Wave function of an anisotropic universe

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The wave function of the Bianchi type-IX universe with small anisotropy is calculated using the Hartle-Hawking prescription.

I. INTRODUCTION

Hartle and Hawking[1] have recently put forward a prescription for calculation of a wave function of the universe. They applied it to the de Sitter model which has only one gravitational degree of freedom, the scale factor $a(t)$. It should be useful to study this prescription using semiclassical models with more degrees of freedom. Here I investigate the wave function of the slightly anisotropic Bianchi type-IX universe. Similar models were considered by Hawking and Luttrell[2] and by Wright and Moss.[3] The authors of the first paper drew their conclusions analyzing properties of the Wheeler-DeWitt equation rather than solving it, while the second paper mentioned above contains the results of numerical integration of this equation (without the condition that anisotropy is small). I think that the explicit semiclassical calculation should be of interest: it fills the gap between these two approaches. The price paid for the analytical expression for the wave function is the condition that the anisotropy is small.

The paper starts with a brief review of the formalism. Section III is devoted to the semiclassical approximation to the wave function of the slightly anisotropic model with cosmological constant. In the next section the classical radiation is added and the numerical solution of the Wheeler-DeWitt equation is used to estimate the ratio of the anisotropy energy density to the energy density of radiation. The result is well below the present experimental limit.

II. ALGORITHM

The wave function of a closed universe depends only on the three-geometry h_{ij} of a spacelike surface. All information about time is hidden in the h_{ij} because in closed spacetime one cannot move a given spacelike surface back and forth in time. One can then calculate the wave function in the manner of Feynman,

$$\Psi[h_{ij}]=N\int_C dg\, e^{iS(g)}\ , \tag{2.1}$$

where $S(g)$ is the action for a given metric $g_{\mu\nu}$ and the integral is over a class C of four-geometries with the three-dimensional boundary on which the induced metric is h_{ij}. Unfortunately, such an integral is not well defined. First, the integrand oscillates rapidly. Second, the class C may contain metrics of an open spacetime and in such a case we need boundary conditions.

We can try to remedy this first problem by going to Euclidean spacetime (cf. Hawking[4]) where (2.1) becomes

$$\Psi[h_{ij}]=N\int_C dg\, e^{-I(g)}\ . \tag{2.2}$$

However, it does not solve the problem completely, because the Euclidean action $I(g)$ is not positive definite. For example, if we make the conformal transformation $g_{\mu\nu}(x)\to\Omega^2(x)g_{\mu\nu}(x)$ keeping h_{ij} fixed then the action contains the term $-\int(\nabla\Omega)^2d^4x$ and if Ω changes rapidly the functional integral blows up. This problem may be solved by "rotating" the conformal factor $\Omega\to i\Omega$. Then the integral can be calculated and the result analytically continued. For details see Refs. 1 and 5.

The solution of the second problem is the cornerstone of the Hartle and Hawking proposal: they suggest to sum over all compact Euclidean metrics. Then the boundary-value problem is solved because there are no boundaries.

When we sum over all compact spacetimes an additional problem may appear—there may be more than one surface with the same h_{ij}, e.g., in a four-sphere there are two spacelike surfaces with the same radii. Therefore it is convenient to change the variables from h_{ij} to the "K representation" in which we use $\tilde{h}_{ij}=h_{ij}(\det h)^{-1/3}$ and the K trace of the external curvature K_i^j. The transformation formulas are

$$\Psi[\tilde{h}_{ij},K]=\int_0^\infty dh\,\exp\left[-\frac{4}{3l^3}\int d^3x\, h^{1/2}K\right]\Psi[h_{ij}]\ , \tag{2.3}$$

$$\Psi[h_{ij}]=\frac{-1}{2\pi i}\int_\Gamma dK\,\exp\left[\frac{4}{3l^3}\int d^3x\, h^{1/2}K\right]\Psi[\tilde{h}_{ij},K]\ , \tag{2.4}$$

where $h=\det h_{ij}$ and Γ goes from $-i\infty$ to $+i\infty$ at the right of all singularities of $\Psi[\tilde{h}_{ij},K]$. l is the Planck length. For details see Ref. 1.

