frac{2}{\sqrt{-g}}\frac{\delta L}{\delta g_{ij}} \quad (2.5)$$

is the dynamical energy-momentum tensor,

$$T_{ij}{}^{k}=S_{ij}{}^{k}+2\delta_{[i}{}^{k}S_{j]l}{}^{l} \quad (2.6)$$

is the modified torsion tensor, and the star derivative is

$$\nabla_i^*\equiv\nabla_i+2S_{ij}{}^{k} \ . \quad (2.7)$$

Variation of the total action with respect to torsion gives, after using (2.3), the algebraic relation

$$T^{ijk}=k\mathcal{T}^{ijk} \ , \quad (2.8)$$

where

$$\mathcal{T}_k{}^{ji}=\frac{1}{\sqrt{-g}}\frac{\delta L}{\delta K_{ij}{}^{k}} \quad (2.9)$$

is the canonical spin tensor.

By using (2.8), we can write the field equation (2.4) as

$$G^{ij}=k\Sigma^{ij} \ , \quad (2.10)$$

where

$$\Sigma^{ij}=\sigma^{ij}+\nabla_k^*(\mathcal{T}^{ijk}-\mathcal{T}^{jki}+\mathcal{T}^{kij}) \quad (2.11)$$

is the canonical (asymmetric) energy-momentum tensor.

It can be proved[5] that Σ^{ij} and $\mathcal{T}^{ijk}$ satisfy the conservation laws

$$\nabla_j^*\Sigma_i{}^{j}=2\Sigma_k{}^{j}S_{ij}{}^{k}+\mathcal{T}_{jk}{}^{l}R_{il}{}^{jk} \quad (2.12)$$

and

$$\nabla_k^*\mathcal{T}_{ij}{}^{k}=\Sigma_{[ij]} \ . \quad (2.13)$$

In Sec. III we introduce a Lagrangian describing the charged spinning fluid with magnetic dipole moment in a U_4 theory.

III. THE CHARGED SPINNING FLUID

We assume the conservation of charge and mass for the spinning fluid, which implies[7] that (charge density)/(mass density) must be constant along the stream lines. It is also assumed that it is not submitted to any dissipative process. Thus we write the charge density current as

$$J^i=e\rho u^i \ . \quad (3.1)$$

Here u^i is the four-velocity of a fluid particle, ρ is the mass density of the fluid, and $e\rho$ is its charge density, with e a constant with dimension of (charge)/(mass).

The magnetic dipole density is

$$M^{ij}=\chi\rho S^{ij} \ , \quad (3.2)$$

where

$$S^{ij}=K(x)(a^{1i}a^{2j}-a^{2i}a^{1j}) \quad (3.3)$$

is the spin tensor of a fluid particle and χ is another constant. a^{μ}, $\mu=1,2,3,4$, are components of an orthonormal tetradic field which is used to dynamically describe the spin.[1] We identify

$$a^{4i}=u^i \ , \quad (3.4)$$

which is sufficient to assure that the constraint

$$S^{ij}u_j=0 \quad (3.5)$$

is satisfied.

The angular velocity[1]

$$\omega_{ij}=\tfrac{1}{2}(\dot a_i{}^{\mu}a_{\mu j}-a_i^{\mu}\dot a_{\mu j}) \quad (3.6)$$

implies that the spin kinetic energy density is written as

$$T_s=\tfrac{1}{2}\rho\omega_{ij}S^{ij}=\rho K a^{1i}\dot a_i^{2} \ . \quad (3.7)$$

The overdot is the proper-time U_4 covariant derivative, $\dot a^{\mu i}=u^j\nabla_j a^{\mu i}$.

The U_4 generalization of the Lagrangian found in Ref. 6 is

$$L/\sqrt{-g}=F+J^iA_i+\tfrac{1}{2}M^{ij}F_{ij}-T_s-\tfrac{1}{4}F_{ij}F^{ij}+\lambda_1(u_iu^i+1)+\lambda_2(\rho u^i)_{;i}+\lambda_3X_{,i}u^i+\lambda_4S_{,i}u^i+\lambda^{11}(a^{1i}a_i^1-1)$$
$$+\lambda^{22}(a^{2i}a_i^2-1)+2\lambda^{12}a^{1i}a_i^2+2\lambda^{14}a^{1i}u_i+2\lambda^{24}a^{2i}u_i \ . \quad (3.8)$$

In expression (3.8),

$$-F=\rho(1+\epsilon) \quad (3.9)$$

is the proper mass-energy density of the fluid. The λ terms are necessary to impose the conservation of the orthonormality of the tetradic field $a^{\mu i}$, the constraint (3.4) and the mass, the entropy S and the particle identity X (Ref. 2). For vanishing EM field, (3.8) is the same Lagrangian as the one proposed by Ray and Smalley.[3] The independent variables we have to take in account in the variation of the total action are g_{ij}, $S_{ij}{}^k$, ρ, S, X, u^i, a^{1i}, a^{2i}, and the Lagrange multipliers λ.

