Quantum radiation in a one-dimensional cavity with moving boundaries

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The quantum theory of a massless free field in a one-dimensional cavity bounded by moving mirrors is formulated in terms of an effective Hamiltonian density which is defined over a fixed length. This effective Hamiltonian is obtained from the original Hamiltonian by the application of a unitary time-dependent transformation with the property that it preserves the reciprocal symmetry of the free field. Using this formulation, the number and the spectrum of the excitations created by the action of the moving mirror on the zero-point energy of the cavity is found in first-order perturbation theory.

I. INTRODUCTION

Creation of particles from the vacuum caused by the motion of boundaries has been the subject of a number of recent studies.[1–7] In most of these works attention has been focused on the problem of a massless field confined to a one-dimensional space. Classically this problem can be solved with the help of the conformal coordinate transformation,[1–4] which enables one to express the wave amplitude in terms of the mode functions of the cavity. From the set of classical solutions one can construct a quantized field theory without making any reference to the Hamiltonian formalism.[1,6] In the present work we discuss an alternative approach based on the Hamiltonian concept and show that it has certain advantages over the conformal-coordinate-transformation method. A review of some of the difficulties associated with the quantization of conformally equivalent systems is given in Sec. II. In Sec. III we discuss the unitary transformation which changes the original Hamiltonian density which is defined over a variable length to a more complicated Hamiltonian defined over a fixed length. This transformation preserves the reciprocal symmetry of the problem, i.e., $\psi\to\pi,\pi\to-\psi$, where ψ and π are the field amplitude and its conjugate momentum density, respectively. The effective Hamiltonian thus obtained can be expanded in terms of the creation and annihilation operators. Two such sets of operators connected to each other by a Bogolubov transformation are considered in Sec. IV, where it is also shown that by applying the Lehmann-Symanzik-Zimmermann (LSZ) method,[8] one can determine the number of the particles which have been created from the vacuum initial state. The resulting equations are solved in first-order perturbation theory in Sec. V and expressions for the number of created particles and the change of the energy of the system are obtained.

II. CONFORMALLY EQUIVALENT SYSTEMS AND THE HAMILTONIAN FORMULATION

Following the pioneering work of Moore,[1] most of the studies of the problem of radiation from moving mirrors have been based on the solution of the conformally equivalent classical wave equation where the boundary conditions are simple. As has been noted by some authors[4] there are certain difficulties associated with the quantization of systems which are conformally equivalent. Among these one can mention the following points: (a) For a given motion of the boundary there is not a unique conformal coordinate transformation which makes the boundary conditions independent of the timelike coordinate. This lack of uniqueness is not problematical as far as the classical solution of the wave equation is concerned; however, there is no reason to believe that these conformally equivalent systems remain equivalent after quantization.[4] (b) In addition to leaving the form of the wave equation unchanged, the conformal transformation has no effect on the classical action; however, the Hamiltonian of the system does not remain invariant under this transformation. In general the Hamiltonian for the new wave equation will not be related to the energy of the system, and therefore, using such a Hamiltonian as the quantal generator of the infinitesimal unfolding of the system in time is not justified. That the acceptable classical Hamiltonians which do not represent the energy of the system will lead to unacceptable quantum-mechanical results is a problem which has been studied in detail in quantum particle dynamics.[8–10] (c) The utility of the conformal transformation is limited to the one-dimensional massless fields. (d) For some simple motions of the mirrors the exact form of the conformal coordinate transformation is known, but for a general motion the transformation can be found only in an approximate form.[1] Let us consider the scalar wave equation

$$\psi_{xx}=\psi_{tt} \tag{2.1}$$

with the boundary conditions

$$\psi(x=0,t)=\psi(x=L(t),t)=0\ , \tag{2.2}$$

where the mirrors are located at the origin and at $x=L(t)$, and they are perfectly reflecting. Now we change the variables from x and t to w and s, where the new variables are defined by

$$w-s=R(t-x)\ , \tag{2.3}$$

$$w+s=R(t+x)\ . \tag{2.4}$$

Here R is a known function which is determined from $L(t)$ (Refs. 1 and 6). A simple calculation shows that Eq. (2.1) expressed in terms of the variables w and s has a form identical to (2.1), viz.,

$$\psi_{ss}=\psi_{ww}\ . \tag{2.5}$$

The wave equation (2.1) is derivable from the Lagrangian density $\mathscr{L}$,

$$\mathscr{L}(\psi_t,\psi_x)=\tfrac{1}{2}(\psi_t{}^2-\psi_x{}^2)\ . \tag{2.6}$$

Using Eqs. (2.3) and (2.4) this Lagrangian density becomes

$$\mathscr{L}(\psi_w,\psi_s)=\frac{1}{2h^2(w,s)}(\psi_w{}^2-\psi_s{}^2)\ , \tag{2.7}$$