In quantum mechanics we have the Schrödinger equation for the wave function. Its analog here is called the Wheeler-DeWitt equation,

$$\left[G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}}-{}^{(3)}R(h)h^{1/2}+2\Lambda h^{1/2}\right]\Psi[h_{ij}]=0\ , \tag{2.5}$$

where

$$G_{ijkl}=\tfrac{1}{2}h^{-1/2}(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl})\ . \tag{2.6}$$

Needless to say, $\Psi[h_{ij}]$ cannot be calculated exactly, or at least nobody knows how to do it. Therefore we have to make an approximation. One possible approach is to narrow the class C to a certain minisuperspace. It is equivalent to freezing all but a few gravitational degrees of freedom. Then it is possible to use a semiclassical approximation to write Ψ as $A\exp(-B)$ where B is given by the classical solution of the Einstein equation. Prefactor A is harder; here we shall concentrate entirely on B. In a minisuperspace the Wheeler-DeWitt equation becomes the usual partial differential equation and one can try to solve it.

III. BIANCHI TYPE IX WITH Λ

As we have already said, we consider the Bianchi type-IX minisuperspace with small anisotropy. The metric of such a model is

$$ds^2=-\sigma^2N^2(t)dt^2+\frac{\sigma^2}{4}a^2(t)(e^{2\beta(t)})_{ij}\sigma^i\sigma^j\ , \tag{3.1}$$

where

$$\begin{aligned}\sigma^1&=\cos\Psi\, d\theta+\sin\Psi\sin\theta\, d\phi\ ,\\ \sigma^2&=\sin\Psi\, d\theta-\cos\Psi\sin\Phi\, d\phi\ ,\\ \sigma^3&=d\Psi+\cos\theta\, d\phi\ ,\end{aligned}\tag{3.2}$$

and constant $\sigma^2=l^2/24\pi^2$ has been introduced for future convenience. $N(t)$ is a lapse function. A time-dependent matrix β is traceless and diagonal. A convenient parametrization is (here, and in what follows, we follow Misner, Thorne, and Wheeler[6])

$$\begin{aligned}\beta_{11}&=\beta_++\sqrt{3}\beta_-\ ,\\ \beta_{22}&=\beta_+-\sqrt{3}\beta_-\ ,\\ \beta_{33}&=-2\beta_+\ .\end{aligned}\tag{3.3}$$

Classical equations of motion can be easily found using the Arnowitt-Deser-Misner (ADM) action principles. In this formalism the action takes the form

$$S_E=\frac{1}{16\pi}\int(p_+d\beta_++p_-d\beta_--p_\alpha d\alpha-N\mathscr{H}dt)\ , \tag{3.4}$$

where $\alpha=\ln a$ and the Hamiltonian $\mathscr{H}$ is

$$\mathscr{H}=\frac{e^{-3\alpha}}{3\pi}\left[-p_\alpha{}^2+p_+{}^2+p_-{}^2-\frac{3\pi^2}{8}e^{6\alpha}\{6e^{-2\alpha}[1-V(\beta_+\beta_-)]-2\Lambda\}\right]. \tag{3.5}$$

The potential $V(\beta_+,\beta_-)$ is a complicated mess but for small $\beta_\pm$ it has the simple form

$$V(\beta_+,\beta_-)=8(\beta_+{}^2+\beta_-{}^2)\ . \tag{3.6}$$

This action is to be varied with respect to N, α, $\beta_\pm$, p_α, $p_\pm$. The equations of motion are

$$\frac{\ddot a}{a}+2\left[\frac{\dot a}{a}\right]^2+\frac{2}{a^2}[1-8(\beta_+{}^2+\beta_-{}^2)]-\Lambda=0\ , \tag{3.7}$$

$$\ddot\beta_\pm+3\left[\frac{\dot a}{a}\right]^2\dot\beta_\pm+\frac{\beta_\pm}{8a^2}=0\ . \tag{3.8}$$

For small anisotropy parameters $\beta_\pm$ we can neglect the influence of anisotropy on the expansion rate. Then (3.7) has the well-known solution

$$a(t)=\frac{\cosh Ht}{H},\quad H=\left[\frac{\Lambda}{3}\right]^{1/2}. \tag{3.9}$$

This solution can be easily "rotated" to Euclidean space, $t\to-i(t-\pi H/2)$:

$$a(t)=\frac{\sinh Ht}{H}\ . \tag{3.10}$$

This solution can be, of course, obtained by considering the Euclidean metric from the beginning. The equation for anisotropy becomes

$$\ddot\beta_\pm+3\coth Ht\,\dot\beta_\pm-\frac{\beta_\pm}{8\sinh^2 Ht}=0\ .$$

After substitution $z=\frac{1}{2}(1-i\coth Ht)$ one gets the hypergeometrical equation

$$z(z-1)\beta_\pm''+(-\tfrac{1}{2}+z)\beta_\pm'+\tfrac{1}{8}\beta_\pm=0\ . \tag{3.11}$$