In what follows we will consider some consequences of choosing (3.8) as the source Lagrangian for the U_4 geometry.

IV. SPIN AND TORSION

From (2.9) and (3.8) we see that the canonical spin tensor is

$$\mathcal{T}^{ijk}=\tfrac{1}{2}\rho S^{ij}u^k , \tag{4.1}$$

which has a Weyssenhoff convective form.[5] The constraint (3.5) implies that

$$\mathcal{T}_{ij}{}^j=\mathcal{T}_{ji}{}^j=0 . \tag{4.2}$$

Also, (2.6) and (2.8) give

$$\mathcal{T}_{ij}{}^k=S_{ij}{}^k=\frac{k}{2}\rho S_{ij}u^k , \tag{4.3}$$

and as the torsion is also trace-free, the star-derivative $\nabla^*=\nabla$, which simplies the field equations.

We note that the Lagrangian (3.8) could be used as the source of any gravitation theory constructed in a U_4 space-time. In any of these theories Eq. (4.1) would be valid. On the contrary, Eq. (4.3) is a specific feature of the Einstein-Cartan theory, since it depends on the specific choice of $\sqrt{-g}\ R$ for the gravitational Lagrangian.

Variation of the action with respect to λ_2 gives the continuity equation

$$(\rho u^i)_{;i}=0 . \tag{4.4a}$$

However, from (2.2), (3.5), and (4.3) it is easy to see that it can be written as

$$\nabla_i(\rho u^i)=0 \tag{4.4b}$$

without loss of mass conservation, which could be associated with torsion.[3]

We emphasize that the two forms of the continuity equation, constructed with the Riemannian and the U_4 connections, respectively, would in principle be inequivalent. It is interesting to note that if one imposes the conservation of the form (4.4b) in place of (4.4a), by using an alternative Lagrangian multiplier term in (3.8), the simple convective form of the canonical spin tensor which appears in (4.1) is lost.

The precession equations are obtained if we make a variation with respect to a^{Bi}, $B=1,2$. They are

$$KP_{il}a^{2l}-\rho K\dot{a}_i^2+2\lambda^{11}a_i^1+2\lambda^{12}a_i^2+2\lambda^{14}u_i=0 \tag{4.5}$$

and

$$-KP_{il}a^{1l}+\rho\dot{K}a_i^1+\rho K\dot{a}_i^1+2\lambda^{22}a_i^2+2\lambda^{12}a_i^1+2\lambda^{24}u_i=0 , \tag{4.6}$$

where

$$P_{il}=\rho(\chi F_{il}-2\partial\epsilon/\partial S^{il}) . \tag{4.7}$$

We observe that a $\dot{\rho}$ term does not appear in (4.5) and (4.6) since it was eliminated, with the aid of the continuity equation (4.4).

By contracting (4.6) and (4.7) with a^{1i}, a^{2i}, and u_i, we obtain the explicit form of the Lagrange multipliers

$$\lambda^{12}=\dot{K}=0 , \tag{4.8a}$$

$$\lambda^{14}=\frac{K}{2}(u^iP_{il}+\rho\dot{u}_l)a^{2l} , \tag{4.8b}$$

$$\lambda^{24}=-\frac{K}{2}(u^1P_{il}+\rho\dot{u}_l)a^{1l} , \tag{4.8c}$$

$$\lambda^{11}=\lambda^{22}=\tfrac{1}{2}T_s-\tfrac{1}{4}P_{il}S^{il} . \tag{4.8d}$$

We can substitute (4.8) in (4.5) and (4.6) to obtain

$$\rho K\dot{a}_i^1=-(T_s+\tfrac{1}{2}P_{kl}S^{kl})a_i^2+\rho KB_{il}a^{1l} \tag{4.9}$$

and

$$\rho K\dot{a}_i^2=+(T_s-\tfrac{1}{2}P_{kl}S^{kl})a_i^1+\rho KB_{il}a^{2l} , \tag{4.10}$$

where

$$B_{il}=(u_iu^j+\delta_i^j)\left[\chi F_{jl}-\frac{\partial\epsilon}{\partial S^{jl}}\right]+u_i\dot{u}_l . \tag{4.11}$$

From (4.9) to (4.10) we obtain the equation of motion for the spin,

$$\dot{S}^{ij}=2S^{[i}{}_lB^{j]l} . \tag{4.12}$$

We note that the spin is Fermi-Walker transported along u^i if we take $P_{il}=0$.