where

$$[h(w,s)]^{-2}=R'[R^{-1}(w-s)]R'[R^{-1}(w+s)]\ . \tag{2.8}$$

R' is the derivative of R with respect to its argument. Equation (2.5) can also be obtained from (2.7), in fact the action integral remains invariant under the conformal coordinate transformation,

$$S=\int L\,dt=\int \mathscr{L}(t,x)dx\,dt$$
$$=\int \mathscr{L}(w,s)h^2(w,s)ds\,dw\ , \tag{2.9}$$

where the Jacobian $D(x,t)/D(s,w)=h^2(w,s)$ has been used to obtain (2.9). The Hamiltonian for Eq. (2.1) is given by

$$H=\tfrac{1}{2}\int_0^{L(t)}[\pi^2(x,t)+\psi_x{}^2(x,t)]dx \tag{2.10}$$

and represents the energy of the system. The transformed wave equation, on the other hand is derivable from

$$H'=\tfrac{1}{2}\int_0^{s_0}[P^2(s,t)+\psi_s{}^2]ds\ , \tag{2.11}$$

where s_0 is a fixed length and ψ_s and $P(s,t)$ are the field and its conjugate momentum, respectively. We note that here H' describes the unfolding of the system in the time-like coordinate w and not in t, and unlike the wave equation or the action integral, H' cannot be derived directly from H. Another problem is that of defining the equal-time commutation relations for P and ψ which is needed for quantizing H'. In the following section we start with the quantum-mechanical Hamiltonian (2.10) and obtain an effective Hamiltonian for the system by means of a unitary transformation.

III. THE EFFECTIVE HAMILTONIAN

From the quantal Hamiltonian (2.10), we find that both $\psi(x,t)$ and $\pi(x,t)$ satisfy the same wave equation,

$$\psi_{xx}=\psi_{tt}\ \text{and}\ \pi_{xx}=\pi_{tt}\ , \tag{3.1}$$

and in addition to the boundary conditions (2.2) we have

$$\pi(0,t)=\pi(x=L(t),t)=0 \tag{3.2}$$

at the positions of the mirrors. The equal-time commutation relation between ψ and π expressed as

$$[\psi(x,t),\pi(x',t)]=i\delta(x-x') \tag{3.3}$$

is also to be satisfied. The equations of motion (3.1), the boundary conditions (2.2) and (3.2), and Eq. (3.3) all remain invariant under the reciprocal transformation

$$\psi(x,t)\rightarrow\pi(x,t),\quad \pi(x,t)\rightarrow-\psi(x,t)\ . \tag{3.4}$$

Our aim is to try to find, by means of a unitary transformation, a new Hamiltonian from (2.10) with the following properties.

(a) By changing the variable x to ξ, without changing t, we want to make the boundary conditions (2.2) and (3.2) only dependent on ξ. (b) In simplifying the boundary conditions, the wave equations (3.1) become more complicated, and there are additional terms in these equations representing the interaction between the moving mirror and the field. (c) The transformation should preserve the symmetry between $\psi(x,t)$ and $\pi(x,t)$ as is given by (3.4).

Let L_0 denote the initial length of the cavity and let $\lambda(t)$ be the dimensionless scale factor

$$\lambda(t)=L(t)/L_0 \tag{3.5}$$

and define $V(t)$ by

$$V(t)=\ln\lambda\int_0^{L(t)}x'\left[\frac{\partial\pi(x')}{\partial x'}\right]\psi(x')dx'\ . \tag{3.6}$$

From the expansion

$$e^{iG}\phi(x)e^{-iG}=\phi(x)+\frac{i}{1!}[G,\phi(x)]+\frac{1}{2!}i^2[G[G,\phi]]+\cdots \tag{3.7}$$

which holds for any two operators $G(t)$ and $\phi(x)$ it follows that

$$e^{iV}\pi(x)e^{-iV}=\exp\left[-\ln\lambda\left[x\frac{\partial}{\partial x}\right]\right]\pi(x)$$
$$=\pi\left[\frac{x}{\lambda}\right] \tag{3.8}$$

and

$$e^{iV}\psi(x)e^{-iV}=\frac{1}{\lambda}\exp\left[-\ln\lambda\left[x\frac{\partial}{\partial x}\right]\right]\psi(x)$$
$$=\frac{1}{\lambda}\psi\left[\frac{x}{\lambda}\right]\ . \tag{3.9}$$