The "hypergeometrical" parameters are[7]

$$a=-1+\frac{3\sqrt{2}}{4},\quad b=-1+\frac{3\sqrt{2}}{4},\quad c=-\tfrac{1}{2}\ . \tag{3.12}$$

z goes from $-i\infty$ to $+i\infty$ and crosses the real line at $\frac{1}{2}$. Therefore we need the solution of (3.11) which is regular at $|z|\to\infty$ $(t\to 0)$. The appropriate choice is

$$\beta_\pm(t)=\beta_\pm^0\,\mathrm{Re}W(\coth Ht)\ , \tag{3.13}$$

where

$$W(z)=z^{-b}F(b,b-c+1,b-a+1,z^{-1})\ . \tag{3.14}$$

Now we can calculate the Euclidean action

$$I=\tfrac{1}{2}\int dt\, a^3\left[-\left[\frac{\dot a}{a}\right]^2+\dot\beta_+{}^2+\dot\beta_-{}^2-\frac{1}{a^2}(1-8\beta_+{}^2-8\beta_-{}^2)+\frac{\Lambda\sigma^3}{3}\right] \tag{3.15}$$

along the classical path which ends at the spacelike surface with a given value of K:

$$I=\frac{-1}{3H^2}\left[1-\frac{\kappa^3}{(1+\kappa^2)^{3/2}}\right]+\frac{\beta_0{}^2}{2H^2}\int_\kappa^\infty dy\, V(y)\ , \tag{3.16}$$

where $\kappa=\sigma K/3H$ and $\beta_0{}^2=(\beta_+^0)^2+(\beta_-^0)^2$,

$$V(y)=\frac{[\mathrm{Re}W'(y)]^2}{(1+y^2)^{1/2}}+\frac{8[\mathrm{Re}W(y)]^2}{(1+y^2)^{3/2}}\ . \tag{3.17}$$

Now, we can write the wave function in the K representation

$$\Psi[K,\beta_\pm]=N\exp(-I^K)\ , \tag{3.18}$$

where

$$I^K = ka^3 + I, \quad k = \frac{H\kappa}{3} . \tag{3.19}$$

Now we can use (2.4) to calculate $\Psi[a,\beta_\pm]$,

$$\Psi[a,\beta_\pm] = \frac{-1}{2\pi i}\int_\Gamma dk\, e^{ka^3}\Psi[k,\beta_\pm] . \tag{3.20}$$

This integral can be calculated using the steepest-descent method. (From now on $H=1$.) If $a<1$ the exponent has two real extrema at

$$\kappa_\pm = \pm\kappa_0 \mp \frac{\beta_0{}^2 V(\kappa_0)}{2a^4(1-a^2)^{1/2}} , \tag{3.21}$$

where

$$\kappa_0 = \left[\frac{1-a^2}{a^2}\right]^{1/2} .$$

The contour of integration Γ can run only through κ_+ and the action is

$$I = -\tfrac{1}{3}[1-(1-a^2)^{3/2}] + \frac{\beta_0{}^2}{2}\left[\frac{\kappa_0}{3a(1-a^2)^{1/2}} + \int_{\kappa_0}^{\infty} dy\, V(y)\right] . \tag{3.22}$$

If $a>1$ we have two complex extrema and the Γ can run through both of them:

$$\tilde\kappa_\pm = \pm i\tilde\kappa_0 \pm i\frac{\beta_0{}^2 V(\pm i\tilde\kappa_0)}{2a^4(a^2-1)^{1/2}} , \tag{3.23}$$

where

$$\tilde\kappa_0 = \left[\frac{a^2-1}{a^2}\right]^{1/2} ,$$

and the action becomes

$$I_\pm = -\tfrac{1}{3}[1 \pm i(a^2-1)^{3/2}] + \frac{\beta_0{}^2}{2}\left[\frac{\mp iV(\pm i\tilde\kappa_0)}{3a(a^2-1)^{1/2}} + \int_{\pm i\tilde\kappa_0}^{\infty} dy\, V(y)\right] . \tag{3.24}$$