The variation of the action with respect to ρ and u^i gives, respectively.

$$\partial F/\partial\rho=T_s/\rho-\frac{\chi}{2}F\cdot S-eA_iu^i+\lambda_{2,i}u^i \tag{4.13}$$

and

$$2\lambda_1u_i-\rho\lambda_{2,i}-\rho Ka^{1j}\nabla_ia_j^2+\lambda_3X_{,i}+\lambda_4S_{,i}+\lambda^{14}a_i^1+\lambda^{24}a_i^2+e\rho A_i=0 , \tag{4.14}$$

which is a potential representation of the four-velocity. By contracting (4.14) with u^i, we get

$$2\lambda^1+T_s+\rho\lambda_{2,i}u^i-e\rho A_iu^i=0 . \tag{4.15}$$

From (4.13) and (4.15) we see that

$$\lambda_1=-(\tfrac{1}{2}M\cdot F+\rho\partial F/\partial\rho) . \tag{4.16}$$

The Maxwell equation[8]

$$F^{ij}{}_{;j}=J^i+M^{ij}{}_{;j} \tag{4.17}$$

is obtained when we perform a variation with respect to A_i. Of course we could have written (4.17) with the U_4-covariant derivatives, but this would bring no new insight.

V. THE ENERGY-MOMENTUM TENSORS AND THE CONSERVATION LAWS

To obtain the dynamical energy-momentum σ^{ij} we have to do a direct but long calculation. From (2.5), (3.8), and (4.7)—(4.16) we arrive at

$$\sigma^{ij}=\rho E u^i u^j+(g^{ij}+u^i u^j)P+\nabla_k[\rho u^{(i}S^{j)k}]$$
$$+[(u^k P_{kl}+\dot{u}_l)u^{(i}-P^{(i}{}_l]S^{j)l}+\sigma^{ij}_{\rm EM}\ , \quad (5.1)$$

where

$$\sigma^{ij}_{\rm EM}=F^{il}F^j{}_l-\tfrac{1}{4}g^{ij}F_{lm}F^{lm} \quad (5.2)$$

is the usual EM energy-momentum tensor,

$$E\equiv 1+\epsilon-(\chi/2)S\cdot F \quad (5.3)$$

and

$$P=\rho^2\frac{\partial\epsilon}{\partial\rho} \quad (5.4)$$

is the pressure.[7]

Expression (5.1) has the same form as the one previously determined in Ref. 6 in the context of a Riemannian space-time.

From (2.11), (4.1), (4.11), and (4.12) we obtain for the canonical energy-momentum tensor the expression

$$\Sigma^{ij}=\sigma^{ij}-\nabla_k(u^{(i}S^{j)k})+\tfrac{1}{2}\rho\dot{S}^{ij}$$
$$=\rho E u_i u_j+(g^{ij}+u^i u^j)P-P^i{}_l S^{jl}$$
$$+(u^k P_{kl}+\rho\dot{u}_l)u^j S^{il}+\sigma^{ij}_{\rm EM}\ . \quad (5.5)$$

We note by (5.3) that

$$\rho E=(\Sigma^{ij}-\sigma^{ij}_{\rm EM})u_i u_j$$
$$\equiv\Sigma^{ij}_{\rm matter}u_i u_j \quad (5.6)$$

and

$$\rho\pi^i=-\Sigma^{ij}_{\rm matter}u_j=\rho[E u^i+S^{il}(\dot{u}_l-P_{lk}u^k/\rho)] \quad (5.7)$$

can be interpreted as the covariant (matter) energy and momentum densities.[9] With (5.7) we can write (4.12) as

$$\frac{DS^{ij}}{D\tau}=2\pi^{[i}u^{j]}+\frac{2}{\rho}S^{[i}{}_l P^{j]l}\ . \quad (5.8)$$