Using these relations we can determine the Hamiltonian H, which is obtained from H, Eq. (2.10), by the transformation e^{iV},

$$H_1 = e^{iV}\left[H_0 - i\frac{\partial}{\partial t}\right]e^{-iV}$$
$$= \int_0^{L(t)}\left[\frac{1}{2}\left\{\frac{1}{\lambda^2}\left[\frac{\partial}{\partial x}\psi\left[\frac{x}{\lambda}\right]\right]^2 + \pi^2\left[\frac{x}{\lambda}\right]\right\} - \frac{\lambda_t}{\lambda^2}x\left[\frac{\partial}{\partial x}\pi\left[\frac{x}{\lambda}\right]\right]\psi\left[\frac{x}{\lambda}\right]\right]dx\ , \tag{3.10}$$

where $\lambda_t = d\lambda/dt$. Because of the asymmetry introduced in the transformed field and its conjugate momenta Eqs. (3.8) and (3.9), we apply a second transformation to the operator ψ, π, and H_1. If $W(t)$ denotes the operator

$$W(t) = \tfrac{1}{2}\ln\lambda \int_0^{L(t)} \pi\left[\frac{x'}{\lambda}\right]\psi\left[\frac{x'}{\lambda}\right]\frac{dx'}{\lambda} \tag{3.11}$$

then using the expression (3.7) we find that

$$e^{iW}\pi\left[\frac{x}{\lambda}\right]e^{-iW} = \frac{1}{\lambda^{1/2}}\pi\left[\frac{x}{\lambda}\right] \tag{3.12}$$

and

$$e^{iW}\frac{1}{\lambda}\psi\left[\frac{x}{\lambda}\right]e^{-iW} = \frac{1}{\lambda^{1/2}}\psi\left[\frac{x}{\lambda}\right]. \tag{3.13}$$

Thus the transformation $e^{iW}e^{iV}$ transforms $\psi(x)$ to $(1/\lambda^{1/2})\psi(x/\lambda)$ and $\pi(x)$ to $(1/\lambda^{1/2})\pi(x/\lambda)$, and is a unitary transformation. The effective Hamiltonian is obtained from H_1, Eq. (3.10), and is given by

$$H_{\rm eff} = e^{iW}\left[H_1 - i\frac{\partial}{\partial t}\right]e^{-iW}\ , \tag{3.14}$$

$$H_{\rm eff} = \int_0^{L_0}\left\{\frac{1}{2}\left[\frac{1}{\lambda^2}\left[\frac{\partial\psi}{\partial\xi}\right]^2 + \pi^2(\xi)\right] - \frac{\lambda_t}{\lambda}\left[\xi\frac{\partial\pi}{\partial\xi}\psi(\xi) + \frac{1}{2}\pi(\xi)\psi(\xi)\right]\right\}d\xi\ . \tag{3.15}$$

From this effective Hamiltonian we find the wave equation for $\psi(\xi,t)$:

$$\left[\frac{1}{\lambda^2} - \frac{\lambda_t^2}{\lambda^2}\xi^2\right]\psi_{\xi\xi} - \psi_{tt} + \frac{2\lambda_t}{\lambda}\xi\psi_{t\xi} + \frac{\lambda_t}{\lambda}\psi_t + \left[\frac{\lambda_{tt}}{\lambda} - \frac{3\lambda_t^2}{\lambda^2}\right]\xi\psi_\xi + \left[\frac{\lambda_{tt}}{2\lambda} - \frac{3}{4}\frac{\lambda_t^2}{\lambda^2}\right]\psi = 0\ . \tag{3.16}$$

The momentum density $\pi(\xi,t)$ satisfied an equation identical to (3.16). In terms of the independent variable ξ, the boundary conditions (2.2) simplify to

$$\psi(\xi=0,t) = \psi(\xi=L_0,t) = 0 \tag{3.17}$$

and a similar condition for $\pi(\xi,t)$. Equation (3.16) is hyperbolic for all values of t, and the characteristic curves are given by the differential equation

$$\frac{d\xi}{dt} = -\frac{\lambda_t}{\lambda}\xi \pm \frac{1}{\lambda} \tag{3.18}$$

which can be integrated to yield

$$\lambda\xi \pm t = \text{ a constant}\ . \tag{3.19}$$

We can transform (3.16) to its canonical form by changing from ξ and t variables to u and v, where

$$u = \lambda\xi - t = x - t\ , \tag{3.20}$$

$$v = \lambda\xi + t = x + t\ , \tag{3.21}$$

and then (3.16) becomes

$$4\psi_{uv} - \frac{\lambda_t}{\lambda}\psi_u + \frac{\lambda_t}{\lambda}\psi_v + \left[\frac{\lambda_{tt}}{2\lambda} - \frac{3}{4}\frac{\lambda_t^2}{\lambda^2}\right]\psi = 0\ , \tag{3.22}$$

where in this relation λ, λ_t, and λ_{tt} are functions of $t = \frac{1}{2}(v-u)$. In Eq. (3.22) if we change u and v to the variables x and t, and replace $\psi(x,t)$ by $\lambda^{1/2}(t)\times\psi(x,t)$ we recover the original wave equation (2.1)