One can write

$$V(\pm i\tilde\kappa_0) = \mathrm{Re}V(i\tilde\kappa_0) \pm i\,\mathrm{Im}V(i\tilde\kappa_0) ,$$

$$\int_{\pm i\tilde\kappa_0}^{\infty} dy\, V(y) = \mathrm{Re}IV(\tilde\kappa_0) \pm i\,\mathrm{Im}IV(\tilde\kappa_0) ,$$

where IV stands for "integral of V." The normalization factor N is given by

$$N^{-2} = \int_{C\cup C'} dg\, \bar\Psi[h_{ij}]\Psi[h_{ij}] ,$$

where the integral is over all compact geometries which belong to the class C in the past of h_{ij} and to the class C' in the future of this surface. This integral can be evaluated in the semiclassical approximation. Since the solution with $\beta=0$ has the lowest energy N is the same as in the isotropic case. Putting it all together we can write the wave function as follows:

$a<1$:

$$\Psi[a,\beta_\pm] = \exp\left\{-\left[\frac{(1-a^2)^{3/2}}{3} + \frac{\beta_0{}^2}{2}\left[\frac{V(\kappa_0)}{3a(1-a^2)^{1/2}} + \int_{\kappa_0}^{\infty} dy\, V(y)\right]\right]\right\} , \tag{3.25}$$

$a>1$:

$$\Psi[a,\beta_\pm] = 2\exp\left\{-\left[\frac{\beta_0{}^2}{2}\left[\frac{\mathrm{Im}V(i\tilde\kappa_0)}{3a(a^2-1)^{1/2}} + \mathrm{Re}IV(\tilde\kappa_0)\right]\right]\right\} \tag{3.26}$$

$$\times\cos\left[\frac{(a^2-1)^{3/2}}{3} + \frac{\beta_0{}^2}{2}\left[\frac{\mathrm{Re}V(i\tilde\kappa_0)}{3a(a^2-1)^{1/2}} - \mathrm{Im}IV(\tilde\kappa_0)\right]\right] . \tag{3.27}$$

The integrals have to be calculated numerically. Figures 1 and 2 show $\Psi[a,\beta_\pm]$ as a function of a for two fixed values of β_0.

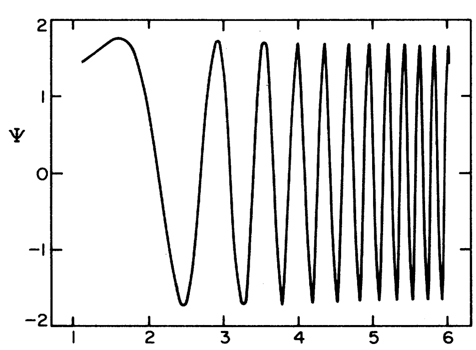

FIG. 1. Wave function as a function of a with $\beta=0.10$ in the semiclassical approximation.

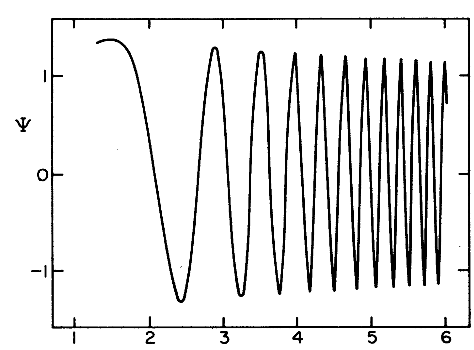

FIG. 2. Wave function as a function of a with $\beta=0.30$ in the semiclassical approximation.

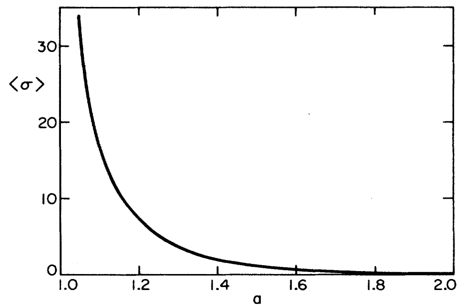

FIG. 3. Energy density of anisotropy $\langle\sigma(a)\rangle$ in the semiclassical approximation.