The conservation law (2.13) is written as the identity

$$\tfrac{1}{2}\rho\dot{S}^{ij}=\nabla_k(\mathcal{T}^{ijk}) \quad (5.9)$$

which can be seen from (4.1) and (4.4). On the contrary (2.12) gives the generalized Euler equation

$$\rho\frac{D\pi^i}{D\tau}=-(g^{ij}+u^i u^j)\nabla_j P+F^{im}J_m+\tfrac{1}{2}M^{kl}\nabla^i F_{kl}$$
$$+\tfrac{1}{2}\rho R^i{}_{jkl}u^j S^{kl}$$
$$+2S^{ijk}S_{jl}P^l{}_k-\nabla_j\left[2\rho\frac{\partial\epsilon}{\partial S_i{}^l}S^{jl}\right]\ , \quad (5.10)$$

which can be seen from (2.12), (3.5), (4.4), (4.11), (4.12), and (4.17). This Euler equation also appeared in Ref. 6 and has the correct limit if spin and torsion vanish.[7] In (5.10) the Riemann tensor term, which is called the Mathisson force, seems to be a peculiarity of the EC theory, as pointed out by de Ritis *et al.*[4] However, we have proved[6] that it appears also in torsion-free theories. There, its origin is the identity

$$|\rho u^{(i}S^{j)k}|_{;kj}=-\frac{\rho}{2}R^{\{\}i}{}_{jkl}u^j S^{kl}+\tfrac{1}{2}\nabla^{\{\}}_j(\rho\dot{S}^{ij})\ , \quad (5.11)$$

in a Riemannian space-time, which can be seen with the aid of the usual Ricci calculus.[9]

VI. SOME COMMENTS ABOUT THE FIRST LAW OF THERMODYNAMICS

When we examine in more detail the Euler equation (5.10) we note that by contracting it with u_i, we obtain the strong relation

$$u_i\nabla_j\left[2\rho\frac{\partial\epsilon}{\partial S_i{}^l}S^{jl}\right]=0\ . \quad (6.1)$$

Assuming the first law of thermodynamics as Ray and Smalley,[3]

$$d\epsilon=T\,dS-Pd(1/\rho)+\tfrac{1}{2}\omega_{ij}dS^{ij}\ , \quad (6.2)$$

we have

$$\left[\frac{\partial\epsilon}{\partial S^{ij}}\right]_{S,\rho}=\tfrac{1}{2}\omega_{ij}\ . \quad (6.3)$$

From (6.3) and (6.1) we have

$$u_i\nabla_j(\rho\omega^{il}S^j{}_l)=0\ , \quad (6.4)$$

which has no clear interpretation.

Also, assuming the first law as

$$d\epsilon=T\,dS-Pd(1/\rho)+\Lambda\,dK\ , \quad (6.5)$$

as in the letter of de Ritis *et al.*,[4] it can be proved from (4.4), (6.1), and (6.5) that

$$\dot{\rho}=-\rho\hat{S}^i\hat{S}^j\nabla_j u_i \quad (6.6)$$

which seems to be a meaningless relation. In (6.6) $\hat{S}^i$ is the unit vector of the spin four-vector $S^i=\eta^{ijkl}u_j S_{kl}$.

However, if we assume

$$\frac{\partial\epsilon}{\partial S^{ij}}=0\ , \quad (6.7)$$

i.e., if we suppress the spin term in (6.2) or (6.5), we do not find these spurious features. We point out that the Lagrangian dependence on the spin is complete in the sense of containing all spin interaction terms and its proper spin kinetic energy, so the energy density represented by $\rho\epsilon$ must be spin free, in order to avoid double variations. We note that in different treatments, interaction terms can appear included in ϵ but then they are explicitly absent from the Lagrangian.[10] Another spurious consequence of (6.2) or (6.5) in this kind of Lagrangian formalism is the loss of free Fermi-Walker transport of the spin, which can be seen from (4.11) and (4.12). This law of motion is usually prescribed by the theories consistent with (3.5).[1,2,11]

VII. CONCLUSIONS

We have shown a complete and full consistent Lagrangian treatment of perfect charged spinning fluids with magnetic dipole moment in an Riemann-Cartan space-time. The spinning fluid is the source of curvature, torsion, and EM fields and the complete differential system of the stream lines has been presented. The field equations and the equations of motion have the expected limit when spin, EM interactions, or torsion vanish. We have also clarified some questions about the dependence of the proper energy density on spin and its relations to the Lagrangian formalism.

A characteristic feature of theories submitted to condition (3.5) is the presence of helicoidal solutions even in the absence of interactions. These solutions are also found here, which can be seen from the Euler equation (5.10).

In order to avoid these helicoidal motions, some authors have used alternative spin constraints,[11–15] but not in fluid media. An open question is to known if it is possible to extend these alternative spin constraints to spinning fluids. Work on this question is in progress and will be reported elsewhere.

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