IV. THE NUMBER OPERATOR

In order to determine the number of the quanta created by the excitation of the vacuum, we need to define the creation and annihilation operators for the ψ field. In the following discussion, for the sake of convenience, we assume that $L_0 = \pi$, and expand the operators ψ and π in terms of q_k and p_k;

$$\psi(\xi,t) = \left[\frac{2\lambda}{\pi}\right]^{1/2}\sum_{k=1}^{\infty} q_k(t)\sin(k\xi) \tag{4.1}$$

and

$$\pi(\xi,t) = \left[\frac{2}{\pi\lambda}\right]^{1/2}\sum_{k=1}^{\infty} p_k(t)\sin(k\xi)\ . \tag{4.2}$$

The factors $\lambda^{1/2}$ and $\lambda^{-1/2}$ in ψ and π are introduced so that the Hamiltonian in terms of p_k and q_k assume the simple form

$$H_{\rm eff} = \frac{1}{2\lambda(t)}\sum_{k=1}^{\infty}(p_k^2 + k^2 q_k^2) + \frac{\lambda_t}{\lambda}\sum_{k=1}^{\infty}\sum_{j\neq k}(-1)^{k+j}\left[\frac{2kj}{j^2-k^2}\right]p_k q_j\ . \tag{4.3}$$

The reciprocity symmetry between ψ and π, Eq. (3.6) imply the following symmetry rule between p_k and q_k:

$$q_k \to p_k/\lambda,\quad p_k \to -\lambda q_k\ , \tag{4.4}$$

and this interchange leaves the canonical commutation relation unchanged. We observe that in (4.3) the total Hamiltonian consists of two parts, the diagonal part which is the first sum and the off-diagonal part [the double sum in (4.3)] and this part is responsible for changing the number of particles. Expressing p_k and q_k in terms of the creation and annihilation operators as

$$p_k = i\left[\frac{k}{2}\right]^{1/2}(a_k^\dagger - a_k) \tag{4.5}$$

and

$$q_k=\left[\frac{1}{2k}\right]^{1/2}(a_k^\dagger+a_k) \tag{4.6}$$

and substituting in (4.3) we find the following expression for $H_{\rm eff}$:

$$H_{\rm eff}=\frac{1}{\lambda(t)}\sum_{k=1}^{\infty}k\left[a_k^\dagger a_k+\frac{1}{2}\right]+\frac{i\lambda_t}{\lambda}\sum_{k=1}^{\infty}\sum_{j\neq k}(-1)^{k+j}\left[\frac{kj}{j^2-k^2}\right]\times\left[\frac{k}{j}\right]^{1/2}(a_k^\dagger a_j-a_j^\dagger a_k+a_k^\dagger a_j^\dagger-a_k a_j)\ . \tag{4.7}$$

The number of quanta associated with this field is an eigenvalue of the operator

$$N_a=\sum_{k=1}^{\infty}a_k^\dagger a_k\ . \tag{4.8}$$

This operator does not commute with $H_{\rm eff}$, and hence it is not a constant of motion. We can also expand $\psi(\xi,t)$ and $\pi(\xi,t)$ as in (4.1) and (4.2) but without the scale factor $\lambda(t)$, i.e.,

$$\psi(\xi,t)=\left[\frac{1}{\pi k}\right]^{1/2}\sum_{k=1}^{\infty}(b_k^\dagger+b_k)\sin(k\xi) \tag{4.9}$$

and

$$\pi(\xi,t)=i\left[\frac{k}{\pi}\right]^{1/2}\sum_{k=1}^{\infty}(b_k^\dagger-b_k)\sin(k\xi)\ , \tag{4.10}$$

where again $b_k^\dagger$ and b_k are the creation and annihilation operators. In this case the reciprocity symmetry implies

$$q_k\to p_k \text{ and } p_k\to -q_k \tag{4.11}$$

and from Eqs. (4.1), (4.2), (4.9), and (4.10) it follows that the a's and b's are related to each other by a Bogolubov transformation,

$$b_k^\dagger=\tfrac{1}{2}(\lambda^{1/2}+\lambda^{-1/2})a_k^\dagger+\tfrac{1}{2}(\lambda^{1/2}-\lambda^{-1/2})a_k\ . \tag{4.12}$$