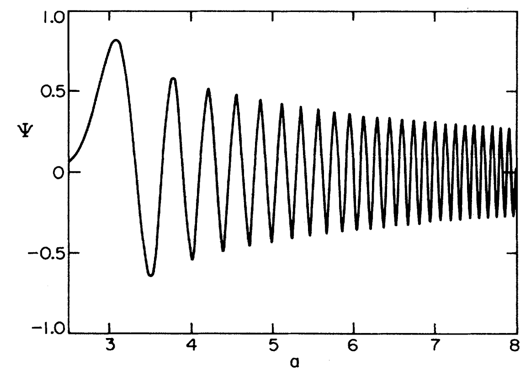

FIG. 4. Wave function for the model with radiation with $\beta=0$ obtained by numerical solution of the Wheeler-DeWitt equation.

We can also calculate the energy density of anisotropy as a function of a. Classically, $\sigma\sim(d\beta/dt)^2$. One can express $\dot{\beta}_\pm$ in terms of canonically conjugated momenta $p_\pm$ and then replace $p_\pm$ by $-i\partial/\partial\beta_\pm$. Doing this we find

$$\hat{\sigma}=\frac{-1}{a^6}\left[\frac{\partial^2}{\partial{\beta_+}^2}+\frac{\partial^2}{\partial{\beta_-}^2}\right] \tag{3.28}$$

and

$$\langle\sigma(a)\rangle=\frac{\int\overline{\Psi}\hat{\sigma}\Psi\,d\beta_+d\beta_-}{\int\overline{\Psi}\Psi\,d\beta_+d\beta_-}\ . \tag{3.29}$$

The result of integration is rather messy:

$$\langle\sigma(a)\rangle=\frac{16}{a^6}\left[A^2-B^2-A+(2AB-A)\times\left[\frac{A\sin 2C+B\cos 2C+[\sin 2C(-A\cos B+B\sin B)-\cos 2C(A\sin B+B\cos B)]}{[(A^2+B^2)/A](e^A-1)+[\cos 2C(-A\cos B+B\sin B)+\sin 2C(A\sin B+B\cos B)]}\right]\right],$$

where

$$A=\frac{1}{2}\left[\frac{\mathrm{Im}V(i\tilde{\kappa}_0)}{3a(a^2-1)^{1/2}}+\mathrm{Re}IV(\tilde{\kappa}_0)\right],$$

$$B=\frac{1}{2}\left[\frac{\mathrm{Re}V(i\tilde{\kappa}_0)}{3a(a^2-1)^{1/2}}-\mathrm{Im}IV(\tilde{\kappa}_0)\right],$$

$$C=\frac{(a^2-1)^{3/2}}{3}-\frac{\pi}{4}\ .$$

$\langle\sigma(a)\rangle$ is plotted on Fig. 3. σ decreases extremely fast, in fact the decrease is quicker than exponential. This can be thought of as an explanation of the isotropy of the universe.

It would be nice to compare the energy density of anisotropy with the energy density of ordinary matter. For this purpose I will now consider the similar model with radiation.

IV. $\Lambda+\gamma$

We can start as previously, simply adding to the action term which describes classical radiation,

$$T_{nn}=\frac{3N}{8\pi}\frac{\gamma}{a^4}\ , \tag{4.1}$$

where γ is a constant which characterizes the density of radiation. The Lorentzian equation for a is (β neglected, as previously)

$$\left[\frac{\dot{a}}{a}\right]^2+\frac{1}{a^2}-H^2-\frac{\gamma}{a^4}=0\ . \tag{4.2}$$

Solutions are

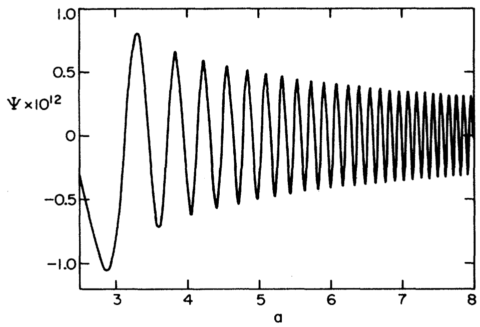

FIG. 5. Wave function for the model with radiation with $\beta=0.45$ obtained by numerical solution of the Wheeler-DeWitt equation.