The effective Hamiltonian obtained by substituting (4.9) and (4.10) in (3.15) is given by

$$H_{\rm eff}=\sum_{k=1}^{\infty}\left[\frac{1}{2}k\left[1+\frac{1}{\lambda^2}\right]\left[b_k^\dagger b_k+\frac{1}{2}\right]+\frac{1}{4}k\left[\frac{1}{\lambda^2}-1\right](b_k^\dagger b_k^\dagger+b_k b_k)\right]+i\frac{\lambda_t}{\lambda}\sum_{k=1}^{\infty}\sum_{j\neq k}(-1)^{k+j}\frac{kj}{(j^2-k^2)}\left[\frac{k}{j}\right]^{1/2}(b_k^\dagger b_j-b_j^\dagger b_k+b_k^\dagger b_j^\dagger-b_k b_j) \tag{4.13}$$

which is equivalent to (4.7) but has an additional term proportional to $(1/\lambda^2-1)$. The number operator in this case is

$$N_b=\sum b_k^\dagger b_k \tag{4.14}$$

which is not the same as N_a, Eq. (4.8).

From the Hamiltonian (4.7) we can determine the equation of motion for a_k:

$$i\frac{d}{dt}a_k=\frac{k}{\lambda(t)}a_k+iR_k(t) \tag{4.15}$$

and the corresponding equation for $a_k^\dagger$. In Eq. (4.15) R_k denotes the following operator:

$$R_k(t)=\frac{\dot\lambda}{\lambda}\sum_{j\neq k}[K(j,k)a_j+G(j,k)a_k^\dagger],\quad \dot\lambda=\lambda_t \tag{4.16}$$

where

$$K(j,k)=(-1)^{j+k+1}\frac{jk}{k^2-j^2}\left[\left[\frac{k}{j}\right]^{1/2}+\left[\frac{j}{k}\right]^{1/2}\right] \tag{4.17}$$

and

$$G(j,k)=(-1)^{k+j}\frac{jk}{k^2-j^2}\left[\left[\frac{j}{k}\right]^{1/2}-\left[\frac{k}{j}\right]^{1/2}\right]\ . \tag{4.18}$$

We utilize the equation of motion (4.15) and its adjoint to calculate the number of excitations of the a or b fields created from the vacuum. We follow the method of LSZ to calculate the expectation values of N_a and N_b, but give the details of the calculation of the number of quanta for the a field. Only the final result will be presented for the b field. Denoting the initial vacuum state of the system at the time $t=0$ by $|0,\text{ in }\rangle$, the number of the quanta at $t=+\infty$ is given by

$$\langle N_a\rangle=\langle 0,\text{in}\,|\,N_a(\text{out})\,|\,0,\text{in}\rangle=\lim_{t\to\infty}\left\langle 0,\text{in}\left|\sum_{k=1}^{\infty}a_k^\dagger(t)a_k(t)\right|0,\text{in}\right\rangle$$

$$=\int_0^\infty\left\langle 0,\text{in}\left|\sum_k\left[\frac{da_k^\dagger}{dt}a_k+a_k^\dagger\frac{da_k}{dt}\right]\right|\text{in},0\right\rangle dt+\lim_{t\to 0}\left\langle 0,\text{in}\left|\sum_k a_k^\dagger(t)a_k(t)\right|\text{in},0\right\rangle\ . \tag{4.19}$$

The last term in (4.19) is zero and therefore

$$\langle N_a \rangle = \int_0^\infty \langle 0,\text{in} \,|\, a_k^\dagger R_k + R_k^\dagger a_k \,|\, \text{in},0\rangle dt \;, \tag{4.20}$$

where we have substituted for da_k/dt and $da_k^\dagger/dt$ from Eq. (4.15) and its adjoint. Noting that R_k is linear in a_j and $a_j^\dagger$, Eqs. (4.15) and the corresponding equation for $a_k^\dagger$ form a set of linear coupled equations which can be solved to yield the time-dependent operator

$$a_k(t) = a_k[t, a_1(0) \cdots a_n(0) \cdots ; a_1^\dagger(0) \cdots a_n^\dagger(0) \cdots] \tag{4.21}$$

and a similar relation for $a_k^\dagger(t)$. By substituting (4.21) in (4.20), we can determine $\langle N_a \rangle$ exactly. However here we will assume that the last term in (4.15) is small compared to $[k/\lambda(t)]a_k$ and solve (4.15) in the first-order perturbation. If R_k is ignored in (4.15), then the zeroth-order solution of $a_k(t)$ is given by

$$a_k^{(0)}(t) = \exp[-ik\Gamma(t)]a_k(0) \;, \tag{4.22}$$

where

$$\Gamma(t) = \int_0^t \frac{dt'}{\lambda(t')} \;. \tag{4.23}$$

For calculating $a_k(t)$ to the first order we evaluate $R_k^{(0)}(t)$ by substituting $a_j^{(0)}(t)$ and $a_j^{\dagger(0)}(t)$ in (4.16):

$$R_k^{(0)} = \sum_{j\neq k}^{\infty} \int_0^t \frac{\dot\lambda(t_1)}{\lambda(t_1)} e^{ik\Gamma(t_1)} [K(j,k) e^{-ij\Gamma(t_1)} a_j(0) + G(j,k) e^{ij\Gamma(t_1)} a_j^\dagger(0)] dt_1, \quad \dot\lambda(t) = \lambda_t(t) \tag{4.24}$$

and then solve the differential equation

$$\frac{da_k^{(1)}}{dt} = -\frac{ik}{\lambda(t)} a_k^{(1)} + R_k^{(0)}(t) \tag{4.25}$$

to obtain

$$a_k^{(1)}(t) = e^{-ik\Gamma(t)}[a_k(0) + \int_0^t \frac{\dot\lambda(t_1)}{\lambda(t_1)} e^{ik\Gamma(t_1)} \sum_{j\neq k} [K(j,k) e^{-ij\Gamma(t_1)} a_j(0) + G(j,k) e^{ij\Gamma(t_1)} a_j^\dagger(0)] dt_1 \tag{4.26}$$

with a similar expression for $a_k^{\dagger(1)}(t)$. The number of excitations $\langle N_a \rangle$ to the first-order perturbation is given by

$$\langle N_a \rangle = \int_0^\infty \langle 0,\text{in} \,|\, a_k^{\dagger(1)} R_k^{(0)} + R_k^{\dagger(0)} a_k^{(1)} \,|\, \text{in},0\rangle dt \;. \tag{4.27}$$

The expectation value in (4.27) can be calculated by substituting (4.24) and (4.26) in (4.27) with the result that

$$\langle N_a \rangle = 2 \sum_{k=1}^{\infty} \sum_{j\neq k} \int_0^\infty \frac{\dot\lambda(t)}{\lambda(t)} \int_0^t \frac{\dot\lambda(t')}{\lambda(t')} G^2(j,k) \cos[(k+j)(\Gamma(t) - \Gamma(t'))] dt' \;. \tag{4.28}$$

We can simplify (4.28) further by noting that $G^2(j,k)$ depends on kj and $(k+j)$, thus if $F(m)$ denotes the following integral

$$F(m) = \int_0^\infty \frac{\dot\lambda(t)}{\lambda(t)} dt \int_0^t \cos[m(\Gamma(t') - \Gamma(t))] \frac{\dot\lambda(t')}{\lambda(t')} dt' \tag{4.29}$$

then by substituting for $G(j,k)$ in (4.28) and carrying out one of the sums we find

$$\langle N_a \rangle = \sum_{m=1}^{\infty} \frac{m^2-1}{3m} F(m) - \sum_{k=1}^{\infty} \tfrac{1}{2} F(2k) \;. \tag{4.30}$$

To see under what conditions the number of quanta $\langle N_a \rangle$ is finite, i.e., the sum in (4.30) is convergent, let us consider those cases where $\lambda(t)$ is expressible in terms of the variable $z = \Gamma(t)$:

$$z = \Gamma(t) = \int_0^t \frac{dt'}{\lambda(t')} \;. \tag{4.31}$$

Using z as the new variable we have the parametric equation for the motion of the mirror,

$$t = \int_0^z \lambda(x') dz' \;, \tag{4.32}$$

then $F(m)$ can be expressed as

$$F(m) = \int_0^\infty dy \cos(my) f(y) \;, \tag{4.33}$$

where

$$f(y) = \int_0^\infty \frac{d}{dz}[\ln\lambda(z+y)] \frac{d}{dz}[\ln\lambda(z)] dz \;. \tag{4.34}$$

In the next section from the properties of the Fourier transform we obtain certain conditions on $\lambda(t)$ and $\dot\lambda(t)$ so that $\langle N_a \rangle$ converges to a finite result. But before discussing this point let us consider the number of quanta of the b field which is created from the vacuum. Following exactly the same method that we have used to calculate $\langle N_a \rangle$ but using the equations of motion for the b field obtained from the Hamiltonian (4.13) we find the analog of Eq. (4.20):

$$\langle N_b \rangle = \int_0^\infty \left\langle 0,\text{in} \left| \sum_{k=1}^{\infty} \left[\frac{i}{2} k \left[1 - \frac{1}{\lambda^2} \right] (b_k^\dagger b_k^\dagger - b_k b_k) + (b_k^\dagger R_k(b) + R_k^\dagger(b) b \right] \right| \text{in},0 \right\rangle dt \ . \tag{4.35}$$