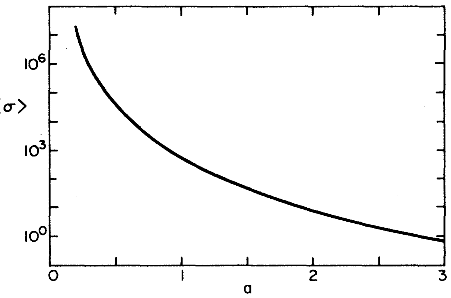

FIG. 6. Energy density of anisotropy $\langle\sigma(a)\rangle$, obtained by numerical integration of the Wheeler-DeWitt equation.

if $\gamma < 1/4H^2$,

$$a(t)=\begin{cases}\dfrac{1}{\sqrt{2}H}[1+(1-4H^2\gamma)^{1/2}\cos 2Ht]^{1/2}, & a>a_+ ,\\ \dfrac{1}{\sqrt{2}H}(\sqrt{\gamma}H\sinh 2Ht-\sinh^2 Ht)^{1/2}, & a<a_- ,\end{cases}$$

$$a_\pm{}^2=[1\pm(1-4H^2\gamma)^{1/2}]/2H^2 ;$$

if $\gamma > 1/4H^2$,

$$a(t)=\begin{cases}\dfrac{1}{H}(\sqrt{\gamma}H\sinh 2Ht-\sinh^2 Ht)^{1/2}, & 0<t<t_0\approx 0.18H^{-1} ,\\ \dfrac{1}{\sqrt{2}H}[1+(4H^2\gamma-1)^{1/2}\sinh H(t-t_0)]^{1/2}, & t>t_0 .\end{cases}$$

We can also find the Euclidean solutions by analytic continuation or directly—by solving the Euclidean equation. However, two serious problems appear. First, technical: It is very difficult to solve the equation for $\beta_\pm(t)$. Second, less technical: The Euclidean solutions are not compact. This difficulty may be circumvented by performing a complex conformal transformation $a=i\alpha$ and solving for α. But even then it is very difficult to calculate the action, rotate it back to a real a, and find extremal points. In such a situation we can try to use the Wheeler-DeWitt equation, which in the present case has the form

$$\left[\frac{\partial^2}{\partial a^2}-\frac{1}{a^2}\left[\frac{\partial^2}{\partial\beta_+{}^2}+\frac{\partial^2}{\partial\beta_-{}^2}\right]+(\Lambda a^4-a^2+\gamma)+8a^2(\beta_+{}^2+\beta_-{}^2)\right]\Psi=0 . \quad (4.3)$$

From the previous calculation we know that Ψ depends on $\beta_\pm$ only through $\beta=(\beta_+{}^2+\beta_-{}^2)^{1/2}$. Therefore we can suppress one variable β and reduce the Wheeler-DeWitt equation to

$$\left[\frac{\partial^2}{\partial a^2}-\frac{1}{a^2}\frac{\partial^2}{\partial\beta^2}+(\Lambda a^4-a^2+\gamma)+8a^2\beta^2\right]\Psi=0 . \quad (4.4)$$

To solve this equation we need the boundary condition, e.g., values of

$$\Psi(0,\beta) \text{ and } \left.\frac{\partial\Psi(a\beta)}{\partial a}\right|_{a=0}$$

which we do not know. We cannot infer them from the path integral since $a=0$ is a turning point and the semiclassical approximation is bound to be wrong. However, if we are interested in the region $a>1$ and we want to estimate $\langle\sigma(a)\rangle$ we do not have to know the boundary condition exactly. $\langle\sigma(a)\rangle$ is rather insensitive to the change of Ψ at $a\approx 0$. The results of the numerical integration are shown in Figs. 4, 5, and 6. The oscillatory behavior is very similar to the result of the semiclassical approximation. The amplitude of Ψ is very strongly damped for bigger values of β. In the region $3<a<10$ the amplitude of Ψ for $\beta=0.5$ is $\approx 10^{11}$ times smaller than the amplitude for $\beta=0$.

Now we can calculate the ratio $\langle\sigma(a)\rangle/\rho_\gamma$. If at present $\rho_\gamma\approx 10^{-33}$ g/cm^3, $\Lambda=3\times 10^{-58}$ cm^2, and $a=10^{30}$ cm then $\langle\sigma\rangle/\rho_\gamma\approx 10^{-78}$ today. Certainly, it does not contradict the present limits on the anisotropy of the background radiation.

V. CONCLUSIONS

It seems to me that these calculations show once again that the Hartle-Hawking algorithm leads to the wave function which agrees with common-sense expectation. However, the study of more complicated models or going beyond the semiclassical approximation requires numerical methods. If they are to be applied to the Wheeler-DeWitt equation the problem of the initial values should be clarified.

ACKNOWLEDGMENTS

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