The expectation value in (4.35) calculated to the first-order perturbation theory yields the following result:

$$\langle N_b \rangle = \sum_{k=1}^{\infty} \tfrac{1}{2} k^2 \int_0^\infty dt \left[1 - \frac{1}{\lambda^2(t)} \right] \int_0^t dt_1 \left[1 - \frac{1}{\lambda^2(t_1)} \right] \cos 2k(t - t_1)$$
$$+ \sum_{k=1}^{\infty} \sum_{j \neq k} \frac{2kj}{(k+j)^2} \int_0^\infty dt \frac{\dot{\lambda}(t)}{\lambda(t)} \int_0^t dt_1 \frac{\dot{\lambda}(t_1)}{\lambda(t_1)} \cos[(k+j)(t_1 - t)] \ . \tag{4.36}$$

Again by changing the variables of integrations, we can simplify (4.36)

$$\langle N_b \rangle = \sum_{k=1}^{\infty} \tfrac{1}{2} k^2 \int_0^\infty dy \cos(2ky) g(y) + \sum_{m=1}^{\infty} \left[\frac{m^2 - 1}{3m} \right] F(m) - \sum_{k=1}^{\infty} \tfrac{1}{2} F(2k) \ , \tag{4.37}$$

where

$$g(y) = \int_0^\infty dt \left[1 - \frac{1}{\lambda^2(t+y)} \right] \left[1 - \frac{1}{\lambda^2(t)} \right] . \tag{4.38}$$

V. RESULTS AND CONCLUSIONS

Under what conditions is the number of quanta created from the vacuum finite? Moore[1] in his work has shown that when there is a discontinuity in the velocity of the mirror, then the number of photons created is infinite; however, if $\lambda(t)$ changes continuously then the number is finite. We will show that the $\langle N_a \rangle$ is infinite if the mirror starts with a nonzero velocity at $t=0$, or if it has any other form of discontinuity in the velocity, otherwise it is finite. As is evident from (4.30), if $F(m)$ goes to zero as $1/m^2$ for large m, then $\langle N_a \rangle$ is logarithmically divergent. On the other hand for $F(m)$ decreasing as $1/m^4$ or faster for large m, $\langle N_a \rangle$ is finite. But $F(m)$ is the Fourier cosine transform of a function of y, Eq. (4.33). According to a result in the theory of Fourier transforms, if $F^{(2r)}(m)$ represents the transform

$$F^{(2r)}(m) = \int_0^\infty dy \cos(my) \frac{d^{(2r)} f(y)}{dy^{(2r)}} \tag{5.1}$$

then

$$F(m) = \int_0^\infty dy \cos(my) f(y) \tag{5.2}$$

can be expressed as[11]

$$F(m) = -\frac{1}{m^2} \left[\frac{df}{dy} \right]_{y=0} + \frac{1}{m^4} \left[\frac{d^3 f}{dy^3} \right]_{y=0} + \cdots$$
$$+ \frac{(-1)^r}{m^{2r}} \left[\frac{d^{2r-1} f}{dy^{2r-1}} \right]_{y=0} + \frac{(-1)^r}{m^{2r}} F^{(2r)}(m) \ . \tag{5.3}$$

From Eqs. (4.30) and (5.3) we conclude that $\langle N_a \rangle$ will be finite if $(df/dy)_{y=0}$ is zero, which in turn implies that

$$\left[\frac{d\lambda}{dz} \right]_{z=0} = 0 \ \text{or} \ \left[\frac{d\lambda(t)}{dt} \right]_{t=0} = 0 \ , \tag{5.4}$$

i.e., if the mirror starts its motion from rest with a finite acceleration. In general if $f(y)$ is continuous but has discontinuous first derivative, then its Fourier transform dies away as $1/m^2$, therefore $df(y)/dy$ must be a continuous function of y for $\langle N_a \rangle$ to be finite. Even if these conditions are met, the change in the energy of the system may or may not be finite. The initial energy of the cavity is just the zero-point energy of the a field,

$$E_i = \sum_{k=1}^{\infty} \tfrac{1}{2} k \tag{5.5}$$

which is quadratically divergent. The final energy is given by

$$E_f = \left\langle 0,\text{in} \left| \sum_{k=1}^{\infty} \frac{k}{\lambda(\infty)} \left[N_k + \frac{1}{2} \right] \right| \text{in},0 \right\rangle$$
$$= \sum_{k=1}^{\infty} \sum_{j \neq k} \frac{2k^2 j}{\lambda(\infty)(j+k)^2} \int_0^\infty \frac{\dot{\lambda}(t)}{\lambda(t)} dt \int_0^t \cos[(k+j)(\Gamma(t') - \Gamma(t))] \frac{\dot{\lambda}(t')}{\lambda(t')} dt' + \sum_{k=1}^{\infty} \frac{k}{2\lambda(\infty)} \ . \tag{5.6}$$

We can write $(E_f - E_i)$ in terms of $F(m)$, Eq. (4.29) as

$$E_f - E_i = \sum_{m=1}^{\infty} \frac{1}{6\lambda(\infty)}(m^2-1)F(m) + \sum_{k=1}^{\infty}\left[\frac{k}{2\lambda(\infty)}(1-F(2k)-\frac{k}{2}\right] . \quad (5.7)$$

Now if the conditions for the finiteness of $\langle N_a \rangle$ are met, and if in addition

$$\lambda(\infty)=\lambda(0)=1 , \quad (5.8)$$

i.e., the initial and the final lengths of the cavity are the same, then $E_f - E_i$ will be finite.

Next let us consider this same question for the b field. In this case we have three infinite sums, Eq. (4.37), with the last two being equal to $\langle N_a \rangle$. Thus for the convergence of $\langle N_b \rangle$ we need to have $\dot{\lambda}(0)=0$, and the additional requirement that $(dg/dy)_{y=0}=0$, where $g(y)$ is defined by (4.38). This last condition implies that $\lambda(\infty)=\lambda(0)$.

When $\lambda(\infty)\neq 1$, then $\langle N_b \rangle$ diverges linearly. The number of excitations for the state of wave number k, $\langle N_b(k) \rangle$ consists of a term which is independent of k, and a number of other terms proportional to $1/k^2$, $1/k^4$, etc. If we omit the term which is independent of k, which in a way is similar to omitting the change in the zero-point excitations, then the result for $\langle N_b \rangle$ will also be finite, but not equal to $\langle N_a \rangle$. Omitting the infinities from the results for $\langle N_b \rangle$ and $E_f - E_i$, we can express these quantities in terms of the second and higher derivatives of λ. Thus from Eqs. (5.3), and the functions $f(y)$ and $g(y)$, Eqs. (4.34) and (4.38), we have

$$\int_0^\infty g(y)\cos(2ky)dy = \frac{2}{(2k)^6}\left[\frac{d^2\lambda}{dt^2}\right]^2_{t=0} - \frac{2}{(2k)^8}\left[\left[\frac{d^3\lambda}{dt^3}\right]^2_{t=0} + \cdots\right] \quad (5.9)$$

and

$$\int_0^\infty f(y)\cos(my)dy = \frac{1}{2m^4}\left[\frac{d^2\lambda}{dt^2}\right]^2_{t=0} + \frac{1}{2m^6}\left[\left[\frac{d^3\lambda}{dt^3}\right]^2_{t=0} + \cdots\right] . \quad (5.10)$$

The contributions from the third- and higher-order derivatives are negligible, since the ratio of two consecutive terms in these expansions is proportional to $1/c^2$. By substituting the leading terms of (5.9) and (5.10) in (4.30), (4.37), and (5.7) we find

$$\langle N_a \rangle = \left[\frac{d^2\lambda}{dt^2}\right]^2_{t=0} [\tfrac{1}{6}(\zeta(3)-\zeta(5)) - \tfrac{1}{64}\zeta(4)] , \quad (5.11)$$

$$\langle N_b \rangle = \frac{1}{6}\left[\frac{d^2\lambda}{dt^2}\right]^2_{t=0} (\zeta(3)-\zeta(5)) \quad (5.12)$$

and

$$E_f - E_i = \frac{1}{12\lambda(\infty)}\left[\frac{d^2\lambda}{dt^2}\right]_{t=0} (\zeta(2)-\zeta(4)) , \quad (5.13)$$

where $\zeta(n)$ is the Riemann ζ function

$$\zeta(n) = \sum_{m=1}^{\infty}\frac{1}{m^n} . \quad (5.14)$$

The result given in Eq. (5.12) is in agreement with the corresponding result given by Moore.[1] Our conclusion about the divergence of the series for $\langle N_a \rangle$, when $\dot{\lambda}(0)\neq 0$, also agrees with Moore's conclusion.

The present method can be used to determine the number of particles created by the moving boundaries in the case of a massive (Klein-Gordon) field. In the nonrelativistic case, i.e., the Schrödinger particles interacting with moving walls, the same technique has been used to show that there is no creation of particles, but the number of particles in a given state, in general, is not a constant of motion, and there are transitions between different energy levels.[12]

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